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Some Results on The Existence of Weak Periodic Solutions For Quasilinear Parabolic Systems With L^1 Data

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ABSTRACT: The aim of this paper is to prove the existence of weak periodic solution and super solution for $M \times M$ reaction diffusion system with L^1 data and nonlinearity on the gradient. The existence is proved by the technique of sub and super solution and Schauder fixed point theorem.

Key Words: Periodic solutions, Systems, Weak solutions.

Contents

1. Introduction

Periodic behavior of solutions of quasilinear parabolic systems intervenes in the mathematical modeling of a large variety of phenomena, no only in the biology but also in natural sciences, chimical, engineering and ecology, such as gas dynamics, fusion processes, certain biological models, cellular processes and disease propagation. The literature of time periodic solutions of ordinary, functional differential equations have a great development, several results has been published not just in pure journals of mathematics but also those of applied and modelling, this is due in part to their wide applicability. Most of the studies are devoted to the existence of global solutions, their periodic behavior and regularity properties, particularly in relation to degenerate and singular systems.
At the same time, the periodicity of solutions for parabolic boundary value problems has also attracted great interests of scientists, and a lots of results have been reported under either Dirichlet or Neumann boundary conditions ([6,10,20,24]) all these papers treat classical solutions. In the last few years attention has been given to the notion of weak solutions for boundary value problems ([3,8,9,14,18]) these works used differents method, topological degree theory, Schauder fixed point theorem, bifurcation theory, method of sub and super solutions.

This work is devoted to study the existence of weak periodic solution for the following reaction-diffusion systems

$$\begin{cases} \dfrac{\partial u_j}{\partial t} - d_j \Delta u_j + G_j(t,x,\nabla u) = f_j & \text{in } \; Q_T, \\ u_j(0,.) = u_j(T,.) & \text{in } \; \Omega, \quad \text{for } j = 1,\dots,M \\ u_j(t,x) = 0 & \text{on } \; \Sigma_T, \end{cases} \tag{1.1}$$

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where $u = (u_1, \dots, u_M)$, $\nabla u = (\nabla u_1, \dots, \nabla u_M)$, $f = (f_1, \dots, f_M)$, $M \geqslant 2$ and Ω is an open regular bounded subset of $\mathbb{R}^N$, $N \geqslant 1$, with smooth boundary $\partial\Omega$, $T > 0$ is the period, $Q_T =]0,T[\times\Omega$, $\Sigma_T =]0,T[\times\partial\Omega$, $-\Delta$ denotes the Laplacian operator on $L^1(\Omega)$ with Dirichlet boundary conditions, d_j are positive constants, G_j is a caratheodory function and f_j is a nonnegative measurable function belongs to $L^1(Q_T)$. The result of this work can be applied to the example like model

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1\Delta u + \alpha_1 \mid \nabla u \mid^{\delta_1} + \beta_1 \mid \nabla v \mid^{\lambda_1} = f_1 & \text{in } Q_T \\ \dfrac{\partial v}{\partial t} - d_2\Delta v + \alpha_2 \mid \nabla u \mid^{\delta_2} + \beta_2 \mid \nabla v \mid^{\lambda_2} = f_2 & \text{in } Q_T \\ u(0,.) = u(T,.) & \text{in } \Omega \\ v(0,.) = v(T,.) & \text{in } \Omega \\ u = v = 0 & \text{on } \Sigma_T \end{cases} \tag{1.2}$$

where d_i, α_i, β_i are positives constants for $i = 1, 2$, to help understand the situation, let us mention some recent works concerning the parabolic systems and periodic problems.

In [6] Amann has been intersted by the problem (1.2) when (f_1, f_2) are regular enough and $1 \leqslant \delta_i, \lambda_i \leqslant 2$, we prove the existence of classical solution in $C^{1,2}(Q_T) \cap C(\overline{Q_T})$ by applying the technics of sub and super-solution and Schauder's fixed point theorem we refer the reader to [10], [20], [24] for more details.

Alaa and M. Iguernane [3] have been considered the problem when the data (f_1, f_2) belongs to $L^2(Q_T)\times L^2(Q_T)$ and $1 \leqslant \delta_i, \lambda_i \leqslant 2$, they show the existence of weak periodic solution in $L^2(0,T; H_0^1(\Omega))\cap \mathcal{C}([0,T], L^2(\Omega))$.

The goal of this paper is to investigate the case when the data are irregular and the nonlinearity has critical growth with respect to the gradient. We have organised this paper as follows, In section 2 we start by defining the notion of weak periodic solution of (1.1), under some hypothesis we prove the existence of weak periodic solution of (1.1). Section 3 is devoted to the application of our result to a periodic class of reaction-diffusion system and section 4 present an existence theorem for the weak periodic super solution of (1.1).

2. Main Result

This section presents two existence results for quasilinear parabolic periodic systems. The first result prove the existence when the nonlinearities are bounded by function L^1. The second result concerns periodic systems with critical growth nonlinearity with respect to the gradient. Let us now introduce the hypothesis which we assume throughout this section.

2.1. Assumptions

For all $j = 1, \dots, M$, we consider that

$$f_j \in L^1(Q_T),\ f_j \geqslant 0 \tag{2.1}$$

$$G_j(t,x,r) \in L^1(Q_T) \ \text{ for all } \ r \in (\mathbb{R}^N)^M \text{and } \ a.e.\,(t,x) \in Q_T \tag{2.2}$$

$$G_j : Q_T \times \Omega\times (\mathbb{R}^N)^M \to [0, +\infty[\text{ a caratheodory function} \tag{2.3}$$

$$G_j(t,x,0) = \min\{G_j(t,x,r), r \in (\mathbb{R}^N)^M\} = 0 \tag{2.4}$$

Before showing the main result, we have to clarify in which sense we want to solve the system (1.1), for which we introduce the notion of weak periodic solution.

