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Hölder Regularity for Degenerate Parabolic Equations with Variable Exponents

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Abstract: In this paper, we discuss a class of degenerate parabolic equations with variable exponents. By using the Steklov average and Young's inequality, we establish energy and logarithmic estimates for solutions to these equations. Then based on the intrinsic scaling method, we prove that local weak solutions are locally continuous.

Key Words: Degenerate PDEs, Regularity theory, Intrinsic scaling.

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1. Introduction

Partial differential equations with nonlinearities involving variable exponents have attracted an increasing amount of attention in recent years. The development, mainly by Ruzicka [9,10], of a theory modeling the behavior of electrorheological fluids, an important class of non-Newtonian fluids, seems to have boosted a still far from completed effort to study and understand this type of equations. Other important applications relate to image processing [5], elasticity [13] or flows in porous media [2,3].

We will consider the parabolic equation in divergence form

$$u_t - \operatorname{div}A(x,t,u,\nabla u) = B(x,t,u,\nabla u) \text{ in } \Omega_T, \tag{1.1}$$

where $\Omega_T \equiv \Omega \times (0,T]$, Ω is a bounded domain in $\mathbb{R}^N$ with $N \geq 2$ and $0 < T < +\infty$. The functions $A : \Omega_T \times \mathbb{R}^{N+1} \longrightarrow \mathbb{R}^N$ and $B : \Omega_T \times \mathbb{R}^{N+1} \longrightarrow \mathbb{R}^N$ are assumed to be measurable and satisfying the following structure conditions

$$|A(x,t,u,\nabla u)| \leq C_1 \left(\phi(x,t) + |u|^{p(x,t)-1} + |\nabla u|^{p(x,t)-1}\right), \tag{1.2}$$

$$|B(x,t,u,\nabla u)| \leq C_2 \left(\phi(x,t) + |u|^{p(x,t)-1} + |\nabla u|^{p(x,t)-1}\right), \tag{1.3}$$

$$A(x,t,u,\nabla u)\nabla u \geq C_3 |\nabla u|^{p(x,t)-1}, \tag{1.4}$$

where $\phi(x,t) \in L^\infty(\Omega_T)$ and C_1, C_2, C_3 are positive constant. Throughout this paper, we will always suppose that $p(x,t)$ is a continuous measurable function such that

$$2 < p^- = \inf_{\Omega} p(x,t) \leq p(x,t) \leq \sup_{\Omega} p(x,t) = p^+ < +\infty. \tag{1.5}$$

Results on the existence and uniqueness of weak solutions of (1.1), together with some localization properties, were obtained by P. Wittbold, A. Zimmermann [12], C. Zhang, S. Zhou [14], and recently S. Ouaro, A. Ouedraogo [8].

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Our aim here is to obtain a local regularity result for local weak solutions of (1.1). In order to achieve this goal, and since the equation is degenerate (in fact, the diffusion coefficient vanishes when $|\nabla u| = 0$), the idea is to study the equation within a geometry that takes this feature into consideration.

The building blocks of Dibenedetto's intrinsic scaling is to show that the continuity of the solution at a point follows from measuring its oscillation in a sequence of nested and shrinking cylinders, with vertex at that point, and showing that the oscillation converge to zero as the cylinders shrink to the point. To fully understand the technical procedure, based on the study of an alternative argument which makes use of energy and logarithmic estimates, one has not only to be familiar with Dibenedetto's technique (see [6,7,11]) but also to overcome the difficulty of having a $(x, t)-$dependence on the exponent p.

2. preliminary and main results

The weak solutions of problem (1.1) is understood in the following way.

Definition 2.1. *A local weak solution of (1.1) is a measurable function $u(x, t)$ defined in Ω_T such that*

1. $u \in L^\infty(0, T, L^\infty(\Omega))$ *with* $\nabla u \in L^{p(x,t)}(\Omega_T)$,
2. *for every subset K of Ω and for every subinterval $[t_1, t_2]$ of $(0, T]$*

$$\begin{aligned}\left[\int_K u\varphi\ dx\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \{-u\varphi_t + A(x,t,u,\nabla u).\nabla\varphi\}dxdt \\ = \int_{t_1}^{t_2}\int_K B(x,t,u,\nabla u)\varphi dxdt,\end{aligned} \tag{2.1}$$

for all locally bounded tested functions $\varphi \in C^1(0, T, C_0^\infty(\Omega))$.

We can write (ii) in an equivalent way that is technically more convenient and involves the discrete time derivative. This can be accomplished by using the Steklov average of a function (see [6] for more details), thereby the equivalent formulation reads

1. for every compact $K \subset \Omega$ and every $0 < t < T - h$

$$\int_{K\times\{t\}} \{u_{h,t}\varphi + [A(x,t,u,\nabla u)]_h .\nabla\varphi - [B(x,t,u,\nabla u)]_h \varphi\}dx = 0, \tag{2.2}$$

for all $\varphi \in C_0^\infty(K)$.

Consider a point $(x_0, t_0) \in \Omega_T$, by translation and to simplify assume $(x_0, t_0) = (0, 0)$. Also, let $0 < R < 1$, be sufficiently small such that the cylinder

$$Q(R^2, R) = K_R \times (-R^2, 0) := \{x : \max_{1\le i\le N} |x_i| < R\} \times (-R^2, 0)$$

is a subset of Ω_T and define

$$\mu^+ = \operatorname*{ess\,sup}_{Q(R^2,R)} u,\ \ \mu^- = \operatorname*{ess\,inf}_{Q(R^2,R)} u \text{ and } \omega = \operatorname*{ess\ osc}_{Q(R^2,R)} u = \mu^+ - \mu^-.$$

Define the positive real number $a_0 = \left(\frac{\omega}{2^\lambda}\right)^{2-p^-}$ for some $\lambda > 1$ to be chosen later, and construct the cylinder

$$Q(a_0R^{p^+}, R) = K_R \times (-a_0R^{p^+}, 0).$$

Assuming that

$$R^{\frac{2-p^+}{2-p^-}} < \frac{\omega}{2^\lambda}, \tag{2.3}$$

consequently the inclusion $Q(a_0R^{p^+}, R) \subset Q(R^2, R)$ holds, and so that

$$\operatorname*{ess\ osc}_{Q(a_0R^{p^+},R)} u \le \omega.$$

Remark 2.2. *if (2.3) does not hold, then the essential oscillation ω goes to zero when the radius R goes to zero, and then there is nothing to prove.*

In order to begin our approach, inside $Q(a_0R^{p^+}, R)$ consider subcylinders of small size constructed as follows

$$(0, t^*) + Q(\theta R^{p^+}, R),\ \ \theta = \left(\frac{\omega}{2}\right)^{2-p^-}.$$

These are contained inside $Q(a_0R^{p^+}, R)$ if

$$(2^{p^- -2} - 2^{\lambda(p^- -2)})\frac{R^{p^+}}{\omega^{p^- -2}} < t^* < 0.$$

Now, given $\nu_0 \in (0, 1)$, to be determined in terms of the data, either

$$\left|\left\{(x,t) \in (0,t^*) + Q(\theta R^{p^+}, R) :\ u(x,t) < \mu^- + \frac{\omega}{2}\right\}\right| \leq \nu_0 \left|Q(\theta R^{p^+}, R)\right| \tag{2.4}$$

or, nothing that $\mu^+ - \frac{\omega}{2} = \mu^- + \frac{\omega}{2}$

$$\begin{aligned}&\left|\left\{(x,t) \in (0,t^*) + Q(\theta R^{p^+}, R) :\ u(x,t) > \mu^+ - \frac{\omega}{2}\right\}\right| \\ &\leq (1-\nu_0)\left|Q(\theta R^{p^+}, R)\right|.\end{aligned} \tag{2.5}$$

The analysis of this alternative leads to the following result.

Proposition 2.1. *There exist positive constants ν_0, $\sigma \in (0, 1)$ depending on the data, such that*

$$\underset{Q(\theta(\frac{R}{8})^{p^+}, \frac{R}{8})}{ess\ osc}\ u \leq \sigma\omega. \tag{2.6}$$

An immediate consequence we state the main result of this work.

