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The Maximum Norm Analysis of a Nonmatching Grids Method for a Class of Parabolic $p(x)$-Laplacian Equation *

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abstract: Motivated by the work of Boulaaras and Haiour in [7], we provide a maximum norm analysis of Schwarz alternating method for parabolic $p(x)$-Laplacian equation, where an optimal error analysis each subdomain between the discrete Schwarz sequence and the continuous solution of the presented problem is established

Key Words: Maximum norm analysis, Nonmatching grids method, Schwarz sequence, $p(x)$-laplace differential equations.

Contents

1. Introduction

The problem: find $u \in L^2\left(0,T;W_0^{1,p(x)}(\Omega)\right) \cap C^2\left(0,T,W^{-1,p(x)}(\Omega)\right)$ solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_{p(x)}u + \alpha u = f(u), \quad \text{in } \Sigma, \\ u = 0 \text{ in } \Gamma, \\ u(.,0) = u_0, \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where Σ is a set in $\mathbb{R}^2 \times \mathbb{R}$ defined as $\Sigma = \Omega \times [0,T]$ with $T < +\infty$, where Ω is a smooth bounded domain of $\mathbb{R}^2$ with boundary Γ, and $2 < p(x), q(x) \in C^1(\bar{\Omega})$ are function with

$$2 < p^- := \inf_{x\in\Omega} \leq p^+ := \sup_{x\in\Omega} < \infty.$$

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Operator $\Delta_{p(x)}$ is called $p(x)$-Laplacian defined as:

$$\Delta_{p(x)}u = div(|\nabla u|^{p(x)-2}\nabla u)$$

The constant α is assumed to be nonnegative satisfies

$$\alpha \geq \frac{1}{\lambda} > 0, \tag{1.2}$$

f is a regular function such that

$$f \in L^2\left(0, T, L^{p(x)}(\Omega)\right) \cap C^1\left(0, T, L^{q(x)}(\Omega)\right) \text{ with } \frac{1}{p(x)} + \frac{1}{q(x)} = 1.$$

In [7] Boulaaras and Haiour provided a maximum norm analysis of a finite element Schwarz alternating method for a nonlinear parabolic partial differential equations on two overlapping subdomains with nonmatching grids. They considered a domain which is the union of two overlapping subdomains where each subdomain has its own independently generated grid. The two meshes being mutually independent on the overlap region,where a triangle belonging to one triangulation does not necessarily belong to the other one. Then according to Lipschitz assumption, they proved that for each subdomain an optimal error has been estimated by applying uniform norm between the discrete Schwarz sequence and the exact solution of a nonlinear parabolic partial differential equations. In this paper, the same approach can be extended to other types as a linear parabolic partial differential equations see [2] and singularly perturbed advection-diffusion equations (see [11]) using the overlapping domain decomposition method, where we applied it in a full discrete (see [7], [5] and [9]).

In [7], the authors studied the overlapping domain decomposition method combined with a finite element approximation for Laplace equation, where an overlapping Schwarz method on nonmatching grids has been used on uniform norm of and they also proved the geometric convergence on every subdomain.

Aforementioned, in this paper, we can extend the study to $p(x)$-Laplacian equation, where we apply a maximum norm analysis of the finite element Schwarz alternating method of the presented problem on two overlapping subdomains with nonmatching grids. and we are following up the same procedures that have been mentioned above in [7] with respect to the stability analysis which has been given by our previous work in [7], we establish on each subdomain, an optimal error analysis between the discrete Schwarz sequence and the continuous solution of $p(x)$-Laplacian equation. In addition the geometric convergence is proved.

2. Nonlinear parabolic equation with function independent with solution

In this section we consider the parabolic problem and transform it into elliptic system and give some definitions and classical results related to nonlinear elliptic equations with the function f is a regular and independent of the solution u .

We define the space

$$C^+(\bar{\Omega}) = \{\text{continuous function } p(\cdot) : \bar{\Omega} \to \mathbb{R}_+ \text{ such that } 2 < p^- < p^+ < \infty\}$$

where

$$p^- = \min_{x\in\bar{\Omega}} p(x) \text{ and } p^+ = \max_{x\in\bar{\Omega}}.$$

Let $p(\cdot) \in C^+(\Omega)$, we define the Lebesgue space with variable exponent

$$L^{p(\cdot)} = \left\{u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega |u(x)|^{p(x)}\, dx\right\}.$$

endowed with Luxembourg norm :

$$\|u\|_{p(\cdot)} = \|u\|_{L^{p(\cdot)}} = \inf\left\{\varepsilon > 0, \int_{\Omega} |\frac{u(x)}{\varepsilon}|^{p(x)}\, dx \leq 1\right\}$$

The space $(L^{p(\cdot)}(\Omega), \|.\|_{p(\cdot)})$ is a reflexive Banach space, uniformly convex and its dual space is isomorphic to $(L^{p(\cdot)}(\Omega), \|.\|_{q(x)})$ where

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1$$

and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega), |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\| = \|u\|_{p(x)} + \|\nabla u\|_{p(x)},\ u \in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of C_0^∞ in $W^{1,p(x)}(\Omega)$.