Definition 2.1. *A function $u = (u_1, \dots, u_M)$ is said to be a weak periodic solution of the system (1.1), if satisfies for all $j = 1, \dots, M$*

$$\begin{cases} u_j \in L^1(0,T; W_0^{1,1}(\Omega)) \cap \mathcal{C}([0,T], L^1(\Omega)), \\ G_j(t,x,\nabla u) \ \in L^1(Q_T), \\ \dfrac{\partial u_j}{\partial t} - d_j\Delta u_j + G_j(t,x,\nabla u) = f_j & \textit{in } \mathcal{D}'(Q_T), \\ u_j(0,.) = u_j(T,.) & \textit{in } L^1(\Omega) \end{cases} \tag{2.5}$$

Definition 2.2. *We call weak periodic super-solution (resp. sub-solution) of (1.1) a function u satisfying (2.5) with " = " remplaced by " $\geq$ " (resp. " $\leq$ ").*

Remark 2.3. *Let $u_j \in \mathcal{C}([0,T], L^1(\Omega))$, we say that $u_j(0,.) = u_j(T,.)$ in $L^1(\Omega)$ if for all $\phi \in L^\infty(\Omega)$,*

$$\lim_{s \mapsto 0} \int_\Omega (u_j(T-s,x) - u_j(s,x))\phi(x)\, dx = 0$$

Theorem 2.4. *Under hypotheses (2.1)-(2.4), and assuming that there exists $w = (w_1, \ldots, w_M)$ a weak periodic super-solution of (1.1) and there exists a function $\theta \in L^1(Q_T)$ such that*

$$\left\{ \begin{array}{l} \mid G_j(t,x,r) \mid \leq \theta(t,x) \ \ a.e. \ \ (t,x) \in Q_T \\ \forall r \in (\mathbb{R}^N)^M, \forall j = 1, \ldots, M \end{array} \right. \tag{2.6}$$

Then the system (1.1) has a weak periodic solution satisfies for all $j = 1, \ldots, M$

$$0 \leq u_j \leq w_j \ \ in \ Q_T$$

Proof. For all $j = 1, \ldots, M$, we approximate f_j as follows, let $h_j^n \in C_0^2(Q_T)$, such that

$$h_j^n \geqslant 0, \qquad \| h_j^n \|_{L^1(Q_T)} \leq \| f_j \|_{L^1(Q_T)}, \tag{2.7}$$

and h_j^n converges to f_j in $L^1(Q_T)^+$, we denote

$$f_j^n = h_j^n 1_{[w_j \leq n]}, \qquad w_j^n = \min(w_j, n).$$

We define the approximate system of (1.1) by

$$\left\{ \begin{array}{ll} u_j^n \in L^2(0,T; H_0^1(\Omega)) \cap \mathcal{C}([0,T], L^2(\Omega)), & \\ \dfrac{\partial u_j^n}{\partial t} - d_j \Delta u_j^n + G_j(t,x,\nabla u_1^{n-1}, \ldots, \nabla u_j^n, \ldots \nabla u_M^{n-1}) = f_j^n & \text{in } \mathcal{D}'(Q_T), \\ u_j^n(0,.) \ = \ u_j^n(T,.) & \text{in } L^2(\Omega). \end{array} \right. \tag{2.8}$$

Since G_j is bounded and $f_j^n \in L^\infty(Q_T)$, the problem (2.8) has a solution u_j^n (see [18]) such that

$$0 \leq u_j^n \leq w_j^n \leq w_j \tag{2.9}$$

We applied the result of [12], to get

$$\begin{aligned} \| u_j^n \|_{L^1(0,T;W_0^{1,1}(\Omega))} &\leq C \Big[\| f_j^n \|_{L^1(Q_T)} + \| G_j \|_{L^1(Q_T)} + \| u_j^n(0) \|_{L^1(\Omega)} \Big] \\ &\leq C \Big[\| f_j \|_{L^1(Q_T)} + \| \theta \|_{L^1(Q_T)} + \| w_j(0) \|_{L^1(\Omega)} \Big]. \end{aligned}$$

The last passage is obtained by using (2.6), (2.7) and (2.9). According to the classical result of [11], the application $(u_j^n(0), \xi_j^n) \longmapsto u_j^n$ is compact from $L^1(\Omega) \times L^1(Q_T)$ into $L^1(0,T; W_0^{1,1}(\Omega))$, where

$$\xi_j^n(t,x) = f_j^n(t,x) - G_j(t,x,\nabla u_1^{n-1}, \ldots, \nabla u_j^n, \ldots, \nabla u_M^{n-1})$$

Then, we can extract a subsequence of u_j^n, still denoted by u_j^n for simplicity, such that

$$\begin{aligned} u_j^n &\longrightarrow u_j \ \text{ in } \ L^1(0,T; W_0^{1,1}(\Omega)) \\ (u_j^n, \nabla u_j^n) &\longrightarrow (u_j, \nabla u_j) \ \text{ a.e. in } \ Q_T. \end{aligned}$$

By applying the dominated convergence theorem, it follows that

$$G_j(t,x,\nabla u_1^{n-1}, \ldots, \nabla u_j^n, \ldots, \nabla u_M^{n-1}) \longrightarrow G_j(t,x,\nabla u) \text{ in } L^1(Q_T)$$

To ensure that u_j is a solution of (1.1), we will show that u_j is periodic in time, to do this we have

$$u_j^n(T) = S_{d_j}(T)u_j^n(0) + \int_0^T S(T-s)\xi_j^n(s,.)ds$$

where $S_{d_j}(t)$ is the semigroup of contractions in $L^1(\Omega)$ generated by the operator $-d_j\Delta$ with Dirichlet boundary condition on $\partial\Omega$. Since $u_j^n(0,.) = u_j^n(T,.)$ in $L^1(\Omega)$, we have for all $\phi \in L^\infty(\Omega)$

$$\begin{aligned}\lim_{n\to+\infty}\int_\Omega u_j^n(0,x)\phi(x)dx &= \lim_{n\to+\infty}\int_\Omega S_{d_j}(T)u_j^n(0,x)\phi(x)dx \\ &\quad + \lim_{n\to+\infty}\int_\Omega\int_0^T S_{d_j}(T-s)\xi_j^n(s,x)\phi(x)dsdx\end{aligned}$$

as we can see $S_{d_j}(t)$ is continuous in $L^1(\Omega)$ and $\xi_j^n \to \xi_j$ strongly in $L^1(Q_T)$, then

$$\begin{aligned}\int_\Omega u_j(0,x)\phi(x)dx &= \int_\Omega S_{d_j}(T)u_j(0,x)\phi(x)dx + \int_\Omega\int_0^T S_{d_j}(T-s)\xi_j(s,x)dsdx, \\ &= \int_\Omega u_j(T,x)\phi(x)dx\end{aligned}$$