Theorem 2.3. *Under assumptions (1.2)-(1.5), any local bounded weak solution of (1.1) is locally Hölder continuous.*

3. Local energy and logarithmic estimates

Let τ and ρ be so small that $Q(\tau, \rho) \subset \Omega_T$. Let ξ denote a piecewise smooth cutoff function in $Q(\tau, \rho)$ such that

$$\xi \in [0, 1],\ |\nabla\xi| < \infty \text{ and } \xi(x,t) = 0 \text{ for } x \text{ outside } K_\rho.$$

Proposition 3.1. *Let u be a local weak solution of (1.1) in Ω_T. There exist positive constants C and C' such that, for every cylinder $Q(\tau, \rho) \subset \Omega_T$ and for every $k \in \mathbb{R}$*

$$\begin{aligned}&\sup_{-\tau<t<0} \int_{K_\rho} (u-k)_\pm^2\, \xi^{p^+}(x,t)dx + C\int_{-\tau}^{t}\int_{K_\rho} |\nabla(u-k)_\pm|^{p^-}\, \xi^{p^+}dxdt \\ &\leq \int_{K_\rho} (u-k)_\pm^2\, \xi^{p^+}(x,-\tau)dx + C'\Bigg[\int_{-\tau}^{t}\int_{K_\rho} (u-k)_\pm^2\, \xi^{p^+-1}\xi_t\, dxdt \\ &+ \int_{-\tau}^{t}\int_{K_\rho} (u-k)_\pm^{p^+}\left(|\nabla\xi|^{p^+} + \xi^{p^+}\right)dxdt \\ &+ \int_{-\tau}^{t}\int_{K_\rho} \chi\left((u-k)_\pm > 0\right)dxdt\Bigg].\end{aligned} \tag{3.1}$$

Proof. In the weak formulation (2.2) take the testing function $\varphi = \pm(u_h - k)_\pm \xi^{p^+}$, where

$$(u_h - k)_- = (k - u_h)_+ = \max\{k - u, 0\}.$$

Integrate over $(-\tau, t)$, $t \in (-\tau, 0)$, estimating the various terms separately, we have first

$$\begin{aligned}
\int_{-\tau}^{t}\int_{K_\rho} u_{h,t}\varphi \, dxdt &= \int_{-\tau}^{t}\int_{K_\rho} u_{h,t}\left(\pm(u_h-k)_\pm \xi^{p^+}\right) \, dxdt \\
&\underset{h\longrightarrow 0}{\longrightarrow} -\frac{p^+}{2}\int_{-\tau}^{t}\int_{K_\rho} (u-k)_\pm^2 \, \xi^{p^+-1}(x,t)\xi_t \, dxdt \\
&+\frac{1}{2}\int_{K_\rho} (u-k)_\pm^2 \, \xi^{p^+}(x,t)dx \\
&-\frac{1}{2}\int_{K_\rho} (u-k)_\pm^2 \, \xi^{p^+}(x,-\tau)dx.
\end{aligned} \tag{3.2}$$

For the remaining terms, let $h \longrightarrow 0$ and then use the structure conditions (1.2) – (1.5), then

$$\begin{aligned}
&\pm\int_{-\tau}^{t}\int_{K_\rho} [A(x,t,u,\nabla u)]_h \nabla\left((u_h-k)_\pm \xi^{p^+}\right) dxdt \underset{h\longrightarrow 0}{\longrightarrow} \\
&\int_{-\tau}^{t}\int_{K_\rho} A(x,t,u,\nabla u)\left[\pm\nabla(u-k)_\pm\xi^{p^+} \pm p^+(u-k)_\pm\xi^{p^+-1}\nabla\xi\right] dxdt \\
&\geq C\Bigg(\int_{-\tau}^{t}\int_{K_\rho} |\nabla(u-k)_\pm|^{p(x,t)} \, \xi^{p^+} \, dxdt \\
&-p^+\Bigg[\int_{-\tau}^{t}\int_{K_\rho} \phi(x,t)(u-k)_\pm\xi^{p^+-1}|\nabla\xi|dxdt \\
&+\int_{-\tau}^{t}\int_{K_\rho} |\nabla(u-k)_\pm|^{p(x,t)-1} \, (u-k)_\pm\xi^{p^+-1}|\nabla\xi|dxdt \\
&+\int_{-\tau}^{t}\int_{K_\rho} (u-k)_\pm^{p(x,t)}\xi^{p^+-1}|\nabla\xi|dxdt\Bigg]\Bigg).
\end{aligned} \tag{3.3}$$

By Young's inequality and using the fact that $0 \leq \xi \leq 1$ and $\frac{p(x,t)}{p(x,t)-1} \geq \frac{p^+}{p^+-1}$ imply that $\xi^{\frac{p(x,t)(p^+-1)}{p(x,t)-1}} \leq \xi^{p^+}$, we get

$$\begin{aligned}
&\int_{-\tau}^{t}\int_{K_\rho} |\nabla(u-k)_\pm|^{p(x,t)-1}(u-k)_\pm\xi^{p^+-1}|\nabla\xi|dxdt \\
&\qquad\qquad \leq \varepsilon\int_{-\tau}^{t}\int_{K_\rho} |\nabla(u-k)_\pm|^{p(x,t)} \, \xi^{p^+} dxdt \\
&\qquad\qquad + C(\varepsilon)\int_{-\tau}^{t}\int_{K_\rho} (u-k)_\pm^{p(x,t)}|\nabla\xi|^{p(x,t)}dxdt,
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&\int_{-\tau}^{t}\int_{K_\rho} \phi(x,t)(u-k)_\pm\xi^{p^+-1}|\nabla\xi|dxdt \\
&\qquad \leq \varepsilon'\int_{-\tau}^{t}\int_{K_\rho} (u-k)_\pm^{p^+} \, |\nabla\xi|^{p^+} dxdt \\
&\qquad + C(\varepsilon')\int_{-\tau}^{t}\int_{K_\rho} |\phi(x,t)|^{\frac{p^+}{p^+-1}} \, \xi^{p^+}\chi\left((u-k)_\pm > 0\right) dxdt
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&\int_{-\tau}^{t}\int_{K_\rho}(u-k)_{\pm}^{p(x)}\xi^{p^+-1}|\nabla\xi|dxdt \\
&\qquad\qquad\leq \varepsilon\text{"}\int_{-\tau}^{t}\int_{K_\rho}(u-k)_{\pm}^{p(x)}|\nabla\xi|^{p(x,t)}dxdt \\
&\qquad\qquad+C(\varepsilon\text{"})\int_{-\tau}^{t}\int_{K_\rho}(u-k)_{\pm}^{p(x,t)}\xi^{p^+}dxdt,
\end{aligned}\tag{3.6}$$

where χ_E denotes the characteristic function of the set E and ε, ε' and ε" are positive constants. combining this in (3.3) we arrive at

$$\begin{aligned}
&\int_{-\tau}^{t}\int_{K_\rho}A(x,t,u,\nabla u)\left[\pm\nabla(u-k)_{\pm}\xi^{p^+}\pm p^+(u-k)_{\pm}\xi^{p^+-1}\nabla\xi\right]dxdt \\
&\quad\geq C_1\bigg(\int_{-\tau}^{t}\int_{K_\rho}|\nabla(u-k)_{\pm}|^{p(x,t)}\,\xi^{p^+}\,dxdt \\
&\quad-\int_{-\tau}^{t}\int_{K_\rho}(u-k)_{\pm}^{p^+}|\nabla\xi|^{p^+}dxdt \\
&\quad-\int_{-\tau}^{t}\int_{K_\rho}(u-k)_{\pm}^{p(x,t)}\left(|\nabla\xi|^{p(x,t)}+\xi^{p^+}\right)dxdt \\
&\quad-\int_{-\tau}^{t}\int_{K_\rho}|\phi(x,t)|^{\frac{p^+}{p^+-1}}\,\xi^{p^+}\chi\left((u-k)_{\pm}>0\right)dxdt\bigg).
\end{aligned}\tag{3.7}$$