2.1. The semi-discrete of parabolic equation

The problem (1.1) can be reformulated into the following continuous parabolic variational equation: find $u \in L^2\left(0, T, W_0^{1,p(x)}(\Omega)\right)$ solution of

$$\begin{cases} \left(\dfrac{\partial u}{\partial t}, v\right) + \mathcal{A}(u,v) + \alpha(u,v) = (f,v), \\ u = 0 \text{ in } \Gamma, \\ u(x,0) = u_0 \text{ in } \Omega, \end{cases} \tag{2.1}$$

where $\mathcal{A}(.,.)$ is the Nonlinear form defined as:

$$u, v \in W_0^{1,p(x)}(\Omega) : \mathcal{A}(u,v) = \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u . \nabla v\, dx + \alpha \int_{\Omega} u.v\, dx. \tag{2.2}$$

The goal of this discretization is transform the parabolic equation into system of the elliptic equations, for this we apply the θ-schema in the equation(2.1). Thus we have, for any $\theta \in [0,1]$ and $k = 1, ..., N$

$$\left(u^k - u^{k-1}, v\right)_{\Omega} + (\Delta t)\,\mathcal{A}\left(u^{\theta,k}, v\right) + \alpha\left(u^{\theta}, v\right) = (\Delta t)\left(f^{\ \theta,k}, v_h\right)_{\Omega}, \tag{2.3}$$

where

$$u^{\theta,k} = \theta u^k + (1-\theta)\, u^{k-1},$$

By multiplying and dividing by θ and by adding $\left(\dfrac{u^{k-1}}{\theta\Delta t}, v\right)$ to both parties of the equalities (2.3), we get

$$\left(\frac{u^{\theta,k}}{\theta\Delta t}, v\right)_{\Omega} + \mathcal{A}\left(u^{\theta,k}, v\right) + \alpha\left(u^{\theta,k}, v\right) = \left(f^{\ \theta,k} + \frac{u^{\theta,k-1}}{\theta\Delta t}, v\right)_{\Omega} \tag{2.4}$$

Then, the problem (2.4) can be reformulated into the following coercive discrete system of elliptic quasi-variational inequalities

$$\mathcal{B}\left(u^{\theta,k}, v\right) + \alpha\left(u^{\theta,k}, v\right) = \left(F^{\theta,k}, v_h\right)_{\Omega},\ v,\ u^{\theta,k} \in V, \tag{2.5}$$

where

$$\begin{cases} \mathcal{B}\left(u^{\theta,k}, v\right) = \mu\left(u_h^{\theta,k}, v_h\right)_{\Omega} + \mathcal{A}\left(u^{\theta,k}, v\right),\ v \in V, \\ \\ \mu = \dfrac{1}{\theta\Delta t} = ... = \dfrac{N}{\theta T} \end{cases} \tag{2.6}$$

and

$$F^{\theta,k} = f^{\theta,k} + \mu u^{\theta,k-1}$$

2.2. Nonlinear elliptic equation

We consider the elliptic problem :find $u^{\theta,k} \in W^{1,p(x)}(\Omega)$

$$\begin{cases} \mathcal{B}\left(u^{\theta,k}, v\right) + \alpha\left(u^{\theta,k}, v\right) = \left(F^{\theta,k}, v\right), \\ u^{\theta,k} = 0 \text{ in } \Gamma, \\ u^{\theta,k}(x,0) = u_0 \text{ in } \Omega, \end{cases} \tag{2.7}$$

Where $\mathcal{B}(u^{\theta,k}, v)$ is Nonlinear form defined as below, and $(F^{\theta,k}, v)$ is linear form defined as $(F^{\theta,k}, v) = \int\limits_{\Omega_i} F^{\theta,k}.v\, dx$,

Remark 2.1. *As well $F^{\theta,k}$ is a regular function, because f it is .*

2.2.1. ***The space discretization.*** Let Ω be decomposed into triangles and τ_h denotes the set of those elements, where $h > 0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform. We consider the usual basis of affine functions φ_i $i = \{1, ..., m(h)\}$ defined by $\varphi_i(M_j) = \delta_{ij}$ where M_j is a vertex of the considered triangulation. We introduce the following discrete spaces V_h of finite element

$$V_h = \left\{ \begin{array}{l} v \in \left(L^2\left(0, T, W_0^{1,p(x)}(\Omega)\right) \cap C\left(0, T, W_0^{1,p(x)}(\bar{\Omega})\right)\right) \\ \text{such that } v_h\,|_K = P_1,\ k \in \tau_h, \\ \\ v_h(.,0) = v_{h0} \text{ in } \Omega, \\ \\ v_h = 0 \text{ in } \Gamma, \end{array} \right\} \tag{2.8}$$

We discretize in space, i.e., we approach the space $W_0^{1,p(x)}$ by a space discretization of finite dimensional $V_h \subset \left(L^2\left(0, T, W_0^{1,p(x)}(\Omega)\right) \cap C\left(0, T, W_0^{1,p(x)}(\bar{\Omega})\right)\right)$, we get the following discrete system of elliptic equations

$$\mathcal{B}\left(u_h^{\theta,k}, v_h\right) + \alpha\left(u_h^{\theta,k}, v_h\right)_\Omega = \left(F^{\theta,k}, v_h\right)_\Omega. \tag{2.9}$$