Then $u_j(0,.) = u_j(T,.)$ in $L^1(\Omega)$. □

Theorem 2.5. *Suppose that (2.1)-(2.4) hold, and assuming that for all $j = 1,\dots,M$*

$$| G_j(t,x,r_1,\dots,r_M) | \le K_j(t,x) + \sum_{j=1}^M C_j \|r_j\|^p, \tag{2.10}$$

for all $p \in [1, \frac{N+2}{N+1}[$, $r_j \in \mathbb{R}^N$, with $K_j \in L^1(Q_T)$ and $C_j > 0$.
Then (1.1) has a weak periodic solution u_j satisfies for all $j = 1,\dots,M$

$$0 \le u_j \le \widehat{w}_j \ \text{ in } \ Q_T$$

where $\widehat{w}_j$ is solution of the following system

$$\begin{cases} \widehat{w}_j \in L^1(0,T;W_0^{1,1}(\Omega)) \cap \mathcal{C}([0,T],L^1(\Omega)), \\ \dfrac{\partial \widehat{w}_j}{\partial t} - d_j\Delta\widehat{w}_j = f_j \ \text{ in } \ \mathcal{D}'(Q_T), \\ \widehat{w}_j(0,.) \ = \ \widehat{w}_j(T,.) \ \text{ in } L^1(\Omega). \end{cases}$$

Remark 2.6. *Thanks to the positivity of the nonlinearities G, we can easily verified that $\widehat{w} = (\widehat{w}_1,\dots,\widehat{w}_M)$ is a weak periodic super solution of (1.1). The existence of $\widehat{w}$ will be proved in the Appendix by Schauder's fixed point theorem.*

2.2. Proof

For all $j = 1,\dots,M$, we approximate G_j as follows

$$G_j^n(t,x,r) = \frac{G_j(t,x,r)}{1 + \frac{1}{n} | G_j(t,x,r) |} 1_{[\widehat{w}_j \leqslant n]}$$

Setting

$$f_j^n = f_j 1_{[\widehat{w}_j \le n]}, \qquad \widehat{w}_j^n = \min(\widehat{w}_j, n)$$

We define the approximate system of (1.1) by

$$\begin{cases} u_j^n \in L^2(0,T;H_0^1(\Omega)) \cap \mathcal{C}([0,T],L^2(\Omega)), \\ \dfrac{\partial u_j^n}{\partial t} - d_j\Delta u_j^n + G_j^n(t,x,\nabla u^n) = f_j^n & \text{in } \mathcal{D}'(Q_T), \\ u_j^n(0,.) \ = \ u_j^n(T,.) & \text{in } L^2(\Omega). \end{cases} \tag{2.11}$$

$\widehat{w}_j^n$ is a weak periodic super-solution of (2.11) and G_j^n is bounded by n, then by applying the result of Theorem 2.4, problem (2.11) has a solution u_j^n such that

$$0 \leq u_j^n \leq \widehat{w}_j^n \leq \widehat{w}_j, \qquad \text{for all } j = 1 \dots M. \tag{2.12}$$

We want to pass to the limit in the approximate system (2.11), for this we need to prove the following lemmas.
Set $X_T = L^p(0,T; W_0^{1,p}(\Omega))$, where $1 \leq p < \frac{N+2}{N+1}$.

Lemma 2.7. *For all $j = 1, \dots, M$.*

i) There exists a constant C depending on $\| f_j \|_{L^1(Q_T)}$ such that

$$\int_{Q_T} | G_j^n(t,x,\nabla u^n) | \leq C$$

ii) There exists a constant C depending on p, T, Ω such that

$$\| u_j^n \|_{X_T} \leq C \Big[2 \| f_j \|_{L^1(Q_T)} + \| \widehat{w}_j(0) \|_{L^1(\Omega)} \Big]$$

Proof. i) Integrate the equation satisfied by u_j^n over Q_T, we get for all $j = 1, \dots, M$.

$$\int_{Q_T} \frac{\partial u_j^n}{\partial t} - \int_{Q_T} d_j \Delta u_j^n + \int_{Q_T} G_j^n(t,x,\nabla u^n) = \int_{Q_T} f_j^n(t,x),$$

since $u_j^n(0,.) = u_j^n(T,.)$ in Ω and $G_j^n \geq 0$, we have

$$\begin{aligned} \int_{Q_T} | G_j^n(t,x,\nabla u^n) | \, dxdt &\leqslant \int_{Q_T} f_j^n(t,x) dxdt, \\ &\leqslant \| f_j \|_{L^1(Q_T)} . \end{aligned}$$

ii) Furthermore, by [12] we have

$$\begin{aligned} \| u_j^n \|_{X_T} &\leq C(p,\Omega) \Big[\| f_j^n \|_{L^1(Q_T)} + \| G_j^n(\nabla u^n) \|_{L^1(Q_T)} + \| u_j^n(0) \|_{L^1(\Omega)} \Big] \\ &\leq C(p,\Omega) \Big[2 \| f_j \|_{L^1(Q_T)} + \| \widehat{w}_j(0) \|_{L^1(\Omega)} \Big] \end{aligned}$$