By the same method, the last term of (2.2) becomes

$$\begin{aligned}
&\int_{-\tau}^{t}\int_{K_\rho}\left|B(x,t,u,\nabla u)(u-k)_{\pm}\xi^{p^+}\right|dxdt \\
&\quad\leq C_2\bigg(\int_{-\tau}^{t}\int_{K_\rho}|\nabla(u-k)_{\pm}|^{p(x,t)}\,\xi^{p^+}\,dxdt+\int_{-\tau}^{t}\int_{K_\rho}(u-k)_{\pm}^{p^+}\xi^{p^+}dxdt \\
&\quad+\int_{-\tau}^{t}\int_{K_\rho}(u-k)_{\pm}^{p(x,t)}\xi^{p^+}dxdt \\
&\quad+\int_{-\tau}^{t}\int_{K_\rho}|\phi(x,t)|^{\frac{p^+}{p^+-1}}\,\xi^{p^+}\chi\left((u-k)_{\pm}>0\right)dxdt\bigg),
\end{aligned}\tag{3.8}$$

where $0<C_2<C_1$.
Using Young's inequality once again we obtain

$$\begin{aligned}
&\int_{-\tau}^{t}\int_{K_\rho}(u-k)_{\pm}^{p(x,t)}\left(|\nabla\xi|^{p(x,t)}+\xi^{p^+}\right)dxdt \\
&\qquad\qquad\leq C\bigg(\int_{-\tau}^{t}\int_{K_\rho}(u-k)_{\pm}^{p^+}\left(|\nabla\xi|^{p^+}+\xi^{p^+}\right)dxdt \\
&\qquad\qquad+\int_{-\tau}^{t}\int_{K_\rho}(1+\xi^{p^+})\chi\left((u-k)_{\pm}>0\right)dxdt\bigg),
\end{aligned}\tag{3.9}$$

and

$$\int_{-\tau}^{t}\int_{K_\rho} |\nabla(u-k)_\pm|^{p^-}\,\xi^{p^+}\,dxdt$$
$$\le C\Big(\int_{-\tau}^{t}\int_{K_\rho} |\nabla(u-k)_\pm|^{p(x,t)}\,\xi^{p^+}dxdt \qquad (3.10)$$
$$+\int_{-\tau}^{t}\int_{K_\rho} \chi\left((u-k)_\pm>0\right)\xi^{p^+}dxdt\Big).$$

Hence, by recalling that $\phi \in L^\infty(\Omega_T)$ and putting (3.7), (3.8), (3.9) and (3.10) into (2.2), we get the desired result. □

Now, introduce the logarithmic function

$$\psi^\pm(u)=\psi\left(H_k^\pm,(u-k)_\pm,c\right)=\left(\ln\left(\frac{H_k^\pm}{H_k^\pm-(u-k)_\pm+c}\right)\right)_+,$$

where $H_k^\pm = \underset{Q(\tau,\rho)}{ess\ \sup}|(u-k)_\pm|$ and $0<c<H_k^\pm$. To avoid the value zero of $\psi^\pm$ we will take our estimates in a smaller sets in K_R where $\psi^\pm$ is a positive function (see sets S_1 in the proof of Lemma 4.2 and S_2 in the proof of Lemma 4.7). In the cylinder $Q(\tau,\rho)$, we take a cutoff function satisfying $\xi\in[0,1]$, $|\nabla\xi|<\infty$ and ξ is independent of $t\in(-\tau,0)$.

Proposition 3.2. *Let u be local weak solution of (1.1) in Ω_T. There exists a positive constant C such that for every cylinder $Q(\tau,\rho)\in\Omega_T$ and for every level $k\in\mathbb{R}$,*

$$\underset{-\tau<t<0}{ess\sup}\int_{K_\rho}\left[\psi^\pm(u)\right]^2\xi^{p^+}dx$$
$$\le \int_{K_\rho\times\{-\tau\}}\left[\psi^\pm(u)\right]^2\xi^{p^+}dx$$
$$+C\Big(\int_{-\tau}^{0}\int_{K_\rho}\psi^\pm(u)\left[(\psi^\pm)'(u)\right]^2\xi^{p^+}dxdt$$
$$+\int_{-\tau}^{0}\int_{K_\rho}\psi^\pm(u)\left[(\psi^\pm)'(u)\right]^{2-p^-}\left(|\nabla u|^{p^+}+1+\xi^{p^+}\right)dxdt \qquad (3.11)$$
$$+\int_{-\tau}^{0}\int_{K_\rho}\psi^\pm(u)\left(|\nabla u|^{p^+}+1+\xi^{p^+}\right)dxdt$$
$$+\int_{-\tau}^{0}\int_{K_\rho}|u|^{p^+}\psi^\pm(u)\left[(\psi^\pm)'(u)\right]^2\xi^{p^+}dxdt\Big).$$

Proof. In (2.2) we take the testing function $\varphi = 2\psi^\pm(u_h)\left[(\psi^\pm)'(u)\right]\xi^{p^+}$. By direct computation we get

$$\left(\psi^\pm(u)\right)''=\left\{\left(\psi^\pm(u)\right)'\right\}^2.$$

Therefore by integrating in time over $(-\tau,t)$ for $t\in(-\tau,0)$, estimate the various terms separately. The first term gives

$$\int_{-\tau}^{t}\int_{K_\rho}u_{h,t}\left\{2\psi^\pm(u_h)\left[(\psi^\pm)'(u_h)\right]\xi^{p^+}\right\}dxdt$$
$$=\int_{-\tau}^{t}\int_{K_\rho}\left(\psi^\pm(u_h)^2\right)_t\xi^{p^+}dxdt \qquad (3.12)$$
$$\underset{h\longrightarrow 0}{\longrightarrow}\int_{K_\rho\times\{t\}}\left[\psi^\pm(u)\right]^2\xi^{p^+}dx-\int_{K_\rho\times\{-\tau\}}\left[\psi^\pm(u)\right]^2\xi^{p^+}dx.$$

For the remaining term, when $h \longrightarrow 0$, we obtain

$$\begin{aligned}
&\int_{-\tau}^{t}\int_{K_\rho} A(x,t,u,\nabla u).\nabla\left(2\psi^{\pm}(u_h)\left[(\psi^{\pm})'(u)\right]\xi^{p^+}\right)dxdt \\
&\quad\geq C\Bigg(\int_{-\tau}^{t}\int_{K_\rho}|\nabla u|^{p(x,t)}\left(1+\psi^{\pm}(u)\right)\left(\left(\psi^{\pm}\right)'(u)\right)^2\xi^{p^+}dxdt \\
&\quad-\int_{-\tau}^{t}\int_{K_\rho}\phi(x,t)\psi^{\pm}(u)\left[(\psi^{\pm})'(u)\right]\xi^{p^+-1}|\nabla\xi|dxdt \\
&\quad-\int_{-\tau}^{t}\int_{K_\rho}|u|^{p(x,t)}\psi^{\pm}(u)\left[(\psi^{\pm})'(u)\right]\xi^{p^+-1}|\nabla\xi|dxdt \\
&\quad-\int_{-\tau}^{t}\int_{K_\rho}|\nabla u|^{p(x,t)-1}\psi^{\pm}(u)\left[(\psi^{\pm})'(u)\right]\xi^{p^+-1}|\nabla\xi|dxdt\Bigg).
\end{aligned} \tag{3.13}$$

Therefore, by the same method we used in Proposition 3.1 and the fact that $\phi \in L^\infty(\Omega_T)$ we obtain that

$$\begin{aligned}
&\int_{-\tau}^{t}\int_{K_\rho} A(x,t,u,\nabla u).\nabla\left(2\psi^{\pm}(u_h)\left[(\psi^{\pm})'(u)\right]\xi^{p^+}\right)dxdt \\
&\quad\geq C_1\Bigg(\int_{-\tau}^{t}\int_{K_\rho}|\nabla u|^{p(x,t)}\left(1+\psi^{\pm}(u)\right)\left(\left(\psi^{\pm}\right)'(u)\right)^2\xi^{p^+}dxdt \\
&\quad-\int_{-\tau}^{t}\int_{K_\rho}\psi^{\pm}(u)\left[(\psi^{\pm})'(u)\right]^2\xi^{p^+}dxdt \\
&\quad-\int_{-\tau}^{t}\int_{K_\rho}|u|^{p^+}\psi^{\pm}(u)\left[(\psi^{\pm})'(u)\right]^2\xi^{p^+}dxdt \\
&\quad-\int_{-\tau}^{t}\int_{K_\rho}\psi^{\pm}(u)\left[(\psi^{\pm})'(u)\right]^{2-p^-}\left(|\nabla\xi|^{p^+}+1\right)dxdt \\
&\quad-\int_{-\tau}^{t}\int_{K_\rho}\psi^{\pm}(u)\left(|\nabla\xi|^{p^+}+1\right)dxdt\Bigg),
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
&\int_{-\tau}^{t}\int_{K_\rho}\left|B(x,t,u,\nabla u)\left(2\psi^{\pm}(u_h)\left[(\psi^{\pm})'(u)\right]\xi^{p^+}\right)\right|dxdt \\
&\leq C_2\Bigg(\int_{-\tau}^{t}\int_{K_\rho}|\nabla u|^{p(x,t)}\psi^{\pm}(u)\left[\left(\psi^{\pm}\right)'(u)\right]^2\xi^{p^+}dxdt \\
&+\int_{-\tau}^{t}\int_{K_\rho}\psi^{\pm}(u)\left[(\psi^{\pm})'(u)\right]^2\xi^{p^+}dxdt \\
&+\int_{-\tau}^{t}\int_{K_\rho}|u|^{p^+}\psi^{\pm}(u)\left[(\psi^{\pm})'(u)\right]^2\xi^{p^+}dxdt \\
&+\int_{-\tau}^{t}\int_{K_\rho}\psi^{\pm}(u)\left[(\psi^{\pm})'(u)\right]^{2-p^-}\xi^{p^+}dxdt+\int_{-\tau}^{t}\int_{K_\rho}\psi^{\pm}(u)\xi^{p^+}dxdt\Bigg),
\end{aligned} \tag{3.15}$$

where $0 < C_2 < C_1$. Hence, putting (3.14) and (3.15) into (2.2) we get the desired result. □

4. continuity of the weak solutions

In this section we analyze the alternative and prove proposition 2.1. By assuming that (2.4) is verified, the following Lemma determine the number ν_0 and guarantee that the solution u is above a smaller level within a smaller cylinder.

