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Certain Results Associated with Hybrid Relatives of the q-Sheffer Sequences

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ABSTRACT: The intended objective of this paper is to introduce a new class of the hybrid q-Sheffer polynomials by means of the generating function and series definition. The determinant definition and other striking properties of these polynomials are established. Certain results for the continuous q-Hermite-Appell polynomials are obtained. The graphical depictions are performed for certain members of the hybrid q-Sheffer family. The zeros of these members are also explored using numerical simulations. Finally, the orthogonality condition for the hybrid q-Sheffer polynomials is established.

Key Words: q-Sheffer polynomials, q-Appell polynomials, Hybrid q-Sheffer polynomials.

Contents

1. Introduction and preliminaries

The advancement of q-calculus stems from the applications in numerous fields such as engineering, economics and mathematics. The q-calculus is a generalization of many research areas including special functions, complex analysis and particle physics. The q-analogues of various orthogonal polynomials and functions presume a very pleasant form reminding directly of their classical counterparts. The q-calculus is mostly being used by physicists. Also, it has served as a bridge between mathematics and physics. In short, q-calculus is quite a popular subject today.

The development in q-calculus is heavily dependent on the use of a proper notation [4]. Throughout the present paper, $\mathbb{C}$ indicates the set of complex numbers, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{N}_0$ indicates the set of non-negative integers. Further, the variable $q \in \mathbb{C}$ such that $|q| < 1$. The following q-standard notations and definitions are taken from [4].

The q-analogues of $a \in \mathbb{C}$ is defined by

$$[a]_q = \frac{1-q^a}{1-q}\ ,\ q \in \mathbb{C} - \{1\} \tag{1.1}$$

and q-factorial function is specified by

$$[n]_q! = \prod_{k=1}^{n} [k]_q,\ \ [0]_q! = 1,\ \ n \in \mathbb{N}. \tag{1.2}$$

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The q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \ k = 0, 1, \ldots, n, \ n \in \mathbb{N}_0. \tag{1.3}$$

and q-exponential function is specified by

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{((1-q)x;q)_\infty}, \ |x| < |1-q|^{-1}. \tag{1.4}$$

The q-analogue of the derivative of a function f at a point $x \in \mathbb{C}$, denoted by $D_q f$ is defined by [12]:

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \ x \neq 0. \tag{1.5}$$

If $f'(0)$ exists, then $D_q f(0) = f'(0)$. As q tends to 1, the q-derivative reduces to the usual derivative. Also, for any two arbitrary functions $f(z)$ and $g(z)$, the q-derivative satisfies the following relation:

$$D_{q,z}(f(z)g(z)) = f(qz)D_{q,z}g(z) + g(z)D_{q,z}f(z). \tag{1.6}$$

The use of polynomials in various fields of science and engineering is quite miraculous. In this article, we shall focus on three special polynomials, namely the q-Appell, q-Sheffer and hybrid q-Sheffer polynomials. In the last few decades, the engrossment in the q-Appell and q-Sheffer sequences and their applications in different fields have moderately incremented. The q-Appell polynomials $A_{n,q}(x)$ are introduced and studied from distinct approaches, see for example [2,13].

Definition 1.1. *For $q \in \mathbb{C}$, $0 < |q| < 1$, the q-Appell polynomials $A_{n,q}(x)$ possess the following generating function [2]:*

$$A_q(t)e_q(xt) = \sum_{n=0}^{\infty} A_{n,q}(x)\frac{t^n}{[n]_q!}, \tag{1.7}$$

where $A_q(t)$ is analytic at $t = 0$ and $A_{n,q} := A_{n,q}(0)$.

Note. Taking $q \to 1$ in definition (1.7), the q-Appell polynomials $A_{n,q}(x)$ reduces to the Appell polynomials $A_n(x)$ [5].

Based on appropriate selection for the function $A_q(t)$, distinct members related to the family of q-Appell polynomials can be obtained. These members along with their names, generating functions, series definitions and related numbers are given in Table 1.