Theorem 2.2. *see ([24] page 54) . Under suitable regularity of the solution of problem (1.1), there exists a constant C independent of h such that*

$$\|u_h - u\| \leq Ch^2 |\log h|. \tag{2.10}$$

Lemma 2.3. *Let $w \in H^1(\Omega) \cap C(\bar{\Omega})$ satisfies $a(w,\phi) + \lambda(w,\phi) \geq 0$ or all nonnegative $\phi \in H^1(\Omega)$ and $w \geq 0$ on Γ, then $w \geq 0$ on $\bar{\Omega}$.*

Proof. The proof is easy, and the similar to that use in [18] □

Notation 2.4. $(F^{\theta,k}, \varphi^{\theta,k}); (\widetilde{F}^{\theta,k}, \widetilde{\varphi}^{\theta,k})$ *be a pair of data and* $\zeta^{\theta,k} = \partial(F^{\theta,k}, \varphi^{\theta,k})$; $\widetilde{\zeta}^{\theta,k} = \partial(\widetilde{F}^{\theta,k}, \widetilde{\varphi}^{\theta,k})$ *the corresponding solutions to (2.5) .*

Proposition 2.5. *Under the previous notation and lemma 1, we have*

$$\left\|\zeta^{\theta,k} - \zeta^{\theta,k}\right\|_{L\infty(\Omega)} \leq \lambda \left\|F^{\theta,k} - \widetilde{F}^{\theta,k}\right\|_{L\infty(\Omega)}. \tag{2.11}$$

Proof. First, putting

$$\mu^{\theta,k} = \left\|F^{\theta,k} - \widetilde{F}^{\theta,k}\right\|_{L^\infty(\Omega)}, \tag{2.12}$$

then

$$\begin{cases} \widetilde{F}^{\theta,k} \leq F^{\theta,k} + \left\| F^{\theta,k} - \widetilde{F}^{\theta,k} \right\|_{L\infty(\Omega)} \\ \leq F^{\theta,k} + \alpha\lambda \left\| F^{\theta,k} - \widetilde{F}^{\theta,k} \right\|_{L\infty(\Omega)} \\ \leq F^{\theta,k} + \alpha\mu^{\theta,k}. \end{cases}$$

So

$$\begin{array}{rcl} \mathcal{B}\left(\widetilde{\zeta}^{\theta,k}, \phi\right) + \alpha\left(\widetilde{\zeta}^{\theta,k}, \phi\right) & \leq & \mathcal{B}\left(\zeta^{\theta,k}, \phi\right) + \alpha\left(\zeta^{\theta,k}, \phi\right) + \alpha\left(\mu^{\theta,k}, \phi\right) \\ & \leq & \mathcal{B}\left(\zeta^{\theta,k} + \mu^{\theta,k}, \phi\right) + \alpha\left(\zeta^{\theta,k} + \mu^{\theta,k}, \phi\right) \\ & = & \left(F^{\theta,k} + \lambda\mu^{\theta,k}, \phi\right), \end{array} \tag{2.13}$$

for all $\phi \geq 0$, $\phi \in W_0^{1,p(x)}(\Omega)$. On the other hand, we have

$$\zeta^{\theta,k} + \phi - \widetilde{\zeta}^{\theta,k} \geq 0 \text{ on } \Gamma. \tag{2.14}$$

So

$$b(\zeta^{\theta,k} + \phi - \widetilde{\zeta}^{\theta,k}) \geq 0. \tag{2.15}$$

By using the result of lemma 1, we get

$$\widetilde{\zeta}^{\theta,k} + \phi - \zeta^{\theta,k} \geq 0 \text{ on } \overline{\Omega} \tag{2.16}$$

Similarly, interchanging the roles of the couples $(F^{\theta,k}, \varphi^{\theta,k})$ and $(\widetilde{F}^{\theta,k}, \widetilde{\varphi}^{\theta,k})$, we get

$$\widetilde{\zeta}^{\theta,k} + \phi - \zeta^{\theta,k} \geq 0 \text{ on } \overline{\Omega}, \tag{2.17}$$

which completes the proof. □

2.2.2. ***The discrete maximum principle assumption (DMP).*** We assume the matrices resulting from the finite element discretization are M-matrix ([12] and [13]). For convenience in all the sequels, C will be a generic constant independent on h. Then we have the following

Lemma 2.6. *Let $w \in V_h$ satisfy $b(w^{\theta,k}, \phi_s) > 0$ for $s = 1, 2...m(h)$and $w^{\theta,k} \geq 0$ on Γ. Then $w^{\theta,k} \geq 0$ on $(\overline{\Omega})$.*

Proof. The proof is similar to that use in lemma 1. □

Notation 2.7. $(F^{\theta,k}, \varphi^{\theta,k})$; $(\widetilde{F}^{\theta,k}, \widetilde{\varphi}^{\theta,k})$ *be a pair of data and* $\zeta_h^{\theta,k} = \partial(F^{\theta,k}, \varphi^{\theta,k})$; $\widetilde{\zeta}_h^{\theta,k} = \partial(\widetilde{F}^{\theta,k}, \widetilde{\varphi}^{\theta,k})$ *the corresponding solutions to (2.5) .*