Lemma 4.1. *There exists $\nu_0 \in (0,1)$ depending on the data, such that if (2.4) holds true then*

$$u(x,t) > \mu^- + \frac{\omega}{4} \ \text{a.e. in} \ (0,t^*) + Q\left(\theta\left(\frac{R}{2}\right)^{p^+}, \frac{R}{2}\right). \tag{4.1}$$

Proof. Up to translation we may assume that $(0,t^*) = (0,0)$. Define two decreasing sequences of positive numbers

$$R_n = \frac{R}{2} + \frac{R}{2^{n+1}}, \ k_n = \mu^- + \frac{\omega}{4} + \frac{\omega}{2^{n+2}}, \ n = 0, 1, ...,$$

construct the family of nested and shrinking cylinders $Q(\theta R_n^{p^+}, R_n)$, and let $0 \le \xi_n(x,t) \le 1$ be a piecewise smooth functions in $Q(\theta R_n^{p^+}, R_n)$ such that

$$\begin{cases} \xi_n = 1 \text{ in } Q(\theta R_{n+1}^{p^+}, R_{n+1}), \ \xi_n = 0 \text{ on } \partial_p Q(\theta R_n^{p^+}, R_n), \\ |\nabla \xi_n| \le \frac{2^{n+1}}{R}, \ 0 < (\xi_n)_t \le \frac{2^{p^+(n+1)}}{\theta R^{p^+}}. \end{cases}$$

Now, by using the energy inequality (3.1) for the functions $(u-k_n)_-$ we get

$$\begin{aligned} &\sup_{-\theta R_n^{p^+} < t < 0} \int_{K_{R_n}} (u-k_n)_-^2 \, \xi_n^{p^+}(x,t) dx \\ &+ \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} |\nabla (u-k_n)_-|^{p^-} \, \xi_n^{p^+} dx dt \\ &\qquad \le C \Bigg[\int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} (u-k_n)_-^2 \, \xi_n^{p^+-1} (\xi_n)_t \, dx dt \\ &\qquad + \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} \chi\left((u-k_n)_- > 0\right) dx dt \\ &\qquad + \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} (u-k_n)_-^{p^+} \left(|\nabla \xi_n|^{p^+} + \xi_n^{p^+} \right) dx dt \Bigg] \\ &\qquad \le C \frac{2^{p^+(n+1)}}{R^{p^+}} \Bigg(\frac{1}{\theta} \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} (u-k_n)_-^2 \, dx dt \\ &\qquad + \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} (u-k_n)_-^{p^+} \, dx dt \\ &\qquad + \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} \chi\left((u-k_n)_- > 0\right) dx dt \Bigg). \end{aligned} \tag{4.2}$$

Using the fact that $(u-k_n)_- = 0$ or

$$(u-k_n)_- = (\mu^- - u) + \frac{\omega}{4} + \frac{\omega}{2^{n+2}} \le \frac{\omega}{2}, \tag{4.3}$$

we get

$$(u-k_n)_-^2 \ge \theta (u-k_n)_-^{p^-}. \tag{4.4}$$

Then the above estimates read

$$\begin{aligned}
&\sup_{-\theta R_n^{p^+}<t<0}\int_{K_{R_n}} (u-k_n)_-^{p^-}\,\xi_n^{p^+}(x,t)dx\\
&+\frac{1}{\theta}\int_{-\theta R_n^{p^+}}^0\int_{K_{R_n}} |\nabla(u-k_n)_-|^{p^-}\,\xi_n^{p^+}dxdt\\
&\le C\frac{2^{p^+(n+1)}}{\theta R^{p^+}}\left\{\left(\frac{\omega}{2}\right)^{p^+}+1\right\}\left(\int_{-\theta R_n^{p^+}}^0\int_{K_{R_n}}\chi\left((u-k_n)_->0\right)dxdt\right)\\
&\le C\frac{2^{np^+}}{\theta R^{p^+}}\left(\frac{\omega}{2}\right)^{p^+}\left(\int_{-\theta R_n^{p^+}}^0\int_{K_{R_n}}\chi\left((u-k_n)_->0\right)dxdt\right).
\end{aligned}\tag{4.5}$$

Here we used Young's inequality for the right terms.

Let us now consider the change of variables $\overline{t}=\frac{t}{\theta}$ and define the functions

$$\overline{u}(.,\overline{t})=u(.,t),\ \overline{\xi_n}(.,\overline{t})=\xi_n(.,t).$$

Then, for

$$A_n=\int_{-R_n^{p^+}}^0\int_{K_{R_n}}\chi\left((\overline{u}-k_n)_->0\right)dxd\overline{t},$$

the inequality (4.5) becomes

$$\begin{aligned}
&\sup_{-R_n^{p^+}<t<0}\int_{K_{R_n}} (\bar{u}-k_n)_-^{p^-}\,\bar{\xi}_n^{\,p^+}(x,t)dx\\
&\qquad+\int_{-R_n^{p^+}}^0\int_{K_{R_n}} |\nabla(\bar{u}-k_n)_-|^{p^-}\,\bar{\xi}_n^{\,p^+}dxd\bar{t}\le C\frac{2^{np^+}}{R^{p^+}}\left(\frac{\omega}{2}\right)^{p^+}A_n.
\end{aligned}\tag{4.6}$$

Next, by using Hölder's inequality, Proposition 3.1 of Chapter I in [6] and (4.6), we get

$$\begin{aligned}
&\int_{-R_n^{p^+}}^0\int_{K_{R_n}} (\bar{u}-k_n)_-^{p^-}\,\bar{\xi}_n^{\,p^+}dxd\bar{t}\\
&\quad\le\left\{\int_{-R_n^{p^+}}^0\int_{K_{R_n}}\left[(\bar{u}-k_n)_-\,\bar{\xi}_n^{\frac{p^+}{p^-}}\right]^{\frac{p^-(N+p^-)}{N}}dxd\bar{t}\right\}^{\frac{N}{N+p^-}}A_n^{\frac{p^-}{N+p^-}}\\
&\quad\le C\left[\sup_{-R_n^{p^+}<\bar{t}<0}\int_{K_{R_n}}(\bar{u}-k_n)_-^{p^-}\,\bar{\xi}_n^{\,p^+}(x,\bar{t})dx\right]^{\frac{p^-}{N+p^-}}\\
&\quad\times\left[\int_{-R_n^{p^+}}^0\int_{K_{R_n}}|\nabla(\bar{u}-k_n)_-|^{p^-}\,\bar{\xi}_n^{\,p^+}dxd\bar{t}\right.\\
&\quad\left.+\int_{-R_n^{p^+}}^0\int_{K_{R_n}}(\bar{u}-k_n)_-^{p^-}\left|\nabla\bar{\xi}_n\right|^{p^-}dxd\bar{t}\right]^{\frac{N}{N+p^-}}A_n^{\frac{p^-}{N+p^-}}\\
&\quad\le C\frac{2^{np^+}}{R^{p^+}}\left(\frac{\omega}{2}\right)^{p^+}A_n^{1+\frac{p^-}{N+p^-}}.
\end{aligned}\tag{4.7}$$

On the other hand

$$\begin{aligned}
\int_{-R_n^{p^+}}^0\int_{K_{R_n}}(\bar{u}-k_n)_-^{p^-}\,\bar{\xi}_n^{\,p^+}dxd\bar{t}&\ge\int_{-R_{n+1}^{p^+}}^0\int_{K_{R_{n+1}}}(\bar{u}-k_n)_-^{p^-}dxd\bar{t}\\
&\ge|k_n-k_{n+1}|^{p^-}A_{n+1}\\
&=\frac{1}{2^{p^+(n+2)}}\left(\frac{\omega}{2}\right)^{p^+}A_{n+1}.
\end{aligned}\tag{4.8}$$