The latter inequality is obtained by using (i) and (2.12). □

According to Lemma 3.3 we have $f_j^n(t,x) - G_j^n(t,x,\nabla u^n)$ bounded in $L^1(Q_T)$, then we can apply the compactness result of [11] to extract a subsequence of u_j^n denoted by u_j^n, such that

$$\begin{gathered} u_j^n \longrightarrow u_j \text{ in } L^1(0,T; W_0^{1,1}(\Omega)) \\ (u_j^n, \nabla u_j^n) \longrightarrow (u_j, \nabla u_j) \text{ a.e. in } Q_T. \end{gathered}$$

To ensure that u_j is a solution of the problem (1.1), it remains to show that u_j^n converges to u_j strongly in X_T. To do this, we write for $m, n \geq 1$ and $0 < \gamma < 1$,

$$\int_{Q_T} | \nabla u_j^n - \nabla u_j^m |^p \leq \left(\int_{Q_T} | \nabla u_j^n - \nabla u_j^m | \right)^\gamma \left(\int_{Q_T} | \nabla u_j^n - \nabla u_j^m |^{\frac{p-\gamma}{1-\gamma}} \right)^{1-\gamma} \tag{2.13}$$

Choose γ such that $\frac{p-\gamma}{1-\gamma} = q \in [1, \frac{N+2}{N+1}[$, then (2.13) gives desired result.
Thanks to the assumption (2.10), we deduce

$$G_j^n(t,x,\nabla u^n) \longrightarrow G_j(t,x,\nabla u) \text{ in } L^1(Q_T)$$

Since the nonlinearities converge strongly in $L^1(Q_T)$, the periodicity of u_j can be obtained by the same reasoning of the first proof.

3. Application to a class of reaction-diffusion systems

In this paragraph we apply the result of the first section to prove the existence of weak periodic solution for the following quasilinear parabolic system

$$\begin{cases} \dfrac{\partial u_j}{\partial t} - d_j \Delta u_j + G_j(t,x,\nabla u) = F_j(t,x,u) + \mu_j & \text{in } Q_T, \\ u_j(0,.) = u_j(T,.) & \text{in } \Omega, \\ u_j(t,x) = 0 & \text{on } \Sigma_T \end{cases} \tag{3.1}$$

where $u = (u_1, \dots, u_M)$, $\nabla u = (\nabla u_1, \dots, \nabla u_M)$, $\mu = (\mu_1, \dots, \mu_M)$, $F(.,u) = (F_1(.,u), \dots, F_M(.,u))$ with $M \geqslant 2$, the nonlinearities G_j and F_j are assumed to be a caratheodory functions and μ_j is a nonnegative measurable function belongs to $L^1(Q_T)$. In order to show the existence of weak solution of (3.1), we will follow the process of approximation, "truncating" the nonlinearities term $G_j(t,x,\nabla u)$ so that it becomes bounded, and applying the result of Theorem 2.4 to study the behavior of a sequence (u_j^n) solutions of the approximated problems. Due to the structure of the approximation, the sequence (u_j^n) will be non decreasing with respect to n, so this monotony will be guaranteed a good compactness of the sequence (u_j^n) in a suitable Banach space.

3.1. Assumptions

For all $j = 1, \dots, M$, we assume that

$$\mu_j \in L^1(Q_T),\ \mu_j \geqslant 0 \tag{3.2}$$

$$F_j : [0,T] \times \Omega \times \mathbb{R}^M \to [0,+\infty[\text{ a caratheodory function} \tag{3.3}$$

$$F_j(t,x,s) \in L^1(Q_T),\ F_j(.,s) \text{ is quasimonotone nondecreasing} \tag{3.4}$$

$$G_j : [0,T] \times \Omega \times (\mathbb{R}^N)^M \to [0,+\infty[\text{ a caratheodory function} \tag{3.5}$$

$$G_j(t,x,r) \leq H_j(t,x) + \sum_{j=1}^{M} C_j \|r_j\|^2 \tag{3.6}$$

with $H_j \in L^1(Q_T)$ and $C_j > 0$. For (3.4) we recall that a function $F_j(.,u)$ is said to be quasimonotone nondecreasing if $F_j(.,u)$ is nondecreasing with respect to all components u_j of u. The notion of weak periodic solution is presented here to clarify in which sense we want to solve the system (3.1).

Definition 3.1. *A function $u = (u_1, \dots, u_M)$ is said to be a weak periodic solution of the system (3.1), if for all $j = 1, \dots, M$, we have*

$$\begin{cases} u_j \in L^1(0,T; W_0^{1,1}(\Omega)) \cap \mathcal{C}([0,T], L^1(\Omega)), \\ G_j(t,x,\nabla u),\ F_j(t,x,u) \in L^1(Q_T), \\ \dfrac{\partial u_j}{\partial t} - d_j \Delta u_j + G_j(t,x,\nabla u) = F_j(t,x,u) + \mu_j & \text{in } \mathcal{D}'(Q_T), \\ u_j(0,.) = u_j(T,.) & \text{in } L^1(\Omega) \end{cases} \tag{3.7}$$

Basing on the result of the first section, we can prove that (3.1) has nonnegative weak solution which is the main result of the following theorem.

Theorem 3.2. *Suppose that the hypotheses (3.2)-(3.6) hold, and assuming that there exists $v = (v_1, \dots, v_M)$ such that, for all $j = 1, \dots, M$*

$$\begin{cases} v_j \in L^1(0,T; W_0^{1,1}(\Omega)) \cap \mathcal{C}([0,T], L^1(\Omega)), \\ F_j(t,x,v) \in L^1(Q_T) \\ \dfrac{\partial v_j}{\partial t} - d_j \Delta v_j = F_j(t,x,v) + \mu_j & \text{in } \mathcal{D}'(Q_T), \\ v_j(0,.) = v_j(T,.) & \text{in } L^1(\Omega). \end{cases}$$

