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The Unique Solution of Some Operator Equations With an Application for Fractional Differential Equations

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ABSTRACT: In this paper we consider the existence and uniqueness of positive solutions to the following operator equation in an ordered Banach space E

$$T_1(x,x)+T_2(x,x)=x,\ x\in P,$$

where P is a cone in E. We study an application for fractional differential equations.

Key Words: Fractional differential equation, Mixed monotone operator, Boundary value problem, Positive solution.

Contents

1. Introduction and preliminaries

In [4] Liu and coauthors supposed the existence and uniqueness of positive solutions to the following operator equation in ordered Banach spaces E,

$$T_1(x,x)+T_2(x,x)=x,\ x\in P, \tag{1.1}$$

where P is a cone in E, and $T_1,T_2:P\times P\to P$ are two mixed monotone operators, which satisfy the following conditions:
(i) for all $t\in(0,1)$, there exists $\psi(t)\in(t,1]$, such that for all $x,y\in P$,

$$T_1(tx,\frac{1}{t}y)\geq\psi(t)T_1(x,y).$$

(ii) for all $t\in(0,1),x,y\in P$,

$$T_2(tx,\frac{1}{t}y)\geq tT_2(x,y).$$

In 2013, Y. Sang [6,7] proved some results on a class of mixed monotone operators with perturbations. In this paper, by applying results of Liu and Sang, we obtain some new results on the existence and uniqueness of positive solutions for operator equation $T_1(x,x)+T_2(x,x)=x,\ x\in P$. In the last section, we study an application for fractional differential equations.
Suppose that $(E,\|\ .\ \|)$ is a Banach space which is partially ordered by a cone $P\subseteq E$, that is, $x\leq y$ if and only if $y-x\in P$. If $x\neq y$, then we denote $x<y$ or $x>y$. We denote the zero element of E by θ. Recall that a non-empty closed convex set $P\subset E$ is a cone if it satisfies (i) $x\in P,\ \lambda\geq0\Longrightarrow\lambda x\in P$, (ii) $x\in P,\ -x\in P\Longrightarrow x=\theta$. A cone P is called normal if there exists a constant $N>0$ such that $\theta\leq x\leq y$ implies $\|x\|\leq N\|y\|$. Also we define the ordered interval $[x_1,x_2]=\{x\in E|x_1\leq x\leq x_2\}$ for all $x_1,x_2\in E$. We say that an operator $T:E\to E$ is increasing whenever $x\leq y$ implies $Tx\leq Ty$. T is called a positive operator if $T(x)\geq\theta$ for any $x\geq\theta$.

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Definition 1.1. [1,2] $T : P \times P \to P$ is said to be a mixed monotone operator if $T(x, y)$ is increasing in x and decreasing in y, i.e., $u_i, v_i \in P$ $(i = 1, 2)$, $u_1 \leq u_2, v_1 \geq v_2$ imply $T(u_1, v_1) \leq T(u_2, v_2)$. The element $x \in P$ is called a fixed point of T if $T(x, x) = x$.

An element $x^* \in D$ is called a fixed point of T if it satisfies $T(x^*, x^*) = x^*$. Let $h > \theta$, write $P_h = \{x \in E | \exists \lambda, \mu > 0 \text{ such that } \lambda h \leq x \leq \mu h\}$.
Throughout this section, we work in the Banach space

$$C[0, 1] = \{x : [0, 1] \to \mathbb{R} \text{ is continuous}\}$$

with the standard norm $\| x \| = \sup\{|x(t)| : t \in [0, 1]\}$. Let

$$P = \{x \in C[0, 1] : x(t) \geq 0, t \in [0, 1]\},$$

then it is a normal cone in $C[0, 1]$ and the normality constant is 1. We know that this space can be equipped with a partial order given by

$$x \leq y, \quad x, y \in C[0, 1] \Leftrightarrow x(t) \leq y(t),\ t \in [0, 1].$$

Definition 1.2. [3,5] The Riemann-Liouville fractional derivative of order α for a continuous function f is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} (\frac{d}{dt})^n \int_0^t \frac{f(s)}{(t - s)^{\alpha - n + 1}} ds,\ (n = [\alpha] + 1)$$

where the right-hand side is pointwise defined on $(0, \infty)$.

Definition 1.3. [3,5] Let $[a, b]$ be an interval in $\mathbb{R}$ and $\alpha > 0$. The Riemann-Liouville fractional order integral of a function $f \in L^1([a, b], \mathbb{R})$ is defined by

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t - s)^{1 - \alpha}} ds,$$

whenever the integral exists.