Combining (4.7) and (4.8), we get that

$$A_{n+1} \leq C\frac{4^{np^+}}{R^{p^+}} A_n^{1+\frac{p^-}{N+p^-}}. \tag{4.9}$$

An direct computation leads to

$$\frac{\left|Q(R_n^{p^+}, R_n)\right|^{1+\frac{p^-}{N+p^-}}}{\left|Q(R_{n+1}^{p^+}, R_{n+1})\right|} \leq 2^{P^+ + N} R^{\frac{p^-(p^+ + N)}{N+p^-}}. \tag{4.10}$$

Next, define the numbers

$$X_n = \frac{A_n}{Q(R_n^{p^+}, R_n)},$$

dividing (4.9) by $Q(R_{n+1}^{p^+}, R_{n+1})$ and using (4.10), we obtain the following recursive relation

$$X_{n+1} \leq C4^{np^+} X_n^{1+\frac{p^-}{N+p^-}}.$$

Therefore, Lemma 4.1 of Chapter I in [6] implies that if

$$X_0 \leq C^{-\frac{N+p^-}{p^-}} 4^{-p^+\left(\frac{p^- + N}{p^-}\right)^2} = \nu_0, \tag{4.11}$$

then

$$X_n \longrightarrow 0. \tag{4.12}$$

However, (4.11) is nothing but the assumption (2.4). Hence, the result easily follows from (4.12). □

Now consider the time level $-\hat{t} = t^* - \theta\left(\frac{R}{2}\right)^{p^+}$. From the conclusion of Lemma 4.1, we have

$$u(x, -\hat{t}) > \mu^- + \frac{\omega}{4} \text{ a.e. in } x \in K_{\frac{R}{2}},$$

we will use this time level as an initial condition to bring the information up to $t = 0$, and therefore to obtain an analogous inequality in a smaller cylinder. A first step in this direction is given by the following result.

Lemma 4.2. *For every $\nu_1 \in (0, 1)$, there exists a positive integer s_1 depending on the data, such that*

$$\left|x \in K_{\frac{R}{4}},\ u(x, t) < \mu^- + \frac{\omega}{2^{s_1}}\right| \leq \nu_1 |K_{\frac{R}{4}}|,\ \forall t \in (-\hat{t}, 0). \tag{4.13}$$

Proof. Consider the cylinder $Q(\hat{t}, \frac{R}{2})$ and write the logarithmic estimate (3.11) over this cylinder for the function $(u - k)_-$ with

$$k = \mu^- + \frac{\omega}{4} \text{ and } c = \frac{\omega}{2^{n+2}},$$

where n is to be chosen later. Defining

$$k - u \leq H_k^- = \operatorname*{ess\,sup}_{Q(\hat{t}, \frac{R}{2})} \left|\left(u - \mu^- - \frac{\omega}{4}\right)_-\right| \leq \frac{\omega}{4} \tag{4.14}$$

Assuming $H_k^- \leq \frac{\omega}{8}$ (if not the result is trivial). Then the logarithmic function ψ^- is well defined and satisfies the inequalities

$$\psi^- \leq n\ln(2),\ \text{since } \frac{H_k^-}{H_k^- + u - k + c} \leq \frac{\frac{\omega}{4}}{c} = 2^n, \tag{4.15}$$

and for $u \neq -k + c$,

$$0 \leq (\psi^-)' \leq \frac{1}{H_k^- + u - k + c} \leq \frac{1}{c}, \tag{4.16}$$

and

$$\left|(\psi^-)'(u)\right|^{2-p^-} = \left(H_k^- + u - k + c\right)^{p^- - 2} \leq \left(\frac{\omega}{2}\right)^{p^- - 2}. \tag{4.17}$$

For $t = -\hat{t}$, by virtue of Lemma 4.1 we have $u(x, -\hat{t}) > k$, and therefore

$$\left[\psi^-(u)\right](x, -\hat{t}) = 0 \text{ for } x \in K_{\frac{R}{2}}.$$

Now, choose a cutoff function $0 < \xi(x) \leq 1$, defined on $K_{\frac{R}{2}}$ such that

$$\xi = 1 \text{ in } K_{\frac{R}{2}} \text{ and } |\nabla \xi| \leq \frac{8}{R}.$$

From Definition 2.1, we know that if u is a weak solution of (1.1), then there exists a positive constant M such that

$$ess \sup_{\Omega_T} u \leq M. \tag{4.18}$$

Gathering these estimates, and using the fact that

$$\hat{t} \leq \left(\frac{\omega}{2^\lambda}\right)^{2-p^-} R^{p^+}, \tag{4.19}$$

we arrive at

$$\begin{aligned}
&ess \sup_{-\hat{t}<t<0} \int_{K_{\frac{R}{2}} \times \{t\}} \left[\psi^-(u)\right]^2 \xi^{p^+} dx \\
&\leq C\Bigg(\int_{-\hat{t}}^0 \int_{K_{\frac{R}{2}}} \psi^-(u) \left[(\psi^-)'(u)\right]^2 \xi^{p^+} dxdt \\
&+ \int_{-\hat{t}}^0 \int_{K_{\frac{R}{2}}} \psi^-(u) \left[(\psi^-)'(u)\right]^{2-p^-} \left(|\nabla u|^{p^+} + 1 + \xi^{p^+}\right) dxdt \\
&+ \int_{-\hat{t}}^0 \int_{K_{\frac{R}{2}}} \psi^-(u) \left(|\nabla u|^{p^+} + 1 + \xi^{p^+}\right) dxdt \\
&+ \int_{-\hat{t}}^0 \int_{K_{\frac{R}{2}}} |u|^{p^+} \psi^-(u) \left[(\psi^-)'(u)\right]^2 \xi^{p^+} dxdt\Bigg) \\
&\leq C\Bigg(n \ln(2) \left(\frac{\omega}{2^{n+2}}\right)^{-2} \left(\frac{\omega}{2^\lambda}\right)^{2-p^-} R^{p^+} \\
&+ n \ln(2) \left(\frac{8}{R}\right)^{p^+} \left(\frac{\omega}{2^{n+2}}\right)^{p^- - 2} \left(\frac{\omega}{2^\lambda}\right)^{2-p^-} R^{p^+} \\
&+ n \ln(2) \left(\frac{8}{R}\right)^{p^+} \left(\frac{\omega}{2^\lambda}\right)^{2-p^-} R^{p^+} \\
&+ M^{p^+} n \ln(2) \left(\frac{\omega}{2^{n+2}}\right)^{-2} \left(\frac{\omega}{2^\lambda}\right)^{2-p^-} R^{p^+}\Bigg) \left|K_{\frac{R}{4}}\right|.
\end{aligned} \tag{4.20}$$

Now, by virtue of Remark 2.2, we can estimate that

$$\left(\frac{\omega}{2^{n+2}}\right)^{-2} \omega^{2-p^-} R^{p^+} \leq 1 \text{ and } \omega^{2-p^-} R^{p^+} \leq 1.$$

Consequently, we obtain

$$\operatorname*{ess\,sup}_{-\hat{t}<t<0} \int_{K_{\frac{R}{2}}\times\{t\}} \left[\psi^-(u)\right]^2 \xi^{p^+} dx \leq Cn2^{\lambda(p^- -2)} \left|K_{\frac{R}{4}}\right|. \tag{4.21}$$

The left hand side of (4.20) is estimated from below considering integration over the smaller set

$$S_1 = \left\{x \in K_{\frac{R}{4}},\ u(x,t) < \mu^- + \frac{\omega}{2^{n+2}}\right\} \subset K_{\frac{R}{2}},\ t \in (-\hat{t}, 0).$$

On such a set

$$\left[\psi^-(u)\right]^2 \geq \ln^2\left\{\frac{H_k^-}{H_k^- + u - k + c}\right\} \geq \ln^2\left\{\frac{\frac{\omega}{4}}{c}\right\} = (n-1)^2 \ln^2(2). \tag{4.22}$$

Putting this into (4.21) gives that for all $t \in (-\hat{t}, 0)$

$$\left|\left\{x \in K_{\frac{R}{4}},\ u(x,t) < \mu^- + \frac{\omega}{2^{n+2}}\right\}\right| \leq C\frac{n}{(n-1)^2} 2^{\lambda(p^- -2)} \left|K_{\frac{R}{4}}\right|.$$

The proof is complete once we choose $s_1 = n + 2$ with $n > 1 + \frac{2C}{\nu_1} 2^{\lambda(p^- -2)}$. □

The conclusion of Lemma 4.2 will be employed to deduce that, within the cylinder $Q\left(\hat{t}, \frac{R}{8}\right)$, the set where u is away from its infimum is arbitrarily small.