Proposition 2.8. *Let DMP hold, we have*

$$\left\| \zeta_h^{\theta,k} - \zeta_h^{\theta,k} \right\|_{L\infty(\Omega)} \leq \lambda \left\| F^{\theta,k} - \widetilde{F}^{\theta,k} \right\|_{L\infty(\Omega)}. \tag{2.18}$$

Proof. The proof is similar to that of the continuous case. □

3. Nonlinear parabolic equation with nonlinear function

Consider the nonlinear elliptic problem :find $u^{\theta,k} \in W_0^{1,p(x)}(\Omega)$

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_{p(x)} u + \alpha u = f(u), & \text{in } \Sigma, \\ u = 0 \text{ in } \Gamma, \\ u(.,0) = u_0, & \text{in } \Omega, \end{cases} \tag{3.1}$$

with Σ defined as blow and the function f is is a nondecreasing nonlinearity, assuming that $f(\cdot)$ is a Lipschitz continuous on $\mathbb{R}$; that is

$$|f(x) - f(y)| \leq K|x - y| \qquad \forall x, y \in \mathbb{R} \tag{3.2}$$

such that

$$\frac{1}{2} < K\lambda < 1 \tag{3.3}$$

Using the semi-discrete parabolic equation and variational formulation of (3.1) we find the following :find $u^{\theta,k} \in W_0^{1,p(x)}(\Omega)$

$$\begin{cases} \mathcal{B}\left(u^{\theta,k}, v\right) + \alpha\left(u^{\theta,k}, v\right) = \left(F^{\theta,k}(u^{\theta,k}), v\right) \\ u^{\theta,k} = 0 \text{ on } \Gamma \end{cases} \tag{3.4}$$

in this case $F^{\theta,k}$ is Nonlinear Lipschitz continuous on R, because f that is.

3.1. Schwarz alternating methods for parabolic equation

We decompose (Ω) in two overlapping smooth subdomain Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$, we denote by $\partial\Omega_i$ the boundary of Ω_i and $\Gamma_i = \partial\Omega_i \cap \Omega_j$ and assume that the intersection of $\overline{\Gamma}_i$ and $\overline{\Gamma}_j; i \neq j$ is empty. Let

$$V_i = \begin{cases} v \in \left(L^2\left(0, T, W_0^{1,p(x)}(\Omega)\right) \cap C\left(0, T, W_0^{1,p(x)}(\bar{\Omega})\right)\right) \\ \text{such that } v = w_j \text{ on } \Gamma_i \text{ and } v = 0 \text{ on } \Gamma \cap \Gamma_i. \end{cases}$$

We associate with problem (3.1) the following system: find $(u_1^{\theta,k}, u_2^{\theta,k}) \in V_1^{\theta,k} \times V_2^{\theta,k}$ solution to

$$\begin{cases} \mathcal{B}_1(u_1^{\theta,k}, v) + \alpha(u^{\theta,k}, v)_{\Omega_1} = (F^{\theta,k}, v)_{\Omega_1}, \\ \mathcal{B}_2(u_2^{\theta,k}, v) + \alpha(u^{\theta,k}, v)_{\Omega_2} = (F^{\theta,k}, v)_{\Omega_2}, \end{cases} \tag{3.5}$$

where

$$u, v \in W_0^{1,p(x)}(\Omega) : \mathcal{B}_i\left(u^{\theta,k}, v\right) = \int_{\Omega_i} |\nabla u^{\theta,k}|^{p(x)-2} \nabla u^{\theta,k} . \nabla v \, dx \tag{3.6}$$

and

$$u_i^{\theta,k} = u^{\theta,k} / \Omega_i; i = 1, 2$$

3.2. The Continuous Schwartz Sequences

Let u_0 be an initialization in $C_0\left(\overline{\Omega}\right)$,i.e., continuous functions vanishing on $\partial\Omega$ such that

$$\mathcal{B}(u_0, v) + \alpha(u_0, v) = (F^{\theta,k}(u_0), v). \tag{3.7}$$

Starting from $u_0 = u_0/\Omega_2$, we respectively define the alternating Schwarz sequences$\left(u_1^{n+1}\right)$on Ω_1 such that $u_1^{\theta,k,n+1} \in V_1^{\left(u_2^{\theta,k,n}\right)}$ solves of

$$\mathcal{B}_1(u_1^{\theta,k,n+1}, v) + \alpha(u_1^{\theta,k,n+1}, k) = (F_1^{\theta,k}(u_1^{\theta,k,n+1}), v), \tag{3.8}$$

and $(u_2^{\theta,k,n+1})$on Ω_2 such that $u_2^{\theta,k,n+1} \in V_2^{\left(\theta,k,u_1^{\theta,k,n+1}\right)}$ solves

$$\mathcal{B}_2(u_2^{\theta,k,n+1},v)+\alpha(u_2^{\theta,k,n+1},v)=(F_1^{\theta,k}(u_2^{\theta,k,n+1}),v), \tag{3.9}$$