Then (3.1) has a weak periodic solution $u = (u_1, \dots, u_M)$ such that for $j = 1, \dots, M$, we have

$$0 \leqslant u_j \leqslant v_j$$

3.2. Approximating Problem

For $j = 1, \dots, M$, we consider the sequence defined by $u_j^0 = v_j$ and for $n \geqslant 1$, u_j^n is the solution of the following system

$$\begin{cases} u_j^n \in L^1(0,T;W_0^{1,1}(\Omega)) \cap \mathcal{C}([0,T],L^1(\Omega)), \\ \dfrac{\partial u_j^n}{\partial t} - d_j \Delta u_j^n + G_j^n(t,x,\nabla u^n) = F_j(t,x,u^{n-1}) + \mu_j & \text{in } \mathcal{D}'(Q_T), \\ u_j^n(0,.) = u_j^n(T,.) & \text{in } L^1(\Omega). \end{cases} \tag{3.8}$$

Where,

$$G_j^n(t,x,r) = \frac{G_j(t,x,r)}{1 + \frac{1}{n} \mid G_j(t,x,r) \mid}$$

by using Theorem 2.4 combined with an induction argument, we prove the existence of u_j^n solution of the approximate system (3.8) such that

$$0 \leqslant u_j^n \leqslant u_j^{n-1} \leqslant v_j. \tag{3.9}$$

3.3. A priori estimates

Before giving the lemmas that will be useful for the proof of Theorem 3.2, let us define the truncation function $T_k \in C^2$ for all real positive number k by,

$$\begin{aligned} T_k(s) &= s \text{ if } 0 \leqslant s \leqslant k, \\ T_k(s) &\leqslant k+1 \text{ if } s \geqslant k, \\ 0 \leqslant T_k'(s) &\leqslant 1 \text{ if } s \geqslant 0, \\ T_k'(s) &= 0 \text{ if } s \geqslant k+1, \\ 0 \leqslant -T_k''(s) &\leqslant C(k). \end{aligned}$$

For example, the function T_k can be defined as

$$\begin{aligned} &T_k(s) = s \text{ in } [0,k], \\ &T_k(s) = \frac{1}{2}(s-k)^4 - (s-k)^3 + s \text{ in } [k,k+1], \\ &T_k(s) = \frac{1}{2}(k+1) \text{ for } s > k+1. \end{aligned}$$

Setting

$$S_k(v) = \int_0^v T_k(s)\, ds.$$

Lemma 3.3. *For $j = 1, \dots, M$.*

i) There exists a constant C depending on $\| \mu_j \|_{L^1(Q_T)}$ and $\| F_j(v) \|_{L^1(Q_T)}$, such that

$$\int_{Q_T} \mid G_j^n(t,x,\nabla u^n) \mid dxdt \leqslant C.$$

ii)

$$\lim_{k \mapsto +\infty} \sup_n \int_{[u_j^n > k]} \mid G_j^n(t,x,\nabla u^n) \mid dxdt = 0.$$

Proof. (i) Integrating the equation satisfies by u_j^n over Q_T,

$$\int_{Q_T} \frac{\partial u_j^n}{\partial t} - \int_{Q_T} d_j \Delta u_j^n + \int_{Q_T} G_j^n(t,x,\nabla u^n) = \int_{Q_T} F_j(t,x,u^{n-1}) + \int_{Q_T} \mu_j,$$

since $u_j^n(0,.) = u_j^n(T,.)$ in $L^1(\Omega)$ and by using the assumptions (3.3), (3.4) and (3.9) we get

$$\int_{Q_T} \mid G_j^n(t,x,\nabla u^n) \mid \leqslant \int_{Q_T} F_j(t,x,v) + \int_{Q_T} \mu_j.$$

(ii) Multiplying the equation satisfies by u_j^n by the truncated function $T_k(u_j^n)$ and integrating over Q_T, we obtain

$$\int_{Q_T} \frac{\partial S_k(u_j^n)}{\partial t} + d_j \int_{Q_T} \mid \nabla T_k(u_j^n) \mid^2 + \int_{Q_T} G_j^n(t,x,\nabla u^n) T_k(u_j^n)$$
$$= \int_{Q_T} F_j(t,x,u^{n-1}) T_k(u_j^n) + \int_{Q_T} \mu_j T_k(u_j^n),$$

the hypothesis on F_j and the periodicity of u_j^n, yields

$$\int_{Q_T} G_j^n(t,x,\nabla u^n) T_k(u_j^n) \leqslant \int_{Q_T} F_j(t,x,v) T_k(u_j^n) + \int_{Q_T} \mu_j T_k(u_j^n),$$

then for every $0 < A < k$, we have

$$k \int_{[u_j^n > k]} G_j^n(t,x,\nabla u^n) \leqslant k \int_{Q_T \cap [u_j^n > A]} \Big(F_j(t,x,v) + \mu_j \Big)$$
$$+ A \int_{Q_T \cap [u_j^n \leqslant A]} \Big(F_j(t,x,v) + \mu_j \Big),$$

consequently,

$$\int_{[u_j^n > k]} G_j^n(t,x,\nabla u^n) \leqslant \int_{Q_T} \Big(F_j(t,x,v) + \mu_j \Big) \chi_{[u_j^n > A]}$$
$$+ \frac{A}{k} \int_{Q_T} \Big(F_j(t,x,v) + \mu_j \Big).$$

To conclude the desired result, it suffices to show that

$$\lim_{k \mapsto +\infty} \sup_n \int_{Q_T} \Big(F_j(t,x,v) + \mu_j \Big) \chi_{[u_j^n > A]} = 0,$$

to do this we remark,

$$\mid [u_j^n > A] \mid \leqslant \frac{1}{A} \parallel u_j^n \parallel_{L^1(Q_T)} \leqslant \frac{1}{A} \parallel v_j \parallel_{L^1(Q_T)},$$

which implies,

$$\lim_{A \mapsto +\infty} \sup_n \mid [u_j^n > A] \mid = 0.$$

Since $(F_j(t,x,v) + \mu_j) \in L^1(Q_T)$, we have for each $\epsilon > 0$ there exists $\delta > 0$ such that for all mesurable $E \subset Q_T$,

$$\mid E \mid < \delta, \quad \int_E \Big(F_j(t,x,v) + \mu_j \Big) \leqslant \frac{\epsilon}{2},$$

according to the previous result, we obtain that for each $\epsilon > 0$, then there exists A_ϵ such that for all $A \geqslant A_\epsilon$

$$\sup_n \left(\int_{Q_T} (F_j(t,x,v) + \mu_j) \chi_{[u_j^n > A]} \right) \leqslant \frac{\epsilon}{2},$$

choosing $A = A_\epsilon$ and letting k tend to infinity, we obtain

$$\lim_{k \mapsto +\infty} \sup_n \left(\int_{[u_j^n > k]} G_j^n(t,x,\nabla u^n) \right) = 0.$$