Lemma 4.3. *There exists $1 < s_2 \in \mathbb{N}$, depending on the data, such that*

$$u(x,t) > \mu^- + \frac{\omega}{2^{s_2+1}} \ \textit{a.e.}\ (x,t) \in Q\left(\hat{t}, \frac{R}{8}\right). \tag{4.23}$$

Proof. Define two decreasing sequences of positive numbers

$$R_n = \frac{R}{8} + \frac{R}{2^{n+1}},\ k_n = \mu^- + \frac{\omega}{2^{s_2+1}} + \frac{\omega}{2^{s_2+1+n}},\ n = 0, 1, \ldots.$$

Construct the family of nested and shrinking cylinders $Q(\theta R_n^{p^+}, R_n)$, and let $0 \leq \xi_n(x) \leq 1$ be a piecewise smooth function in K_{R_n} that equals one on $K_{R_{n+1}}$ and $|\nabla \xi_n| \leq \frac{2^{n+4}}{R}$. Lemma 4.2 implies that

$$(u - k_n)_-(x, -\hat{t}) = 0 \text{ in } K_{R_n}.$$

Now, since $(u - k_n)_- \leq \frac{\omega}{2^{s_2}}$, using (4.19) and letting $s_2 > \lambda + \frac{p^+}{p^- -2}$ we get

$$(u - k_n)_-^2 \geq \frac{\hat{t}}{\left(\frac{R}{2}\right)^{p^+}} (u - k_n)_-^{p^-}.$$

Therefore, with these choices and by dividing the local energy estimates (3.1) for $(u - k_n)_-$ by $\dfrac{\hat{t}}{\left(\frac{R}{2}\right)^{p^+}}$ we get

$$\begin{aligned}
&\sup_{-\hat{t}<t<0} \int_{K_{R_n}\times\{t\}} (u - k_n)_-^{p^-} \xi_n^{p^+} dx \\
&+ \frac{\left(\frac{R}{2}\right)^{p^+}}{\hat{t}} \int_{-\hat{t}}^0 \int_{K_{R_n}} |\nabla (u - k_n)_-|^{p^-} \xi_n^{p^+} dxdt \\
&\leq C\frac{2^{np^+}}{\hat{t}} \left(\int_{-\hat{t}}^0 \int_{K_{R_n}} (u - k_n)_-^{p^+} dxdt + \int_{-\hat{t}}^0 \int_{K_{R_n}} \chi\left((u - k_n)_- > 0\right) dxdt\right) \\
&\leq C\frac{2^{np^+}}{\hat{t}} \left(\frac{\omega}{2^{s_2}}\right)^{p^+} \int_{-\hat{t}}^0 \int_{K_{R_n}} \chi\left((u - k_n)_- > 0\right) dxdt.
\end{aligned} \tag{4.24}$$

Introducing the change of variables $\bar{t} = t\dfrac{\left(\frac{R}{2}\right)^{p^+}}{\hat{t}}$ and defining the new function $\bar{u}(x,\bar{t}) = u(x,t)$. Accordingly, by using the same argument we used in the proof of Lemma 4.1, we get

$$\frac{1}{2^{p^+(n+2)}}\left(\frac{\omega}{2^{s_2}}\right)^{p^+} A_{n+1} \leq C\frac{2^{np^+}}{\left(\frac{R}{2}\right)^{p^+}}\left(\frac{\omega}{2^{s_2}}\right)^{p^+} A_n^{1+\frac{p^-}{n+p^-}}, \tag{4.25}$$

where

$$A_n = \int_{-\left(\frac{R}{2}\right)^{p^+}}^{0}\int_{K_{R_n}} \chi\left((\bar{u}-k_n)_- > 0\right) dx d\bar{t}.$$

Next, define the numbers

$$X_n = \frac{A_n}{Q\left(\left(\frac{R}{2}\right)^{p^+}, R_n\right)},$$

dividing (4.25) by $Q\left(\left(\frac{R}{2}\right)^{p^+}, R_{n+1}\right)$, we obtain the following recursive relation

$$X_{n+1} \leq C4^{np^+} X_n^{1+\frac{p^-}{N+p^-}}.$$

Therefore, Lemma 4.1 of Chapter I in [6] implies that if

$$X_0 \leq C^{-\frac{N+p^-}{p^-}} 4^{-p^+\left(\frac{p^-+N}{p^-}\right)^2} = \nu_1, \tag{4.26}$$

then

$$X_n \longrightarrow 0. \tag{4.27}$$

By applying Lemma 4.2 with $s_1 := s_2$ we get easily (4.26). Hence, the result follows from (4.27). □

As an immediate consequence we get the reduction of the oscillation of u.

Corollary 4.4. *There exists a constant $\sigma_0 \in (0,1)$ depending only on the data, such that if (2.4) holds then*

$$\underset{Q\left(\theta\left(\frac{R}{8}\right)^{p^+}, \frac{R}{8}\right)}{ess\ osc}\ u \leq \sigma_0\ \omega. \tag{4.28}$$

Proof. The proof follows since $Q\left(\theta\left(\frac{R}{8}\right)^{p^+}, \frac{R}{8}\right) \subset Q\left(\hat{t}, \frac{R}{8}\right)$, where we have $\sigma_0 = 1 - \frac{1}{2^{s_2+1}}$. □

Assume that (2.4) does not hold. Then, (2.5) is in force. Even in this case, we are able to deduce a result analogous to Corollary 4.4.

Lemma 4.5. *Assume that (2.5) holds true. there exists a time level*

$$t_0 \in \left[t^* - \theta R^{p^+}, t^* - \frac{\nu_0}{2}\theta R^{p^+}\right] \tag{4.29}$$

such that

$$\left|\left\{x \in K_R,\ u(x,t_0) > \mu^+ - \frac{\omega}{2}\right\}\right| \leq \left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right)|K_R|. \tag{4.30}$$

Proof. In fact, if (4.30) does not hold, then also (2.5) does not hold. □

This lemma shows that at the time level t_0, the portion of the cube K_R where $u(x)$ is close to its supremum is small. The next lemma claims that this indeed occurs for all time levels near the top of the cylinder $(0,t^*) + Q(\theta R^{p^+}, R)$

Lemma 4.6. *There exists* $1 < s_3 \in \mathbb{N}$ *depending on the data such that, for all* $t \in \left[t^* - \frac{\nu_0}{2}\theta R^{p^+}, t^*\right]$

$$\left|\left\{x \in K_R,\ u(x,t) > \mu^+ - \frac{\omega}{2^{s_3}}\right\}\right| \le \left(1 - \left(\frac{\nu_0}{2}\right)^2\right) |K_R|. \tag{4.31}$$

Proof. Consider the cylinder $K_R \times (t_0, t^*)$ and the level $k = \mu^+ - \frac{\omega}{2}$. Define

$$u - k \le H_k^+ = \underset{K_R \times (t_0,t^*)}{ess\sup} \left|(u - \mu^+ + \frac{\omega}{2})\right| \le \frac{\omega}{2} \tag{4.32}$$

Assuming that $H_k^+ > \frac{\omega}{4}$ (otherwise there will be nothing to prove). Select $n \in \mathbb{N}$ big enough so that

$$0 < c = \frac{\omega}{2^{n+1}} < H_k^+.$$

Then the logarithmic function ψ^+ is well defined and satisfies the followings.

$$\psi^+ \le n\ln(2) \text{ since } \frac{H_k^+}{H_k^+ - u + k + c} \le \frac{\frac{\omega}{4}}{c} = 2^n, \tag{4.33}$$

and, for $u \ne k + c$,

$$0 \le (\psi^+)' \le \frac{1}{H_k^+ - u + k + c} \le \frac{1}{c} \tag{4.34}$$

and,

$$\left|\left(\psi^+\right)'(u)\right|^{2-p^-} = \left(H_k^+ - u + k + c\right)^{p^- - 2} \le \left(\frac{\omega}{2}\right)^{p^- - 2}. \tag{4.35}$$