Theorem 3.1. *[7] The sequences $(u_h^{n+1});(u_h^{n+1})$, $n\geq 0$ produced by the Schwarz alternating method converge geometrically to a solution u of the elliptic obstacle problem. More precisely, there exist $\delta_1,\delta_2 \in (0,1)$ which depend on (Ω_1,Γ_2) and $(\Omega_2,\Gamma 1)$ such that for all $n\geq 0$,*

$$\sup_{\overline{\Omega}_1}\left|u_h-u^{2n+1}\right|\leq \delta_1^n\delta_2^n\ \sup_{\Gamma_1}\left|u_h-u_h^0\right| \tag{3.10}$$

and

$$\sup_{\overline{\Omega}_2}\left|u_h-u^{2n}\right|\leq \delta_1^n\delta_2^{n-1}\sup_{\Gamma_2}\left|u_h-u_h^0\right|. \tag{3.11}$$

3.3. The discrete Schwartz sequences

As we have defined before, for $i=1,2$, let τ^{h_i} be a standard regular and quasiuniform finite element triangulation in $\Omega_i;h_i$, being the mesh size. The two meshes being mutually independent $\Omega_1\cap\Omega_2$, a triangle belonging to one triangulation does not necessarily belong to the other and for every $w\in C\left(\Omega_i\right)$, we set

$$V_{hi}^{(w_j^{\theta,k})}=\left\{\begin{array}{l} v\in\left(L^2\left(0,T,W_0^{1,p(x)}\left(\Omega\right)\right)\cap C\left(0,T,W_0^{1,p(x)}\left(\bar{\Omega}\right)\right)\right) \\ \\ \text{such that } v=0 \text{ on } \Gamma\cap\Gamma_i;\ v=\pi_{h_i}\left(w\right)\ \text{ on } \Gamma_i, \end{array}\right\}.$$

where π_{h_i}denote an interpolation operator on Γ_{0i}.

Now, we define the discrete counterparts of the continuous Schwarz sequences defined in (3.8) and (3.9) .

Indeed, let u_{0h} be the discrete analog of u_0, defined in (3.7), we respectively, define by $u_{1h}^{\theta,k,n+1}\in V_{h1}^{\left(u_{2h}^{\theta,k,n}\right)}$ such that

$$\mathcal{B}_1(u_{1h}^{\theta,k,n+1},v)+\alpha(u_{1h}^{\theta,k,n+1},v)=(F^{\theta,k}(u_{1h}^{\theta,k,n+1}),v),\forall v\in V_h^{(\varphi)};\ n\geq 0 \tag{3.12}$$

and $u_{2h}^{\theta,k,n+1}\in V_{h2}^{\left(u_{1h}^{\theta,k,n+1}\right)}$ such that

$$\mathcal{B}_2(u_{2h}^{\theta,k,n+1},v)+\alpha(u_{2h}^{\theta,k,n+1},v)=(F^{\theta,k}(u_{2h}^{\theta,k,n+1}),v),\forall v\in V_h^{(\varphi)};\ n\geq 0. \tag{3.13}$$

4. Maximum norm analysis of asymptotic behavior

We begin by introducing two discrete auxiliary sequences and prove a fundamental lemma.

4.1. Two auxiliary Schwarz sequences

For $w_{2h}^0=u_{2h}^0$, we define the sequences $w_{1h}^{\theta,\infty,n+1}$ and $w_{2h}^{\theta,n+1}$ such that $u_{1h}^{\theta,n+1}\in V_{h1}^{\left(u_2^{\theta,n}\right)}$ solves

$$\mathcal{B}_1(w_{1h}^{\theta,n+1},v)+\alpha(w_{1h}^{\theta,\infty,n+1},v)=(F^{\theta,k}(u_{1h}^{\theta,n+1}),v),\forall v\in V_{h1}^{(\varphi)};n\geq 0, \tag{4.1}$$

and $w_{2h}^{\theta,n+1}\in V_{2h}^{\left(u_{1h}^{\theta,n+1}\right)}$ solves

$$\mathcal{B}_2(w_{2h}^{\theta,n+1},v)+\alpha(w_{2h}^{\theta,n+1},v)=(F^{\theta,k}(u_{2h}^{\theta,n+1}),v),\forall v\in V_{h2}^{(\varphi)};n\geq 0, \tag{4.2}$$

respectively. It is then clear that $w_{1h}^{\theta,\infty,n+1}$ and $w_{2h}^{\theta,\infty,n+1}$ are the finite element approximation of $u_1^{\theta,\infty,n+1}$ and $u_2^{\theta,\infty,n+1}$ defined in (4.1), (4.2), respectively. Then, as $F^{\theta,k}\left(.\right)$ is continuous,$\left\|F^{\theta,k}\left(u_i^{\theta,k,n+1}\right)\right\|_\infty$

$\leq \lambda \left\| u_i^{\theta,k,n+1} \right\|_\infty$, (independent i of n). Therefore, making use of standard maximum norm estimates for nonlinear parabolic problems with f independent of solution, we have

$$\left\| u_i^{\theta,k,n} - u_{ih}^{\theta,k,n} \right\|_{L\infty(\Omega_i)} \leq Ch^2 \left| \log h \right| \tag{4.3}$$

where C is a constant independent of both h and n.

Notation 4.1. *From now on, we shall adopt the following notations:* $|.|_1 = |.|_{L\infty(\Gamma_1)}$, $|.|_2 = |.|_{L\infty(\Gamma_2)}$, $\|.\|_1 = \|.\|_{L\infty(\Gamma_1)}$, $\|.\|_2 = \|.\|_{L\infty(\Gamma_2)}$, *and we set* $\pi_{h_1} = \pi_{h_2} = \pi_h$.