2. Main results

Lemma 2.1. *Let P be a normal cone in E. Assume that $T : P \times P \to P$ is a mixed monotone operator and satisfies:*
(A_1): there exists $h \in P$ with $h \neq \theta$ such that $T(h, h) \in P_h$;
(A_2): for any $u, v \in P$ and $t \in (a, b)$, there exists $\psi(t) \in (0, 1]$, $\tau(t) : (a, b) \to (0, 1)$ ($\tau(t)$ is surjection), with $\psi(t) > \tau(t)$ such that

$$T(\tau(t)u, \frac{1}{\tau(t)} v) \geq \psi(t) T(u, v), \quad \forall u, v \in P.$$

Then
(i) $T : P_h \times P_h \to P_h$;
(ii) there exist $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that

$$r v_0 \leq u_0 \leq v_0, \quad u_0 \leq T(u_0, v_0) \leq T(v_0, u_0) \leq v_0;$$

(iii) T has a unique fixed point x^* in P_h;
(iv) for any initial values $x_0, y_0 \in P_h$, by constructing successively the sequences as follows

$$x_n = T(x_{n-1}, y_{n-1}), \quad y_n = T(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots$$

we have $x_n \to x^*$, $y_n \to y^*$.

Proof: Firstly, from condition (A_2) we get

$$T(\frac{1}{\tau(t)}x,\tau(t)y)\le \frac{1}{\psi(t)}T(x,y),\quad \forall t\in(0,1),x,y\in P. \tag{2.1}$$

For any $u,v\in P_h$ there exist $\tau(t_1),\tau(t_2)\in(0,1)$ such that

$$\tau(t_1)h\le u\le \frac{1}{\tau(t_1)}h,\quad \tau(t_2)h\le v\le \frac{1}{\tau(t_2)}h.$$

Let $\tau(t)=\min\{\tau(t_1),\tau(t_2)\}$. Then $\tau(t)\in(0,1)$. From (2.1) and the mixed monotone properties of operator T, we have

$$T(u,v)\le T(\frac{1}{\tau(t_1)}h,\tau(t_2)h)\le T(\frac{1}{\tau(t)}h,\tau(t)h)\le \frac{1}{\psi(t)}T(h,h),$$

$$T(u,v)\ge T(\tau(t_1)h,\frac{1}{\tau(t_2)}h)\ge T(\tau(t)h,\frac{1}{\tau(t)}h)\ge \psi(t)T(h,h).$$

It follows from $T(h,h)\in P_h$ that $T(u,v)\in P_h$. Hence we have $T:P_h\times P_h\to P_h$. Since $T(h,h)\in P_h$, we can choose a sufficiently small number $t_0\in(a,b)$ and function $\tau(t_0)\in(0,1)$ such that

$$\tau(t_0)h\le T(h,h)\le \frac{1}{\tau(t_0)}h. \tag{2.2}$$

Noting that $\tau(t_0)<\psi(t_0)\le 1$, we can take a positive integer k such that

$$(\frac{\psi(t_0)}{\tau(t_0)})^k\geqslant \frac{1}{\tau(t_0)}. \tag{2.3}$$

Put $u_0=(\tau(t_0))^kh, v_0=\frac{1}{(\tau(t_0))^k}h$. Evidently, $u_0,v_0\in P_h$ and $u_0=(\tau(t_0))^{2k}v_0<v_0$. Take, $r\in\big(0,(\tau(t_0))^{2k}\big]$, then $r\in(0,1)$ and $u_0\geqslant rv_0$. By the mixed monotone properties of T, we have $T(u_0,v_0)\le T(v_0,u_0)$. Further, combining condition (A_2) with (2.2) and (2.3) we have

$$\begin{aligned}
T(u_0,v_0)&=T((\tau(t_0))^kh,\frac{1}{(\tau(t_0))^k}h)=T((\tau(t_0)).(\tau(t_0))^{k-1}h,\frac{1}{\tau(t_0)}.\frac{1}{(\tau(t_0))^{k-1}}h)\\
&\geqslant \psi(t_0)T((\tau(t_0))^{k-1}h,\frac{1}{(\tau(t_0))^{k-1}}h)\\
&=\psi(t_0)T((\tau(t_0)).(\tau(t_0))^{k-2}h,\frac{1}{(\tau(t_0))}.\frac{1}{(\tau(t_0))^{k-2}}h)\\
&\geqslant \psi(t_0).\psi(t_0)T((\tau(t_0))^{k-2}h,\frac{1}{(\tau(t_0))^{k-2}}h)\\
&\geqslant\ldots\geqslant(\psi(t_0))^kT(h,h)\geqslant(\psi(t_0))^k\tau(t_0)h\geqslant(\tau(t_0))^kh=u_0.
\end{aligned}$$