In the logarithmic inequality (3.11) applied to the function $(u-k)_+$, let $x \longmapsto \xi(x)$ be a smooth cutoff function defined in K_R such that for some $\pi \in (0,1)$

$$\begin{cases} 0 \le \xi \le 1 \text{ in } K_R,\ \xi = 0 \text{ on } K_{(1-\pi)R}, \\ |\nabla \xi| \le (\pi R)^{-1}. \end{cases}$$

Gathering these estimates, using Lemma 4.5 and the fact that

$$t^* - t \le \theta R^{p^+}, \tag{4.36}$$

we arrive at

$$\begin{aligned} &\underset{t_0<t<t^*}{ess\sup} \int_{K_R \times \{t\}} \left[\psi^+(u)\right]^2 \xi^{p^+} dx \\ &\quad \le n^2 (\ln 2)^2 \left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right) |K_R| + C\Bigg[n\ln 2 \left(\frac{\omega}{2^{n+1}}\right)^{-2} \theta R^{p^+} \\ &\quad + \frac{n\ln 2}{\pi^{p^+}} + n\ln 2 \left(\frac{1}{\pi R}\right)^{p^+} \theta R^{p^+} \\ &\quad + M^{p^+} n \ln 2 \left(\frac{\omega}{2^{n+1}}\right)^{-2} \theta R^{p^+}\Bigg] |K_R| \end{aligned} \tag{4.37}$$

Now, by virtue of Remark 2.2, we can estimate

$$\left(\frac{\omega}{2^{n+2}}\right)^{-2} \theta R^{p^+} \le 1 \text{ and } \theta R^{p^+} \le 1.$$

Consequently, we get

$$\underset{t_0<t<t^*}{ess\sup} \int_{K_R \times \{t\}} \left[\psi^+(u)\right]^2 \xi^{p^+} dx \le \left[n^2 (\ln 2)^2 \left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right) + C\frac{n}{\pi^{p^+}}\right] |K_R|. \tag{4.38}$$

The left hand side is estimated below by integrating over the smaller set

$$S_2 = \left\{ x \in K_{(1-\pi)R} : \ u(x,t) > \mu^+ - \frac{\omega}{2^{n+1}} \right\} \subset K_R.$$

On such a set, $\xi = 1$ and $\psi^+ \geq (n-1)\ln 2$, because

$$\frac{H_k^+}{H_k^+ - u + k + c} \geq \frac{\frac{\omega}{2}}{\frac{\omega}{2} - u + k + \frac{\omega}{2^{n+1}}} \geq \frac{\frac{\omega}{2}}{\frac{\omega}{2^n}} \geq 2^{n-1},$$

since one has $-u + \mu^+ < \frac{\omega}{2^n}$. Therefore for all $t \in (t_0, t^*)$

$$|S_2| \leq \left\{ \left(\frac{n}{n-1}\right)^2 \left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right) + \frac{C}{n\pi^{p^+}} \right\} |K_R|.$$

Therefore, for all $t \in (t_0, t^*)$

$$\begin{aligned} \left| \left\{ x \in K_R, \ u(x,t) > \mu^+ - \frac{\omega}{2^{n+1}} \right\} \right| &\leq |S_2| + N\pi |K_R| \\ \leq \left\{ \left(\frac{n}{n-1}\right)^2 \left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right) + \frac{c}{n\pi^{p^+}} + N\pi \right\} |K_R|. \end{aligned} \tag{4.39}$$

The proof is complete once we choose π so small that $N\pi \leq \frac{3}{8}\nu_0^2$, then n so large that

$$\frac{C}{n\pi^{p^+}} \leq \frac{3}{8}\nu_0^2 \text{ and } \left(\frac{n}{n-1}\right)^2 \leq (1 - \frac{\nu_0}{2})(1+\nu_0) > 1,$$

and finally take $s_3 = n + 1$. □

Recalling that $t_0 \in \left[t^* - \theta R^{p^+}, t^* - \frac{\nu_0}{2}\theta R^{p^+}\right]$ and choosing λ such that $2^{(\lambda-1)(p^- -2)} \geq 2$, the previous lemma immediately implies the following lemma.

Lemma 4.7. *There exists $1 < s_3 \in \mathbb{N}$ depending on the data, such that for all $t \in \left(-\frac{a_0}{2}R^{p^+}, 0\right)$,*

$$\left| \left\{ x \in K_R, \ u(x,t) > \mu^+ - \frac{\omega}{2^{s_3}} \right\} \right| \leq \left(1 - \left(\frac{\nu_0}{2}\right)^2\right) |K_R|. \tag{4.40}$$

From Lemma 4.7 we deduce that within the cylinder $Q(a_0 R^{p^+}, R)$, the set where u is close to its supremum is arbitrarily small.

Lemma 4.8. *For every $v_1 \in (0,1)$, there exists $s_3 \leq \lambda \in \mathbb{N}$ depending on the data, such that*

$$\left| \left\{ (x,t) \in Q\left(\frac{a_0}{2}R^{p^+}, R\right), \ u(x,t) > \mu^+ - \frac{\omega}{2^\lambda} \right\} \right| \leq \nu_1 \left| Q\left(\frac{a_0}{2}R^{p^+}, R\right) \right|. \tag{4.41}$$

Proof. Consider the cylinder $Q(a_0 R^{p^+}, 2R)$ and the levels $k = \mu^+ - \frac{\omega}{2^s}$, for $s_3 \leq s \leq \lambda$. Next, consider the local energy estimates (3.1) for the functions $(u-k)_+$, where $0 \leq \xi(x,t) \leq 1$ is a smooth cutoff function defined in $Q(a_0 R^{p^+}, 2R)$ and satisfying

$$\begin{cases} \xi = 1 \text{ in } Q(\frac{a_0}{2}R^{p^+}, R), \ \xi = 0 \text{ on } \partial_p Q(a_0 R^{p^+}, 2R), \\ |\nabla \xi| \leq \frac{1}{R}, \ 0 < \xi_t \leq \frac{2}{a_0 R^{p^+}}. \end{cases}$$

Neglecting the first term on the left hand side of (3.1), and using the indicated choices, we obtain

$$\begin{aligned}
\iint\limits_{Q(\frac{a_0}{2}R^{p^+},R)} |\nabla(u-k)_+|^{p^-}\,\xi^{p^+}dxdt &\le C\Bigg(\frac{2}{a_0R^{p^+}}\iint\limits_{Q(a_0R^{p^+},2R)}(u-k)_+^2dxdt\\
&\quad+\frac{1}{R^{p^+}}\iint\limits_{Q(a_0R^{p^+},2R)}(u-k)_+^{p^+}dxdt\\
&\quad+\iint\limits_{Q(a_0R^{p^+},2R)}\chi\left((u-k)_+>0\right)dxdt\Bigg)\\
&\le C\left(\frac{2}{a_0R^{p^+}}\left(\frac{\omega}{2^s}\right)^2+\frac{1}{R^{p^+}}\left(\frac{\omega}{2^s}\right)^{p^+}+2^{N+1}\right)\left|Q\left(\frac{a_0}{2}R^{p^+},R\right)\right|\\
&\le C\Bigg(\frac{2}{R^{p^+}}\left(\frac{\omega}{2^s}\right)^{p^--p^+}\left(\frac{\omega}{2^s}\right)^{p^+}+\frac{1}{R^{p^+}}\left(\frac{\omega}{2^s}\right)^{p^+}\\
&\quad+\frac{2^{N+1}}{R^{p^+}}\left(\frac{\omega}{2^s}\right)^{p^+}\left(\frac{\omega}{2^s}\right)^{-p^+}R^{p^+}\Bigg)\left|Q\left(\frac{a_0}{2}R^{p^+},R\right)\right|,
\end{aligned}\tag{4.42}$$

here, we used the fact that $s\le\lambda$. Now, by virtue of Remark 2.2 we can estimate

$$\left(\frac{\omega}{2^s}\right)^{p^--p^+}R^{p^+}\le 1 \text{ and } \left(\frac{\omega}{2^s}\right)^{-p^+}R^{p^+}\le 1.$$

Consequently, we get

$$\iint\limits_{Q(\frac{a_0}{2}R^{p^+},R)}|\nabla u|^{p^-}\,\xi^{p^+}dxdt\le\frac{C}{R^{p^+}}\left(\frac{\omega}{2^s}\right)^{p^+}\left|Q\left(\frac{a_0}{2}R^{p^+},R\right)\right|.\tag{4.43}$$