4.2. Iterative discrete algorithm

We give our following discrete algorithm

$$u_{ih}^{\theta,k,n+1} = T_h u_{ih}^{k-1,n+1}, k = 1, ..., p,\ u_{ih}^{\theta,k,n+1} \in V_{hi}^{(u_2^{\theta,k,n})} \tag{4.4}$$

where $u_h^{\theta,k}$ is the solution of the problem (3.4) and the first iteration u_h^0 is solution of (3.7).

Lemma 4.2. *under assumption (3), there exists a constant C independent of both h and n such that*

$$\left\| u_i^{\theta,k,n+1} - u_{ih}^{\theta,k,n+1} \right\|_i \leq \frac{Ch^2 \left| \log h \right|}{1-\rho}, \quad i = 1, 2. \tag{4.5}$$

Proof. We know from standard error estimate on uniform norm for linear problem [24] that there exists a constant C independent of h such that

$$\left\| u^0 - u_h^0 \right\|_{L=(\Omega)} \leq Ch^2 \left| \log h \right|. \tag{4.6}$$

Let us now prove (4.5) by induction. Indeed for $n = 1$, using the result of Proposition 1, we have in Ω_1

$$\begin{aligned}
\|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\|_1 &\leq \|u_1^{\theta,k,1} - w_{1h}^{\theta,k,1}\|_1 + \|w_{1h}^{\theta,k,1} - u_{1h}^{\theta,k,1}\|_1 \\
&\leq Ch^2|\log h| + \|w_{1h}^{\theta,k,1} - u_{1h}^{\theta,k,1}\|_1 \\
&\leq Ch^2|\log h| + \lambda\|F^{\theta,k}(u_1^{\theta,k,1}) - F^{\theta,k}(u_{1h}^{\theta,k,1})\|_1 \\
&\leq Ch^2|\log h| + K\lambda\|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\|_1
\end{aligned}$$

Which give

$$\|u_1^{\theta,k,1} - u_{1h}^{\theta,k,1}\|_1 \leq Ch^2|\log h| \tag{4.7}$$

Similar for Ω_2

$$\|u_2^{\theta,k,1} - u_{2h}^{\theta,k,1}\|_2 \leq Ch^2|\log h| \tag{4.8}$$

Now, let assume that

$$\|u_1^{\theta,k,n} - u_{1h}^{\theta,k,n}\|_1 \leq Ch^2|\log h|$$

and prove that,

$$\|u_1^{\theta,k,n+1} - u_{1h}^{\theta,k,n+1}\|_1 \leq Ch^2|\log h|$$

So,

$$\begin{aligned}
\|u_1^{\theta,k,n+1} - u_{1h}^{\theta,k,n+1}\|_1 &\leq \|u_1^{\theta,k,n+1} - w_{1h}^{\theta,k,n+1}\|_1 + \|w_{1h}^{\theta,k,n+1} - u_{1h}^{\theta,k,n+1}\|_1 \\
&\leq Ch^2|\log h| + \|w_{1h}^{\theta,k,n+1} - u_{1h}^{\theta,k,n+1}\|_1 \\
&\leq Ch^2|\log h| + \lambda\|F^{\theta,k}(u_1^{\theta,k,n+1}) - F^{\theta,k}(u_{1h}^{\theta,k,n+1})\|_1 \\
&\leq Ch^2|\log h| + K\lambda\|u_1^{\theta,k,n+1} - u_{1h}^{\theta,k,n+1}\|_1
\end{aligned}$$

Which give that,

$$\|u_1^{\theta,k,n+1} - u_{1h}^{\theta,k,n+1}\|_1 \leq Ch^2|\log h|. \tag{4.9}$$

Similar assume that

$$\|u_2^{\theta,k,n} - u_{2h}^{\theta,k,n}\|_2 \leq Ch^2|\log h|$$

and prove that,

$$\|u_2^{\theta,k,n+1} - u_{2h}^{\theta,k,n+1}\|_2 \leq Ch^2|\log h|.$$

□

Putting an fundamental theorem in present paper,

Theorem 4.3. *Let $h = max\,(h_1, h_2)$. Then, for n large enough, there exists a constant C independent of both h and n such that*

$$\left\|u_i^{\theta,k,n+1} - u_{ih}^{\theta,k,n+1}\right\|_1 \leq Ch^2 \left|\log h\right|, \quad \forall i = 1, 2. \tag{4.10}$$

Proof. Let us give the proof for $i = 1$. The one for $i = 2$ is similar and so will be omitted. Indeed, Let $\delta = \delta_1\delta_2$, then making use of Theorem 2 and Lemma 3, we get

$$\begin{aligned}
\left\|u_1^{\theta,k} - u_{1h}^{\theta,k,n+1}\right\|_1 &\leq \left\|u_1^{\theta,k} - u_1^{\theta,k,n+1}\right\|_1 + \left\|u_1^{\theta,k,n+1} - u_{1h}^{\theta,k,n+1}\right\|_1 \\
&\leq \delta_1^n\delta_2^n \left|u^0 - u\right|_1 + \frac{Ch^2\left|\log h\right|}{1-\rho} \\
&\leq \delta^{2n}\left|u^0 - u\right|_1 + \frac{Ch^2\left|\log h\right|}{1-\rho}.
\end{aligned}$$