From (2.1) we get

$$\begin{aligned}
T(v_0,u_0)&=T(\frac{1}{(\tau(t_0))^k}h,(\tau(t_0))^kh)=T(\frac{1}{(\tau(t_0))}.\frac{1}{(\tau(t_0))^{k-1}}h,(\tau(t_0)).(\tau(t_0))^{k-1}h)\\
&\leqslant \frac{1}{\psi(t_0)}T(\frac{1}{(\tau(t_0))^{k-1}}h,(\tau(t_0))^{k-1}h)\\
&=\frac{1}{\psi(t_0)}T(\frac{1}{(\tau(t_0))}.\frac{1}{(\tau(t_0))^{k-2}}h,(\tau(t_0)).(\tau(t_0))^{k-2}h)\\
&\leqslant \frac{1}{\psi(t_0)}.\frac{1}{\psi(t_0)}T(\frac{1}{(\tau(t_0))^{k-2}}h,(\tau(t_0))^{k-2}h)\\
&\leqslant\ldots\leqslant\frac{1}{(\psi(t_0))^k}T(h,h)\leqslant\frac{1}{\tau(t_0)(\psi(t_0))^k}h\leqslant\frac{1}{(\tau(t_0))^k}h=v_0.
\end{aligned}$$

Thus we have $u_0 \leq T(u_0, v_0) \leq T(v_0, u_0) \leq v_0$.
Construct successively the sequences

$$u_n = T(u_{n-1}, v_{n-1}), \quad v_n = T(v_{n-1}, u_{n-1}), \quad n = 1, 2, ...$$

Evidently, $u_1 \leqslant v_1$. By the mixed monotone properties of T, we obtain $u_n \leqslant v_n, n = 1, 2, ...$ It also follows from the mixed monotone properties of T that

$$u_0 \leq u_1 \leq \ldots \leq u_n \leq \ldots \leq v_n \leq \ldots \leq v_1 \leq v_0. \tag{2.4}$$

Noting that $u_0 \geqslant rv_0$, we can get $u_n \geqslant u_0 \geqslant rv_0 \geqslant rv_n, n = 1, 2, ...$ Let

$$r_n = \sup\{r > 0 | u_n \geq rv_n\}, \quad n = 1, 2, \ldots.$$

Thus we have $u_n \geq rv_n, n = 1, 2, \ldots.$ and then $u_{n+1} \geq u_n \geq r_n v_n \geq r_n v_{n+1}, n = 1, 2, \ldots.$ Therefore, $r_{n+1} \geq r_n, i.e., \{r_n\}$ is increasing in $(0, 1]$. Suppose $r_n \to r^*$ as $n \to \infty$, then $r^* = 1$. Otherwise, $0 < r^* < 1$, by (A_2) there exists $z_1 \in (a, b)$ such that $\tau(z_1) = r^*$. Consider the following two cases:
Cases i: There exists an integer N such that $r_N = r^*$. In this case, we have $r_n = r^*$ and $u_n \geq r^* v_n$ for all $n \geq N$ hold. Hence

$$\begin{aligned} u_{n+1} &= T(u_n, v_n) \geq T(r^* v_n, \frac{1}{r^*} u_n) = T(\tau(z_1) v_n, \frac{1}{\tau(z_1)} u_n) \\ &\geq \psi(z_1) T(v_n, u_n) = \psi(z_1) v_{n+1} \quad n \geq N. \end{aligned}$$

By the definition of r_n, we have $r_{n+1} = r^* \geq \psi(z_1) > \tau(z_1) = r^*, \quad n \geq N$, which is a contradiction.
Case ii: For all integers n, $r_n < r^*$. Then we obtain $0 < \frac{r_n}{r^*} < 1$. By (A_2), there exist $z_n \in (a, b)$ such that $\tau(z_n) = \frac{r_n}{r^*}$. Hence

$$\begin{aligned} u_{n+1} &= T(u_n, v_n) \geq T(r_n v_n, \frac{1}{r_n} u_n) = T(\frac{r_n}{r^*} r^* v_n, \frac{1}{\frac{r_n}{r^*} r^*} u_n) \\ &\geq T(\tau(z_n) r^* v_n, \frac{1}{\tau(z_n) r^*} u_n) \\ &\geq \psi(z_n) T(r^* v_n, \frac{1}{r^*} u_n) = \psi(z_n) T(\tau(z_1) v_n, \frac{1}{\tau(z_1)} u_n) \geq \psi(z_n) \psi(z_1) T(v_n, u_n) \\ &\geq \psi(z_n) \psi(z_1) v_{n+1} \end{aligned}$$