Now, we consider the levels $k_1=\mu^+-\frac{\omega}{2^s}$, $k_2=\mu^+-\frac{\omega}{2^{s+1}}$, $k_2-k_1=\frac{\omega}{2^{s+1}}$, and define, for $t\in\left(-\frac{a_0}{2}R^{p^+},0\right)$

$$A_s(t)=\left\{x\in K_R,\ u(x,t)>\mu^+-\frac{\omega}{2^s}\right\} \text{ and } A_s=\int_{-\frac{a_0}{2}R^{p^+}}^0|A_s(t)|\,dt.$$

Using Lemma 2.2 and Remarks 2.2 and 2.3 of [6,p.5] applied to the function $u(.,t)$ for all times $t\in\left(-\frac{a_0}{2}R^{p^+},0\right)$, we get

$$\begin{aligned}
\left(\frac{\omega}{2^{s+1}}\right)|A_{s+1}| &\le C\frac{R^{N+1}}{|K_R-A_s(t)|}\int_{-\frac{a_0}{2}R^{p^+}}^0\int_{A_s(t)-A_{s+1}(t)}|\nabla u|dx\\
&\le C\frac{R^{N+1}}{\left(\frac{\nu_0}{2}\right)^2|K_R|}\int_{-\frac{a_0}{2}R^{p^+}}^0\int_{A_s-A_{s+1}}|\nabla u|dx\\
&\le\frac{C}{\nu_0^2}\left(\frac{\omega}{2^s}\right)\left|Q\left(\frac{a_0}{2}R^{p^+},R\right)\right|^{\frac{1}{p^-}}|A_s(t)-A_{s+1}(t)|^{\frac{p^--1}{p^-}},
\end{aligned}\tag{4.44}$$

here, we used Lemma 4.7, Hölder's inequality and (4.43). According to the previous energy estimates we get, for $s=s_3,\ s_3+1,...,\lambda-1$

$$|A_{s+1}|^{\frac{p^-}{p^--1}}\le C(\nu_0)^{\frac{-2p^-}{p^--1}}\left|Q\left(\frac{a_0}{2}R^{p^+},R\right)\right|^{\frac{1}{p^--1}}|A_s-A_{s+1}|,$$

and we then add these inequalities for $s=s_3,\ s_3+1,...,\lambda-1$. Since $\mu^+-\frac{\omega}{2^{s+1}}\le\mu^+-\frac{\omega}{2^\lambda}$, the quantities $A_{s+1}\ge A_\lambda$. we combine this fact to obtain

$$\sum_{s=s_3}^{\lambda-1}A_{s+1}^{\frac{p^-}{p^--1}}\ge(\lambda-s_3)A_\lambda^{\frac{p^-}{p^--1}}.$$

Note, also that $\sum_{s=s_3}^{\lambda-1}|A_s-A_{s+1}|\leq \left|Q\left(\frac{a_0}{2}R^{p^+},R\right)\right|$. Collecting results, we arrive at

$$A_\lambda \leq \frac{C}{(\lambda-s_3)^{\frac{p^--1}{p^-}}}(\nu_0)^{-2}\left|Q\left(\frac{a_0}{2}R^{p^+},R\right)\right|$$

and the proof is complete once we choose $s_3<\lambda\in\mathbb{N}$ sufficiently large so that

$$\frac{C}{(\lambda-s_3)^{\frac{p^--1}{p^-}}}(\nu_0)^{-2}\leq \nu_1.$$

□

Lemma 4.9. *The number $v_1\in(0,1)$ can be chosen (and consequently, so λ), such that*

$$u(x,t)\leq \mu^+-\frac{\omega}{2^{\lambda+1}}\ a.e.\ (x,t)\in Q\left(\frac{a_0}{2}R^{p^+},R\right). \tag{4.45}$$

Proof. Define two decreasing sequences of positive numbers

$$R_n=\frac{R}{2}+\frac{R}{2^{n+1}},\ k_n=\mu^+-\frac{\omega}{2^{\lambda+1}}-\frac{\omega}{2^{\lambda+1+n}},\ n=0,1,....$$

Now, consider the local energy estimates (3.1) for the functions $(u-k_n)_+$ over the constructed family of nested and shrinking cylinders $Q\left(\frac{a_0}{2}R_n^{p^+},R_n\right)$, where $0\leq\xi_n(x,t)\leq 1$ are smooth functions defined in $Q\left(\frac{a_0}{2}R_n^{p^+},R_n\right)$ such that

$$\begin{cases}\xi_n=1 \text{ in } Q\left(\frac{a_0}{2}R_{n+1}^{p^+},R_{n+1}\right),\ \xi_n=0 \text{ on } \partial_pQ\left(\frac{a_0}{2}R_n^{p^+},R_n\right),\\ |\nabla\xi_n|\leq\frac{2^{n+1}}{R},\ 0<(\xi_n)_t\leq\dfrac{2^{p^+(n+1)}}{\frac{a_0}{2}R^{p^+}}.\end{cases}$$

Once again, performing the same calculation used in the proof of Lemma 4.1, we get

$$\begin{aligned}&\operatorname*{ess\,sup}_{-\frac{a_0}{2}R_n^{p^+}<t<0}\int_{K_{R_n}\times\{t\}}(u-k_n)_+^{p^+}\xi_n^{p^+}dx\\&\quad+\frac{1}{a_0}\iint\limits_{Q(\frac{a_0}{2}R_n^{p^+},R_n)}|\nabla(u-k_n)_+|^{p^-}\xi_n^{p^+}dxdt\\&\quad\leq C\frac{2^{np^+}}{a_0R^{p^+}}\left(\frac{\omega}{2^\lambda}\right)^{p^+}\iint\limits_{Q(\frac{a_0}{2}R_n^{p^+},R_n)}\chi\left((u-k_n)_+>0\right)dxdt.\end{aligned}\tag{4.46}$$

Introducing the change of variables $\bar{t}=\dfrac{t}{\frac{a_0}{2}}$ and defining

$$\bar{u}(x,\bar{t})=u(x,t) \text{ and } \bar{\xi_n}(x,\bar{t})=\xi_n(x,t).$$

Therefore, the previous estimates implies

$$\frac{1}{2^{p^+(n+2)}}\left(\frac{\omega}{2^\lambda}\right)^{p^+}A_{n+1}\leq C\frac{2^{np^+}}{R^{p^+}}\left(\frac{\omega}{2^\lambda}\right)^{p^+}A_n^{1+\frac{p^-}{N+p^-}}, \tag{4.47}$$

where, A_n is defined as

$$A_n=\iint\limits_{Q(R_n^{p^+},R_n)}\chi\left((\bar{u}-k_n)_+>0\right)dxd\bar{t}.$$

Next, defining $X_n = \frac{A_n}{\left|Q(R_n^{p^+}, R_n)\right|}$, we arrive at

$$X_{n+1} \leq C 4^{np^+} X_n^{1+\frac{p^-}{N+p^-}}.$$

Therefore, using Lemma 4.1 of Chapter I in [6], the result is proved if we can assume that

$$X_0 \leq C^{-\frac{N+p^-}{p^-}} 4^{-p^+\left(\frac{p^-+N}{p^-}\right)^2} = \nu_1. \tag{4.48}$$

For this value of ν_1, Lemma 4.8 implies that $X_0 \leq \nu_1$. Hence, we can conclude that $X_n \longrightarrow 0$ where $n \longrightarrow +\infty$ and the result follows. □

As an immediate consequence we get the reduction of the oscillation of u in the second case

Corollary 4.10. *There exists a constant $\sigma_1 \in (0,1)$ depending only on the data, such that if (2.5) holds then*

$$\underset{Q\left(\frac{a_0}{2}\left(\frac{R}{2}\right)^{p^+}, \frac{R}{2}\right)}{ess\ osc} u \leq \sigma_1\ \omega. \tag{4.49}$$

Proof. The proof follows by choosing $\sigma_1 = 1 - \frac{1}{2^{s_\lambda+1}}$. □

Now, we are able to prove Proposition 2.1, recalling the conclusions of Corollaries 4.4 and 4.10 and since $\theta\left(\frac{R}{8}\right)^{p^+} \leq \frac{a_0}{2}\left(\frac{R}{2}\right)^{p^+}$, we get that

$$\underset{Q\left(\theta\left(\frac{R}{8}\right)^{p^+}, \frac{R}{8}\right)}{ess\ osc} u \leq \sigma\ \omega,$$

where $\sigma = \max\{\sigma_0, \sigma_1\}$.

The proof of Theorem 2.3 follows from a slight modification of the arguments in Proposition 9 in [7]. From (2.6) one defines recursively a sequence Q_n of nested and shrinking cylinders and a sequence ω_n converging to zero, such that

$$\underset{Q_n}{ess\ osc}\, u \leq \omega_n.$$

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