So, for n large enough, we have

$$\delta^{2n} \leq h^2 \tag{4.11}$$

and thus

$$\begin{aligned}
\left\|u_1^{\theta,k} - u_{1h}^{\theta,k,n+1}\right\|_1 &\leq Ch^2 + Ch^2\left|\log h\right| \\
&\leq Ch^2\left|\log h\right|,
\end{aligned}$$

which is the desired result. □

4.3. Asymptotic behavior

This section is devoted to the proof of main result of the present paper, where we prove the theorem of the asymptotic behavior in L^∞-norm for parabolic variational inequalities, where we evaluate the variation in L^∞ between $u_h(T)$, the discrete solution calculated at the moment $T = p\Delta t$ and u^∞, the asymptotic continuous solution of (3.4). We begin by introducing new two discrete auxiliary sequences and prove a new fundamental lemma

4.3.1. New two auxiliary Schwarz sequences. For $w_{2h}^0 = u_{2h}^0$, we define the sequences $w_{1h}^{\theta,k,n+1}$ and $w_{2h}^{\theta,k,n+1}$ such that $u_{1h}^{\theta,n+1} \in V_{h1}^{(u_2^{\theta,n})}$ solves

$$\mathcal{B}_1(w_{1h}^{\theta,n+1}, v) + \alpha(w_{1h}^{\theta,\infty,n+1}, v) = (F^{\theta,k}(u_{1h}^{\theta,n+1}), v), \forall v \in V_{h1}^{(\varphi)}; n \geq 0, \tag{4.12}$$

and $w_{2h}^{\theta,n+1} \in V_{2h}^{(u_{1h}^{\theta,n+1})}$ solves

$$\mathcal{B}_2(w_{2h}^{\theta,n+1}, v) + \alpha(w_{2h}^{\theta,n+1}, v) = (F^{\theta,k}(u_{2h}^{\theta,n+1}), v), \forall v \in V_{h2}^{(\varphi)}; n \geq 0, \tag{4.13}$$

respectively. It is then clear that $w_{1h}^{\theta,\infty,n+1}$ and $w_{2h}^{\theta,\infty,n+1}$ are the finite discritisation of $u_1^{\infty,n+1}$ and $u_2^{\infty,n+1}$ defined in (4.12), (4.13), respectively.

Proposition 4.4. *[3]Under the previous hypotheses and notations, we have the following estimate of convergence if* $\theta \geq \frac{1}{2}$

$$\left\| u_h^{\theta,k,n+1} - u_h^{\infty} \right\|_{\infty} \leq \left(\frac{1}{1+\theta \Delta t} \right)^k \left\| u_h^{\infty} - u_{h0} \right\|_{\infty}, \tag{4.14}$$

if $0 \leq \theta < \frac{1}{2}$, *we have*

$$\left\| u_h^{\theta,k,2n+1} - u_h^{\infty} \right\|_{\infty} \leq \left(\frac{2}{2+\theta\left(1-2\theta\right)\rho\left(A\right)} \right)^k \left\| u_h^{\infty} - u_{h0} \right\|_{\infty}, \tag{4.15}$$

where $\rho(A)$ *is the spectral radius of the elliptic operator.*

Remark 4.5. *the last proposition stay true in the case of the function* $f^{\theta,k}$ *independent with the solution* $u^{\theta,k}$

So, for this case (the function $F^{\theta,k}$ dependent of the solution $u^{\theta,k}$) we need the following lemma

Lemma 4.6. *Under the assumption (3), we have the following estimate of convergence if* $\theta \geq \frac{1}{2}$

$$\left\| u_h^{\theta,k,n+1} - u_h^{\infty} \right\|_{\infty} \leq \left(\frac{1}{1+\theta \Delta t} \right)^k \left\| u_h^{\infty} - u_{h0} \right\|_{\infty}, \tag{4.16}$$

if $0 \leq \theta < \frac{1}{2}$, *we have*

$$\left\| u_h^{\theta,k,2n+1} - u_h^{\infty} \right\|_{\infty} \leq \left(\frac{2}{2+\theta\left(1-2\theta\right)\rho\left(A\right)} \right)^k \left\| u_h^{\infty} - u_{h0} \right\|_{\infty}, \tag{4.17}$$

where $\rho(A)$ *is the spectral radius of the elliptic operator.*

Proof. Using the proposition (1) and (3) , the assumption (3) which give that:

Case (1) if $\theta \geq \frac{1}{2}$

$$\begin{aligned}
\left\| u_h^{\theta,k,n+1} - u_h^{\infty} \right\|_{\infty} &\leq \left\| u_h^{\theta,k,n+1} - w_h^{\infty} \right\|_{\infty} + \left\| w_h^{\theta,k,n+1} - u_h^{\infty} \right\|_{\infty} \\
&\leq \left(\frac{1}{1+\theta \Delta t} \right)^k \left\| u_h^{\infty} - u_{h0} \right\|_{\infty} \\
&\quad + \lambda \| F^{\theta,k}(w_h^{\theta,k,n+1}) - F^{\theta,k}(u_h^{\infty}) \| \\
&\leq \left(\frac{1}{1+\theta \Delta t} \right)^k \left\| u_h^{\infty} - u_{h0} \right\|_{\infty} + K\lambda \left\| u_h^{\theta,k,n+1} - u_h^{\infty} \right\|_{\infty} \\
&\leq \frac{1}{1-K\lambda} \left(\frac{1}{1+\theta \Delta t} \right)^k \left\| u_h^{\infty} - u_{h0} \right\|_{\infty}.
\end{aligned} \tag{4.18}$$