By the definition of r_n, we have $r_{n+1} \geq \psi(z_n)\psi(z_1) \geq \tau(z_n)\psi(z_1) = \frac{r_n}{r^*}\tau(z_1)$. Let $n \to \infty$, we have $r^* \geq \psi(z_1) > \tau(z_1) = r^*$ which is also a contradiction. Thus, $\lim_{n\to\infty} r_n = 1$. For any natural number p we have

$$\theta \leq u_{n+p} - u_n \leq v_n - u_n \leq v_n - r_n v_n = (1 - r_n) v_n \leq (1 - r_n) v_0,$$

$$\theta \leq v_n - v_{n+p} \leq v_n - u_n \leq v_n - r_n v_n \leq (1 - r_n) v_0.$$

Since the cone P is normal, we have

$$\| u_{n+p} - u_n \| \leq N(1 - r_n) \| v_0 \| \to 0, \quad (as \quad n \to \infty),$$

$$\| v_n - v_{n+p} \| \leq N(1 - r_n) \| v_0 \| \to 0, \quad (as \quad n \to \infty),$$

where N is the normality constant of P. So we can claim that $\{u_n\}$ and $\{v_n\}$ are cauchy sequences. Because E is complete, there exist u^*, v^* such that $u_n \to u^*, v_n \to v^* as \quad n \to \infty$. By (2.4), we know that $u_n \leq u^* \leq v^* \leq v_n$ with $u^*, v^* \in P_h$ and we have $\theta \leq v^* - u^* \leq v_n - u_n \leq (1 - r_n) v_0$. Further

$$\| v^* - u^* \| \leq N(1 - r_n) \| v_0 \| \to 0, (as \quad n \to \infty).$$

Thus $u^* = v^*$. Let $x^* := u^* = v^*$ and then we obtain

$$u_{n+1} = T(u_n, v_n) \leq T(x^*, x^*) \leq T(v_n, u_n) = v_{n+1},$$

Let $n \to \infty$, we get $x^* = T(x^*, x^*)$. That is, x^* is a fixed point of T in P_h . In the following, we prove that x^* is the unique fixed point of T in P_h. In fact, suppose $\bar{x}$ is a fixed point of T in P_h. Since $x^*, \bar{x} \in P_h$, there exist positive numbers $\bar{\mu_1}, \bar{\mu_2}, \bar{\lambda_1}, \bar{\lambda_2} > 0$ such that

$$\bar{\mu_1} h \leqslant x^* \leqslant \bar{\lambda_1} h, \quad \bar{\mu_2} h \leqslant \bar{x} \leqslant \bar{\lambda_2} h.$$

Then we obtain

$$\bar{x} \leqslant \bar{\lambda_2} h = \frac{\bar{\lambda_2}}{\bar{\mu_1}} \bar{\mu_1} h \leqslant \frac{\bar{\lambda_2}}{\bar{\mu_1}} x^*, \quad \bar{x} \geqslant \bar{\mu_2} h = \frac{\bar{\mu_2} h}{\bar{\lambda_1}} \bar{\lambda_1} h \geqslant \frac{\bar{\mu_2}}{\bar{\lambda_1}} x^*.$$

Let $e_1 = sup\{0 < e \leq 1 | ex^* \leq \bar{x} \leq e^{-1} x^*\}$. Evidently, $0 < e_1 \leq 1$, $e_1 x^* \leq \bar{x} \leq e_1^{-1} x^*$. If $0 < e_1 < 1$, according to (H_1) there exists $z_2 \in (a, b)$ such that $\tau(z_2) = e_1$. then

$$\begin{aligned} \bar{x} &= T(\bar{x}, \bar{x}) \geq T(e_1 x^*, e_1^{-1} x^*) = T(\tau(z_2) x^*, \tau^{-1}(z_2) x^*) \geq \psi(z_2) T(x^*, x^*) \\ &= \psi(z_2) x^*. \end{aligned}$$

and

$$\begin{aligned} \bar{x} &= T(\bar{x}, \bar{x}) \leq T(e_1^{-1} x^*, e_1 x^*) = T(\tau^{-1}(z_2) x^*, \tau(z_2) x^*) \leq \frac{1}{\psi(z_2)} T(x^*, x^*) \\ &= \frac{1}{\psi(z_2)} x^*. \end{aligned}$$

we have

$$\psi(z_2) x^* \leq \bar{x} \leq \frac{1}{\psi(z_2)} x^*.$$

Hence $e_1 \geq \psi(z_2) > \tau(z_2) = e_1$ which is a contradiction. Thus we have $e_1 = 1$ i.e. $\bar{x} = x^*$. Therefore, T has a unique fixed point x^* in P_h. Note that $[u_0, v_0] \subset P_h$, then we know that x^* is the unique fixed point of T in $[u_0, v_0]$. Now we construct successively the sequences $x_n = T(x_{n-1}, y_{n-1}), y_n = T(y_{n-1}, x_{n-1}), n = 1, 2, ...,$ for any initial points $x_0, y_0 \in P_h$. Since $x_0, y_0 \in P_h$, we can choose numbers $e_2, e_3 \in (0, 1)$ such that

$$e_2 h \leqslant x_0 \leqslant \frac{1}{e_2} h, \quad e_3 h \leqslant y_0 \leqslant \frac{1}{e_3} h.$$