□

Lemma 3.4. *Let (u_j^n) be the sequence defined as above. Then for $j = 1, \dots, M$.*

i) (u_j^n) *converges to* u_j *strongly in* $L^1(0,T; W_0^{1,1}(Q_T))$,

ii) $\| T_k(u_j^n) \|_{L^2(0,T;H_0^1)} \leqslant C\Big[\| F_j(v) \|_{L^1(Q_T)} + \| \mu_j \|_{L^1(Q_T)} \Big]$.

Proof. (i) Letting

$$\eta_j^n = F_j(t,x,u^{n-1}) + \mu_j - G_j^n(t,x,\nabla u^n),$$

from the result (i) of Lemma 3.3, (3.4) and (3.9), it follows that η_j^n bounded in $L^1(Q_T)$ and according to [11], the application

$$\begin{aligned} L^1(\Omega) \times L^1(Q_T) &\longrightarrow L^1(0,T; W_0^{1,1}(Q_T)) \\ (u_j^n(0), \eta_j^n) &\longmapsto u_j^n \end{aligned}$$

is compact. Then, we can extract a subsequence of (u_j^n), still denoted by (u_j^n) for simplicity, such that

$$\begin{aligned} u_j^n &\longrightarrow u_j \text{ in } L^1(0,T; W_0^{1,1}(\Omega)) \\ (u_j^n, \nabla u_j^n) &\longrightarrow (u_j, \nabla u_j) \text{ a.e. in } Q_T \end{aligned}$$

(ii) Multiplying by $T_k(u_j^n)$ the equation satisfies by u_j^n, we obtain

$$\int_{Q_T} \frac{\partial S_k(u_j^n)}{\partial t} + d_j \int_{Q_T} | \nabla T_k(u_j^n) |^2 + \int_{Q_T} G_j^n(t,x,\nabla u^n) T_k(u_j^n)$$
$$= \int_{Q_T} F_j(t,x,u^{n-1}) T_k(u_j^n) + \int_{Q_T} \mu_j T_k(u_j^n),$$

the periodicity implies,

$$\int_{Q_T} \frac{\partial S_k(u_j^n)}{\partial t} = 0.$$

We use (3.5) and (3.9), to get

$$\int_{Q_T} G_j^n(t,x,\nabla u^n) T_k(u_j^n) \geqslant 0,$$

finally by application of (3.4) and (3.9) we get,

$$\int_{Q_T} | \nabla T_k(u_j^n) |^2 \leqslant C\Big[\int_{Q_T} F_j(t,x,v) + \int_{Q_T} \mu_j \Big].$$

□

Lemma 3.5. *Let (u_j^n) be the sequence defined as above. Then for all $j = 1, \dots, M$.*

$$T_k(u_j^n) \text{ converges to } T_k(u_j) \text{ strongly in } L^2(0,T; H_0^1(\Omega)).$$

Proof. To prove this lemma, we consider for all $j = 1, \dots, M$,

$$\begin{aligned} z_j^{n,k} &= T_k(v_j - u_j^n), \\ z_j^k &= T_k(v_j - u_j), \\ z_j^{n,k,h} &= (T_k(v_j - u_j^n))^h, \end{aligned}$$

where σ^h denotes the Lebesgue steklov regularization defined for $h>0$ by

$$\sigma^h(t,x)=\frac{1}{h}\int_t^{t+h}\sigma(s,x)ds.$$

To prove $T_k(u_j^n)$ converges strongly to $T_k(u_j)$ in $L^2(0,T;H_0^1(\Omega))$, it suffices to prove that

$$\lim_{n\to\infty}\int_{Q_T}\left\|\nabla z_j^{n,k}\right\|^2 dxdt\leq\int_{Q_T}\left\|\nabla z_j^k\right\|^2 dxdt.$$

For $h>0$, we have

$$\begin{aligned}\lim_{n\mapsto+\infty} d_j\int_{Q_T}\left\|\nabla z_j^{n,k}\right\|^2 dxdt &= \lim_{h\to 0}\lim_{n\mapsto+\infty} d_j\int_{Q_{T-h}}\left\|\nabla z_j^{n,k,h}\right\|^2 dxdt\\ &=\lim_{h\to 0}\lim_{n\mapsto+\infty}\int_0^{T-h}< z_j^{n,k,h},-d_j\Delta z_j^{n,k,h}> dxdt\\ &\leq\lim_{h\to 0}\lim_{n\mapsto+\infty}\int_0^{T-h}< z_j^{n,k,h},\frac{\partial z_j^{n,k,h}}{\partial t}-d_j\Delta z_j^{n,k,h}> dxdt,\end{aligned}$$

we remark that,

$$\frac{\partial z_j^{n,k,h}}{\partial t}-d_j\Delta z_j^{n,k,h}\geqslant 0,$$

and according to (3.9), we have $0\leqslant z_j^{n,k,h}\leqslant z_j^{k,h}$, then

$$\begin{aligned}&\lim_{n\mapsto+\infty} d_j\int_{Q_T}\left\|\nabla z_j^{n,k}\right\|^2 dxdt\\ &\leqslant\lim_{h\to 0}\lim_{n\mapsto+\infty}\int_0^{T-h}< z_j^{k,h},\frac{\partial z_j^{n,k,h}}{\partial t}-d_j\Delta z_j^{n,k,h}> dxdt,\\ &\leqslant\lim_{h\to 0}\lim_{n\mapsto+\infty}\left[\int_0^{T-h}< z_j^{k,h},\frac{\partial z_j^{n,k,h}}{\partial t}> dt+d_j\int_{Q_{T-h}}\nabla z_j^{n,k,h}\nabla z_j^{k,h}dxdt\right].\end{aligned}$$