Case (2) if $0 \leq \theta \leq \frac{1}{2}$

$$\begin{aligned}\left\|u_h^{\theta,k,2n+1}-u_h^{\infty}\right\|_{\infty} &\leq \left\|u_h^{\theta,k,2n+1}-w_h^{\theta,k,2n+1}\right\|_{\infty}+\left\|w_h^{\theta,k,2n+1}-u_h^{\infty}\right\|_{\infty}\\ &\leq \left(\frac{2}{2+\theta\left(1-2\theta\right)\rho\left(A\right)}\right)^{k}\left\|u_h^{\infty}-u_{h_0}\right\|_{\infty}\\ &\quad+\lambda\|F^{\theta,k}(w_h^{\theta,k,2n+1})-F^{\theta,k}(u_h^{\infty})\|\\ &\leq \left(\frac{2}{2+\theta\left(1-2\theta\right)\rho\left(A\right)}\right)^{k}\left\|u_h^{\infty}-u_{h_0}\right\|_{\infty}\\ &\quad+K\lambda\left\|u_h^{\theta,k,2n+1}-u_h^{\infty}\right\|_{\infty}\\ &\leq \frac{1}{1-K\lambda}\left(\frac{2}{2+\theta\left(1-2\theta\right)\rho\left(A\right)}\right)^{k}\left\|u_h^{\infty}-u_{h_0}\right\|_{\infty}.\end{aligned} \tag{4.19}$$

□

Theorem 4.7. *According to the results of lemma 4 and theorem 3,there exist C independent of both h and $n$such that*
case (1) *if $\theta \geq \frac{1}{2}$*

$$\left\|u_{1h}^{\theta,p,n+1}-u^{\infty}\right\|_{\infty}\leq C\left[h^2\left|\log h\right|+\left(\frac{1}{1+\theta\Delta t}\right)^{p}\right], \tag{4.20}$$

and

$$\left\|u_{2h}^{\theta,p,n+1}-u^{\infty}\right\|_{\infty}\leq C\left[h^2\left|\log h\right|+\left(\frac{1}{1+\theta\Delta t}\right)^{p}\right], \tag{4.21}$$

case (2) *for $0\leq\theta<\frac{1}{2}$*

$$\left\|u_{1h}^{\theta,p,n+1}-u^{\infty}\right\|_{\infty}\leq C\left[h^2\left|\log h\right|+\left(\frac{2}{2+\theta\left(1-2\theta\right)\rho\left(A\right)}\right)^{p}\right] \tag{4.22}$$

and

$$\left\|u_{2h}^{\theta,p,n+1}-u^{\infty}\right\|_{\infty}\leq C\left[h^2\left|\log h\right|+\left(\frac{2}{2+\theta\left(1-2\theta\right)\rho\left(A\right)}\right)^{p}\right], \tag{4.23}$$

where C is a constant independent of h and k.

Proof. We have

$$\left\|u_h^{\theta,p,2n+1}-u^{\infty}\right\|_{\infty}\leq\left\|u_h^{\theta,p,2n+1}-u_h^{\infty}\right\|_{\infty}+\left\|u_h^{\infty}-u^{\infty}\right\|_{\infty}.$$

Using the lemma 4 and theorem 3, we have for $\theta\geq\frac{1}{2}$

$$\left\|u_h^{\theta,p,2n+1}-u^{\infty}\right\|_{\infty}\leq C\left[h^2\left|\log h\right|^3+\left(\frac{1}{1+\theta\Delta t}\right)^{p}\right],$$

and for $0\leq\theta<\frac{1}{2}$ we have

$$\left\|u_h^{\theta,p,2n+1}-u^{\infty}\right\|_{\infty}\leq C\left[h^2\left|\log h\right|^3+\left(\frac{2}{2+\theta\left(1-2\theta\right)\rho\left(\Delta\right)}\right)^{p}\right]$$

In the same proofing that (4.22) and (4.23) . □

Remark 4.8. *It can be seen in the previous estimates (4.20) up to (4.23),* $\left(\frac{1}{1+\beta\theta\Delta t}\right)^p$, $\left(\frac{2}{2+\theta\left(1-2\theta\right)\rho\left(\Delta\right)}\right)^p$ *goes to* 0 *when* p *tend to infinity. Therefore, the estimation order for both the coercive and noncoercive problems is*

$$\left\|u^{\infty}-u_{1h}^{\infty,n+1}\right\|_{L^{\infty}\left(\bar{\Omega}_1\right)}\leq Ch^2\left|\log h\right|^3$$

and

$$\left\|u^{\infty}-u_{2h}^{\infty,n+1}\right\|_{L^{\infty}\left(\bar{\Omega}_2\right)}\leq Ch^2\left|\log h\right|^3.$$

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