From (H_1) there is $z_3 \in (a, b)$ such that Let $\tau(z_3) = e^* = \min\{e_2, e_3\}$. Then $e^* \in (0, 1)$ and $e^* h \leqslant x_0$, $y_0 \leqslant \frac{1}{e^*} h$. We can choose a sufficiently large positive integer m such that $(\frac{\psi(z_3)}{\tau(z_3)})^m \geqslant \frac{1}{\tau(z_3)}$. Put $\bar{u_0} = \tau(z_3) h, \bar{v_0} = \frac{1}{\tau(z_3)} h$. It is easy to see that $\bar{u_0}, \bar{v_0} \in P_h$ and $\bar{u_0} \leq x_0$, $y_0 \leq \bar{v_0}$. Constructing successively the sequences

$$\begin{aligned} x_n &= T(x_{n-1}, y_{n-1}), y_n = T(y_{n-1}, x_{n-1}), n = 1, 2, ..., \\ \bar{u_n} &= T(\bar{u_{n-1}}, \bar{v_{n-1}}), \bar{v_n} = T(\bar{v_{n-1}}, \bar{u_{n-1}}), n = 1, 2, ..., \end{aligned}$$

By using the mixed monotone properties of operator T, we have $\bar{u_n} \leq x_n$, $y_n \leq \bar{v_n}, n = 1, 2, ...,$ similarly, it follows that there exists $y^* \in P_h$ such that $T(y^*, y^*) = y^*$,$\lim_{n\to\infty} \bar{u_n} = \lim_{n\to\infty} \bar{v_n} = y^*$. By the uniqueness of fixed points of operator T in P_h, we get $x^* = y^*$. Since cone P is normal, we have $\lim_{n\to\infty} \bar{u_n} = \lim_{n\to\infty} \bar{v_n} = x^*$. This completes the proof. □

Theorem 2.1. *Let P be a normal cone in E, and $T_1, T_2 : P \times P \to P$ be two mixed monotone operators. Assume that for all $a < t < b$, there exist two positive-valued functions $\tau(t), \varphi(t,x,y)$ (with $\varphi(t,x,y) \leq 1$) on an interval (a,b) such that;*
(H_{11}) *$\tau : (a,b) \to (0,1)$ is a surjection;*
(H_{12}) *there exists a constant $\delta > 0$ such that $T_1(x,y) \geq \delta T_2(x,y)$ for all $x \in P$;*
(H_{13}) *$T_1(\tau(t)x, \frac{1}{\tau(t)}y) \geq \varphi(t,x,y)T_1(x,y)$, $T_2(\tau(t)x, \frac{1}{\tau(t)}y) \geq \tau(t)T_2(x,y)$ and $\tau(t) \leq (1-\varphi(t,x,y))\delta + 1$ for all $t \in (a,b)$, $x, y \in P$;*
(H_{14}) *there is $h \in P$ with $h > \theta$ such that $T_1(h,h) \in P_h, T_2(h,h) \in P_h$.*
Then the operator equation $T_1(x,x) + T_2(x,x) = x$ has a unique solution x^* in P_h. Moreover, for any initial values $x_0, y_0 \in P_h$, by constructing successively the sequence as follows

$$x_n = T_1(x_{n-1}, y_{n-1}) + T_2(x_{n-1}, y_{n-1})$$
$$y_n = T_1(y_{n-1}, x_{n-1}) + T_2(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ and $y_n \to x^*$ in E as $n \to \infty$.

Proof: Firstly, from (H_{13}) for any $t \in (a,b)$ and $x, y \in P$, we have

$$T_1(\frac{1}{\tau(t)}x, \tau(t)y) \leq \frac{1}{\varphi(t,x,y)}T_1(x,y) \tag{2.5}$$

and

$$T_2(\frac{1}{\tau(t)}x, \tau(t)y) \leq \frac{1}{\tau(t)}T_2(x,y). \tag{2.6}$$

Since $T_1(h,h) \in P_h,\ T_2(h,h) \in P_h$, there exist constants $a_i > 0,\ b_i > 0\ (i = 1,2)$ such that

$$a_1 h \leq T_1(h,h) \leq b_1 h, \tag{2.7}$$

$$a_2 h \leq T_2(h,h) \leq b_2 h. \tag{2.8}$$

Next we show $T_1 : P_h \times P_h \to P_h$. For any $x, y \in P_h$, we can choose two sufficiently functions $\tau(t_1), \tau(t_2) : (a,b) \to (0,1)$ such that

$$\tau(t_1)h \leq x \leq \frac{1}{\tau(t_1)}h, \quad \tau(t_2)h \leq y \leq \frac{1}{\tau(t_2)}h. \tag{2.9}$$