Since $z_j^{n,k}$ converges to z_j^k weakly in $L^2(0,T;H_0^1(\Omega))$, then $z_j^{n,k,h}$ converges to $z_j^{k,h}$ weakly in $L^2(0,T;H_0^1(\Omega))$, we obtain

$$\begin{aligned}&\lim_{n\mapsto+\infty} d_j\int_{Q_T}\left\|\nabla z_j^{n,k}\right\|^2 dxdt\\ &\leqslant\lim_{h\to 0}\left[\int_0^{T-h}< z_j^{k,h},\frac{\partial z_j^{k,h}}{\partial t}> dt+d_j\int_{Q_{T-h}}\left\|\nabla z_j^{k,h}\right\|^2 dxdt\right]\\ &\leqslant\lim_{h\to 0}\left[\frac{1}{2}\int_\Omega[(z_j^{k,h})^2]_0^{T-h}dx+d_j\int_{Q_{T-h}}\left\|\nabla z_j^{k,h}\right\|^2 dxdt\right]\\ &\leqslant d_j\int_{Q_T}\left\|\nabla z_j^k\right\|^2 dxdt.\end{aligned}$$

$\square$

3.4. Passing to the Limit

According to lemma (3.4) there exists a mesurable fonction

$$u_j \in L^1(0,T;W_0^{1,1}(\Omega))$$

and a subsequence still denoted (u_j^n) for simplicity, such that

$$u_j^n \longrightarrow u_j \text{ in } L^1(0,T;W_0^{1,1}(\Omega)),$$
$$(u_j^n,\nabla u_j^n) \longrightarrow (u_j,\nabla u_j) \text{ a.e. in } Q_T,$$

then,

$$F_j(t,x,u^{n-1}) \longrightarrow F_j(t,x,u) \text{ a.e. in } Q_T,$$

thanks to Lebesgue theorem, we have

$$F_j(t,x,u^{n-1}) \longrightarrow F_j(t,x,u) \text{ in } L^1(Q_T).$$

By the previous result of Lemmas (3.3) and (3.4), we have

$$G_j^n(t,x,\nabla u^n) \longrightarrow G_j(t,x,\nabla u) \text{ a.e. in } Q_T, \tag{3.10}$$

It remains to show that

$$G_j^n(t,x,\nabla u^n) \to G_j(t,x,\nabla u) \text{ in } L^1(Q_T),$$

using (3.10) it suffices to prove that $G_j^n(t,x,\nabla u^n)$ is equi-integrable in $L^1(Q_T)$ namely

$$\forall \varepsilon > 0, \ \exists\, \delta > 0, \ \forall\, E \subset Q_T, \ \text{ if } |E| < \delta \text{ then } \int\limits_E G_j^n(t,x,\nabla u^n)\,dx\,dt \le \varepsilon.$$

Let E be a mesurable subset of Q_T, $\varepsilon > 0$, and $k > 0$. We have for all $j = 1,\dots,M$,

$$\int\limits_E G_j^n(t,x,\nabla u^n)\,dx\,dt = I_{j,1} + I_{j,2}.$$

Where

$$I_{j,1} = \int\limits_{E\cap[u_j^n > k]} G_j^n(t,x,\nabla u^n)\,dx\,dt,$$

and

$$I_{j,2} = \int\limits_{E\cap[u_j^n \le k]} G_j^n(t,x,\nabla u^n)\,dx\,dt.$$

The first integral $I_{j,1}$ verify the following inequality

$$I_{j,1} \le \int\limits_{[u_j^n > k]} G_j^n(t,x,\nabla u^n)\,dx\,dt,$$

we obtain from the Lemma (3.3) the existence of $k^* > 0$, such that, for all $k \geqslant k^*$, we have

$$I_{j,1} \le \frac{\epsilon}{3}.$$

Concerning $I_{j,2}$ we use the assumption (3.6), we obtain for all $k \ge k^*$

$$I_{j,2} \le \int\limits_E \left(H_j(t,x) + \sum_{j=1}^{M} C_j \mid \nabla T_k(u_j^n) \mid^2 \right) dx\,dt.$$

Since $H_j \in L^1(Q_T)$, then H_j is equi-integrable in $L^1(Q_T)$, there exists $\delta_1 > 0$, such that, if $\mid E \mid\leq \delta_1$, then

$$\int_E H_j(t,x)\,dx\,dt \leq \frac{\epsilon}{3}.$$

We have also from the lemma (3.5) the sequence $(\mid \nabla T_k(u_j^n) \mid^2)_n$ is equi-integrable in $L^1(Q_T)$, which implies the existence of $\delta_2 > 0$, such that, if $\mid E \mid\leq \delta_2$, we have

$$\sum_{j=1}^{M} C_j \int_E \mid \nabla T_k(u_j^n) \mid^2 \,dx\,dt \leq \frac{\epsilon}{3}.$$

Finally, by choosing $\delta^* = \inf(\delta_1, \delta_2)$, if $\mid E \mid\leq \delta^*$, we obtain

$$\int_E G_j^n(t,x,\nabla u^n)\,dx\,dt \,\leq\, \varepsilon.$$

On the other hand,

$$u_j^n(T) = S_{d_j}(T)u_j^n(0) + \int_0^T S(T-s)\eta_j^n(s,.)ds,$$

where,

$$\eta_j^n(t,x) = F_j(t,x,u^{n-1}) + \mu_j(t,x) - G_j^n(t,x,\nabla u^n).$$

Since $u_j^n(0,.) \;=\; u_j^n(T,.)$ in $L^1(\Omega)$, we have for all $\phi \in L^\infty(\Omega)$

$$\begin{aligned}\lim_{n\to+\infty}\int_\Omega u_j^n(0,x)\phi(x)dx &= \lim_{n\to+\infty}\int_\Omega S_{d_j}(T)u_j^n(0,x)\phi(x)dx \\ &\quad + \lim_{n\to+\infty}\int_\Omega\int_0^T S_{d_j}(T-s)\eta_j^n(s,x)\phi(x)dsdx.\end{aligned}$$

As well known $S_{d_j}(t)$ is continuous in $L^1(\Omega)$ and $\eta_j^n \to \eta_j$ strongly in $L^1(Q_T)$, then

$$\begin{aligned}\int_\Omega u_j(0,x)\phi(x)dx &= \int_\Omega S_{d_j}(T)u(0,x)\phi(x)dx + \int_\Omega\int_0^T S_{d_j}(T-s)\eta_j(s,x)dsdx, \\ &= \int_\Omega u_j(T,x)\phi(x)dx.\end{aligned}$$

Then $u_j(0,.) = u_j(T,.)$ in $L^1(\Omega)$.