Let $\tau(t) = \min\{\tau(t_1), \tau(t_2)\}$, then $\tau(t) : (a,b) \to (0,1)$, by (2.5), (2.7) and (2.9), we have

$$T_1(x,y) \leq T_1(\frac{1}{\tau(t)}h, \tau(t)h) \leq \frac{1}{\varphi(t,x,y)}T_1(h,h) \leq \frac{b_1 h}{\varphi(t,x,y)},$$

$$T_1(x,y) \geq T_1(\tau(t)h, \frac{1}{\tau(t)}h) \geq \varphi(t,x,y)T_1(h,h) \geq \varphi(t,x,y)a_1 h.$$

Evidently $\frac{b_1}{\varphi(t,x,y)},\ \varphi(t,x,y)a_1 > 0$. thus $T_1(x,y) \in P_h$; that is, $T_1 : P_h \times P_h \to P_h$. Finally, we show $T_2 : P_h \times P_h \to P_h$. for any $x, y \in P_h$, we can choose two sufficiently function $\tau(t_3), \tau(t_4) : (a,b) \to (0,1)$ such that

$$\tau(t_3)h \leq x \leq \frac{1}{\tau(t_3)}h, \quad \tau(t_4)h \leq y \leq \frac{1}{\tau(t_4)}h. \tag{2.10}$$

Let $\tau(t') = \min\{\tau(t_3), \tau(t_4)\}$, then $\tau(t') \in (0,1)$, by (2.6), (2.8) and (2.10), we have

$$T_2(x,y) \leq T_2(\frac{1}{\tau(t')}h, \tau(t')h) \leq \frac{1}{\tau(t')}T_2(h,h) \leq \frac{1}{\tau(t')}b_2 h,$$
$$T_2(x,y) \geq T_2(\tau(t')h, \frac{1}{\tau(t')}h) \geq \tau(t')T_2(h,h) \geq \tau(t')a_2 h.$$

Evidently, $\frac{1}{\tau(t')}b_2, \tau(t')a_2 > 0$. Thus $T_2(x,y) \in P_h$, that is, $T_2 : P_h \times P_h \to P_h$. Now we define the operator $T = T_1 + T_2 : P_h \times P_h \to P_h$ by

$$T(x,y) = T_1(x,y) + T_2(x,y), \quad x, y \in P_h. \tag{2.11}$$

Then $T : P_h \times P_h \to P_h$ is a mixed monotone operator since $T_1(h,h) \in P_h, T_2(h,h) \in P_h$, we get $T(h,h) = T_1(h,h) + T_2(h,h) \in P_h$. In the following, we show that for any $t \in (a,b)$, there exists $\psi(t) \in (0,1]$ such that for all $x, y \in P$,

$$T(\tau(t)x, \frac{1}{\tau(t)}y) \geq \psi(t)T(x,y),$$

For any $x, y \in P$, by (H_{12}), we have

$$T_1(x,y) + \delta T_1(x,y) \geq \delta T_2(x,y) + \delta T_1(x,y). \tag{2.12}$$

It follows from (2.12) that

$$T_1(x,y) \geq \frac{T_1(x,y) + T_2(x,y)}{1+\delta^{-1}} = \frac{T(x,y)}{1+\delta^{-1}}. \tag{2.13}$$

By (H_{13}), for all $x, y \in P$, we have

$$\begin{aligned}
T(\tau(t)x, \tau^{-1}(t)y) - tT(x,y) =& T_1(\tau(t)x, \tau^{-1}(t)y) + T_2(\tau(t)x, \tau^{-1}(t)y) \\
& - t(T_1(x,y) + T_2(x,y)) \\
\geq& \varphi(t,x,y)T_1(x,y) + \tau(t)T_2(x,y) \\
& - t(T_1(x,y) + T_2(x,y)) \\
\geq& (\varphi(t,x,y) - t)T_1(x,y) + (\tau(t) - t)T_2(x,y) \\
\geq& (\varphi(t,x,y) - t)T_1(x,y) + (\tau(t) - t)\delta^{-1}T_1(x,y) \\
\geq& (\varphi(t,x,y) - t)\frac{T(x,y)}{1+\delta^{-1}} + (\tau(t) - t)\delta^{-1}\frac{T(x,y)}{1+\delta^{-1}} \\
\geq& \frac{(\varphi(t,x,y) - t) + (\tau(t) - t)\delta^{-1})T(x,y)}{1+\delta^{-1}}
\end{aligned}$$

It follows from up that for all $x, y \in P$,

$$\begin{aligned}
T(\tau(t)x, \tau^{-1}(t)y) &\geq tT(x,y) + \frac{(\varphi(t,x,y) - t) + (\tau(t) - t)\delta^{-1})}{1+\delta^{-1}}T(x,y) \\
&\geq (t + \frac{(\varphi(t,x,y) - t) + (\tau(t) - t)\delta^{-1})}{1+\delta^{-1}})T(x,y)
\end{aligned}$$

Let $\psi(t) = (t + \frac{(\varphi(t,x,y)-t)+(\tau(t)-t)\delta^{-1})}{1+\delta^{-1}}) = \dfrac{\varphi(t,x,y) + \tau(t)\delta^{-1}}{1+\delta^{-1}}$, then $\psi(t) \in (0,1]$, $\tau(t) \in (0,1), t \in (a,b)$ and

$$T(\tau(t)x, \tau^{-1}(t)y) \geq \psi(t)T(x,y),$$

By Lemma 2.1 the conclusions of Theorem 2.1 holds. □

3. Applications

In this section, we apply the results in Section 2 to study nonlinear fractional differential equations with two-point boundary conditions. We here consider the existence and uniqueness of positive solutions for the following fractional boundary value problem (FBVP for short):

$$\begin{aligned}
& -D^{\alpha}_{0+}u(w) = F_1(w, u(w)) + F_2(w, u(w)), \quad w = \tau(t), 0 < w < 1, n-1 < \alpha \leq n \\
& u^i(0) = 0, \quad 0 \leq i \leq n-2, \\
& [D^{\beta}_{0+}u(w)]_{w=1} = 0, \quad 1 \leq \beta \leq n-2,
\end{aligned} \tag{3.1}$$

where $D^{\alpha}_{0+}u(w)$ is the Riemann-Liouville fractional derivative of order $\alpha, n > 2, n \in \mathbb{N}$.

Theorem 3.1. *Assume that* $F_1(w,x)=f_1(w,x,x), F_2(w,x)=f_2(w,x,x)$ *and satisfying the following conditions* H_1-H_4*:*
(H_1) $f_1,f_2:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ *are continuous, and for all* $w\in[0,1], f_2(w,0,1)\not\equiv 0$*;*
(H_2) *for fixed* $w\in[0,1]$, $y\in[0,+\infty)$, $f_1(w,x,y), f_2(w,x,y)$ *are increasing in* $x\in[0,+\infty)$*; for fixed* $w\in[0,1]$ *and* $x\in[0,+\infty)$, $f_1(w,x,y), f_2(w,x,y)$ *are decreasing in* $y\in[0,+\infty)$*;*
(H_3) *for all* $\lambda\in(a,b)$*, there exist* $\tau(\lambda)\in[0,1]$ *(*$\tau(t):(a,b)\to[0,1]$ *is a surjection) such that for all* $w\in[0,1]$, $x,y\in[0,+\infty)$, $f_1(w,\tau(\lambda)x,\tau^{-1}(\lambda)y)\geq\varphi(t,x,y)f_1(w,x,y)$, $f_2(w,\tau(\lambda)x,\tau^{-1}(\lambda)y)\geq\tau(\lambda)f_2(w,x,y)$*;*
(H_4) *there exists a constant* $\delta>0$*, such that for all* $w\in[0,1]$, $x,y\in[0,+\infty)$, $f_1(w,x,y)\geq\delta f_2(w,x,y)$*.*
Then the problem (3.1) has a unique positive solution u^* *in* P_h*, where* $h(t)=\tau^{\alpha-1}(t), w=\tau(t)\in[0,1]$*, and for any* $u_0,v_0\in P_h$*, by constructing successively the sequences as follows*

$$u_{n+1}(w)=\int_0^1 G(w,s)[f_1(s,v_n(s),u_n(s))+f_2(s,v_n(s),u_n(s))]ds,\quad n=0,1,...$$
$$v_{n+1}(w)=\int_0^1 G(w,s)[f_1(s,v_n(s),u_n(s))+f_2(s,v_n(s),u_n(s))]ds,\quad n=0,1,...,$$

we have $u_n(w)\rightrightarrows u^*(w), w\in[0,1]$ *and* $v_n(w)\rightrightarrows u^*(w), w\in[0,1]$ *that is,* $\{u_n(w)\}$ *and* $\{v_n(w)\}$ *both converges to* $u^*(w)$ *uniformly for all* $w\in[0,1]$*.*