4. Appendix

Theorem 4.1. *Let $f = (f_1,\dots,f_M)$ be a nonnegative function belongs to $[L^1(Q_T)]^M$. Then there exists $\widehat{w} = (\widehat{w}_1,\dots,\widehat{w}_M)$ a nonnegative weak periodic solution of the following system*

$$\left\{\begin{array}{l} \widehat{w}_j \in L^1(0,T;W_0^{1,1}(\Omega)) \cap \mathcal{C}([0,T],L^1(\Omega)), \\ \dfrac{\partial \widehat{w}_j}{\partial t} - d_j\Delta\widehat{w}_j = f_j \;\; in\;\; \mathcal{D}'(Q_T), \quad for\; j=1,\dots,M \\ \widehat{w}_j(0,.) \;=\; \widehat{w}_j(T,.)\;\; in\;\; L^1(\Omega). \end{array}\right. \tag{4.1}$$

Remark 4.2. *As well known (4.1) is linear and the second membre $f = (f_1,\dots,f_M)$ does not depend on the components of the solution $\widehat{w} = (\widehat{w}_1,\dots,\widehat{w}_M)$, consequently it suffices to prove the result of Theorem 4.1 just for one equation.*

Proof. First of all, we define the following solution operator

$$\begin{aligned} \mathcal{S} : L^1(\Omega) &\to L^1(\Omega) \\ v &\mapsto \widehat{w}(T,.), \end{aligned}$$

where $\widehat{w}$ is the unique solution of the following system

$$\begin{cases} \widehat{w} \in L^1(0,T; W_0^{1,1}(\Omega)) \cap \mathcal{C}([0,T], L^1(\Omega)), \\ \dfrac{\partial \widehat{w}}{\partial t} - d\Delta \widehat{w} = f \text{ in } \mathcal{D}'(Q_T), \\ \widehat{w}(0,.) \ = v \text{ in } L^1(\Omega). \end{cases} \tag{4.2}$$

the existence and uniqueness of $\widehat{w}$ solution of (4.2) can be deduced from [12]. To use the Schauder fixed point theorem, we prove that $\mathcal{S}$ is continuous and compact. The compacity is a direct consequence of [11] and for the continuity, we conside a sequence (v_n) in $L^1(\Omega)$, such that (v_n) converges strongly to v in $L^1(\Omega)$ and let $\widehat{w}_n = \mathcal{S}(v_n)$, $\widehat{w} = \mathcal{S}(v)$. From [12] we have the following estimate

$$\| \widehat{w}_n - \widehat{w} \|_{L^\infty(0,T;L^1(\Omega))} + \| \widehat{w}_n - \widehat{w} \|_{L^1(0,T;W_0^{1,1}(\Omega))} \leqslant C \| v_n - v \|_{L^1(\Omega)}$$

which implies,

$$\| \widehat{w}_n(T,.) - \widehat{w}(T,.) \|_{L^1(\Omega)} \leqslant C \| v_n - v \|_{L^1(\Omega)}$$

then the continuity is achieved. It remains to prove the existence of a radius $R_0 > 0$, such that the ball $B(0, R_0)$ of $L^1(\Omega)$ is invariant for $\mathcal{S}$. To do this, we take y solution of the following problem

$$\begin{cases} y \in L^1(0,T; W_0^{1,1}(\Omega)) \cap \mathcal{C}([0,T], L^1(\Omega)), \\ \dfrac{\partial y}{\partial t} - d\Delta y = f \text{ in } \mathcal{D}'(Q_T), \\ y(0,.) \ = 0 \text{ in } L^1(\Omega). \end{cases} \tag{4.3}$$

from (4.2) and (4.3), we get

$$\begin{cases} (\widehat{w} - y) \in L^1(0,T; W_0^{1,1}(\Omega)) \cap \mathcal{C}([0,T], L^1(\Omega)), \\ \dfrac{\partial (\widehat{w} - y)}{\partial t} - d\Delta (\widehat{w} - y) = 0 \text{ in } \mathcal{D}'(Q_T), \quad \text{for } j = 1, \ldots, M \\ (\widehat{w} - y)(0,.) \ = v \text{ in } L^1(\Omega). \end{cases} \tag{4.4}$$

According to the classical result of [7], the solution of (4.4) satisfies the following estimate

$$\begin{aligned} \| \widehat{w}(T,.) - y(T,.) \|_{L^1(\Omega)} &\leqslant \| \widehat{w}(0,.) - y(0,.) \|_{L^1(\Omega)} \exp(-\lambda_1 T) \\ &\leqslant \| v \|_{L^1(\Omega)} \exp(-\lambda_1 T). \end{aligned}$$

Consequently,

$$\| \mathcal{S}(v)(T,.) \|_{L^1(\Omega)} \leqslant \| y(T,.) \|_{L^1(\Omega)} + \| v \|_{L^1(\Omega)} \exp(-\lambda_1 T),$$

where λ_1 is the first eigenvalue of $-d\Delta$ with Dirichlet boundary condition. To get the desired result it suffices to choose

$$R_0 \geqslant \frac{\| y(T,.) \|_{L^1(\Omega)}}{1 - \exp(-\lambda_1 T)}.$$

This ends the proof. $\square$

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