Proof: The proof is similar with the proof of the Theorem 4.4 in [4]. □

Example 3.1. *Consider the following two-point boundary value problem*

$$\begin{aligned}&-D^{\alpha}_{0+}u(w)=2w^3+\sqrt[3]{u}+\frac{1}{\sqrt[3]{u+1}}+\frac{\sqrt{u+1}}{\sqrt{u+1}}\quad 0<w<1, n-1<\alpha\leq n\\&u^i(0)=0,\quad 0\leq i\leq n-2,\\&[D^{\beta}_{0+}u(w)]_{w=1}=0,\quad 1\leq\beta\leq n-2.\end{aligned}\tag{3.2}$$

The above equations can be written in the form of problem with the functions $f_1,f_2:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ *defined by*

$$f_1(w,x,y)=w^3+\sqrt[3]{x}+\frac{1}{\sqrt[3]{y+1}},\quad w=\tau(t)\in[0,1],\quad x,y\geq 0$$
$$f_2(w,x,y)=w^3+\frac{\sqrt{x+1}}{\sqrt{y+1}},\quad w=\tau(t)\in[0,1],\quad x,y\geq 0.$$

Now we show in the following that all the conditions of Theorem 3.1 are satisfied

1) *Clearly, the functions* $f_1,f_2:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ *are continuous with* $f_2(w,0,1)\not\equiv 0$
2) *We observe that for fixed* $w=\tau(t)\in[0,1]$ *and* $y\in[0,+\infty)$, $f_1(w,x,y)$, $f_2(w,x,y)$ *are increasing in* $x\in[0,+\infty)$*; for fixed* $\tau(t)\in[0,1]$ *and* $x\in[0,+\infty)$, $f_1(w,x,y), f_2(w,x,y)$ *are decreasing in* $y\in[0,+\infty)$*;*
3) *For all* $\lambda\in(a,b), t\in(a,b), \tau(\lambda)\in[0,1]$ *and* $x\geq 0, y\geq 0$*, taking* $\varphi(t,x,y)=\sqrt[3]{\tau(\lambda)}$*, we have*

$$\begin{aligned}f_1(w,\tau(\lambda)x,\tau^{-1}(\lambda)y)&=(w^3+\sqrt[3]{\tau(\lambda)x}+\frac{1}{\sqrt[3]{\tau^{-1}(\lambda)y+1}})\\&=(w^3+\sqrt[3]{\tau(\lambda)x}+\frac{\sqrt[3]{\tau(\lambda)}}{\sqrt[3]{y+\tau(\lambda)}})\\&\geq\sqrt[3]{\tau(\lambda)}(w^3+\sqrt[3]{x}+\frac{1}{\sqrt[3]{1+y}})\\&=\sqrt[3]{\tau(\lambda)}f_1(w,x,y)\\&=\varphi(t,x,y)f_1(w,x,y).\end{aligned}$$

For all $\lambda\in(a,b), t\in(a,b), \tau(\lambda)\in[0,1]$ *and* $x\geq 0, y\geq 0$*, we have*

$$\begin{aligned}f_2(w,\tau(\lambda)x,\tau^{-1}(\lambda)y)&=(w^3+\frac{\sqrt{\tau(\lambda)x+1}}{\sqrt{\tau^{-1}(\lambda)y+1}})\geq(w^3+\frac{\sqrt{\tau(\lambda)x+\tau(\lambda)}}{\sqrt{\tau^{-1}(\lambda)y+\tau^{-1}(\lambda)}})\\&=(w^3+\frac{(\tau^{\frac{1}{2}}(\lambda))\sqrt{x+1}}{(\tau^{\frac{-1}{2}}(\lambda))\sqrt{y+1}})\geq(\tau(\lambda)w^3+\tau(\lambda)\frac{\sqrt{x+1}}{\sqrt{y+1}})\\&=\tau(\lambda)f_2(w,x,y).\end{aligned}$$

4) Taking $\delta = 1$, *for all* $w = \tau(t) \in [0,1]$ *and* $x \geq 0, y \geq 0$, *we have*

$$\begin{aligned} f_1(w,x,y) &= w^3 + \sqrt[3]{\tau(\lambda)x} + \frac{1}{\sqrt[3]{\tau^{-1}(\lambda)y+1}} \\ &\geq w^3 + \frac{\sqrt{\tau(\lambda)x+1}}{\sqrt{\tau^{-1}(\lambda)y+1}} \\ &= f_2(w,x,y). \end{aligned}$$

Thus we have proved that all the conditions of Theorem 3.1 are satisfied. Hence we deduce that (3.2) has one and only one positive solution $x^* \in P_h$, where $h(t) = \tau^{\alpha-1}(t), \tau(t) \in [0,1]$.

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