Table 1: Certain members belonging to the q-Appell family

S. No.	$A_q(t)$	Name of the q-special polynomials and related numbers	Generating function	Series definition
I.	$\left(\frac{t}{e_q(t)-1}\right)$	q-Bernoulli polynomials and numbers [10]	$\left(\frac{t}{e_q(t)-1}\right) e_q(xt) = \sum\limits_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{[n]_q!}$ $\left(\frac{t}{e_q(t)-1}\right) = \sum\limits_{n=0}^{\infty} B_{n,q}\ \frac{t^n}{[n]_q!}$ $B_{n,q} := B_{n,q}(0)$	$B_{n,q}(x) = \sum\limits_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_{k,q}\ x^{n-k}$
II.	$\left(\frac{2}{e_q(t)+1}\right)$	q-Euler polynomials and numbers [10]	$\left(\frac{2}{e_q(t)+1}\right) e_q(xt) = \sum\limits_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{[n]_q!}$ $\left(\frac{2}{e_q(t)+1}\right) = \sum\limits_{n=0}^{\infty} E_{n,q}\ \frac{t^n}{[n]_q!}$ $E_{n,q} := E_{n,q}(0)$	$E_{n,q}(x) = \sum\limits_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q}x^{n-k}$
III.	$\left(\frac{2t}{e_q(t)+1}\right)$	q-Genocchi polynomials and numbers [6]	$\left(\frac{2t}{e_q(t)+1}\right) e_q(xt) = \sum\limits_{n=0}^{\infty} G_{n,q}(x)\ \frac{t^n}{[n]_q!}$ $\left(\frac{2t}{e_q(t)+1}\right) = \sum\limits_{n=0}^{\infty} G_{n,q}\ \frac{t^n}{[n]_q!}$ $G_{n,q} := G_{n,q}(0)$	$G_{n,q}(x) = \sum\limits_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q G_{k,q}x^{n-k}$

Definition 1.2. *For $q \in \mathbb{C}$, $0 < |q| < 1$, the q-Sheffer polynomials $s_{n,q}(x)$ are defined by the following generating function [9]:*

$$\mathcal{G}_q(t)e_q[xH_q(t)] = \sum_{n=0}^{\infty} s_{n,q}(x)\frac{t^n}{[n]_q!}, \tag{1.8}$$

where $\mathcal{G}_q(t) = \sum\limits_{n=0}^{\infty} \mathcal{G}_n \frac{t^n}{[n]_q!} \in \mathcal{E}_q(0)$, $H_q(t) \in \mathcal{E}_q(1)$ and $\mathcal{E}_q(n)$ be the set of Eulerian generating functions of the form:

$$a_n\frac{t^n}{[n]_q!} + a_{n+1}\frac{t^{n+1}}{[n+1]_q!} + a_{n+2}\frac{t^{n+2}}{[n+2]_q!} + \cdots\ , \qquad a_n = 1. \tag{1.9}$$

Remark 1.3. *For $H_q(t) = t$, generating function (1.8) of the q-Sheffer polynomials $s_{n,q}(x)$ yields generating function (1.7) for the q-Appell polynomials $A_{n,q}(x)$. By taking $q \to 1$ in equation (1.8), the q-Sheffer polynomials $s_{n,q}(x)$ reduces to the Sheffer polynomials $s_n(x)$ [16].*

Very recently, a new type of q-Hermite polynomials are considered in [9], which can be considered as a special member of the q-Sheffer family and has the following definition:

Definition 1.4. *The continuous q-Hermite polynomials $H_{n,q}(x;s), 0 < q < 1, 0 \neq s \in \mathbb{R}$ are specified by the following generating function:*

$$e_q\left(xt - \frac{st^2}{1+q}\right) = \sum_{n=0}^{\infty} H_{n,q}(x;s)\frac{t^n}{[n]_q!}, \tag{1.10}$$

where $H_{n,q}(s) := H_{n,q}(0;s)$ are the q-Hermite numbers defined by

$$e_q\left(-\frac{st^2}{1+q}\right) = \sum_{n=0}^{\infty} H_{n,q}(0;s)\frac{t^n}{[n]_q!}. \tag{1.11}$$

To study the hybrid forms of the q-polynomials by different means is an interesting approach. In this article, the hybrid q-Sheffer polynomials are constructed by means of the generating function and series representation. The determinant definition and certain results for these polynomials are derived. The continuous q-Hermite-Bernoulli, continuous q-Hermite-Euler and continuous q-Hermite-Genocchi polynomials are studied as special members of this family and analogues results for these polynomials are also established. The plots of certain polynomials belonging to the hybrid q-Sheffer family are demonstrated using computer experiment for suitable values of the indices. The fascinating zeros of these hybrid q-special polynomials are explored and nature of these zeros are displayed using computer software. In addition, the orthogonality property of the hybrid q-Sheffer polynomials is established.

2. Hybrid q-Sheffer polynomials

In this section, the generating function, series expansion and determinant forms of the hybrid q-Sheffer (q-Sheffer-Appell) polynomials, denoted by ${}_sA_{n,q}(x)$ are obtained.

Using expansion (1.4) in generating equation (1.7) and then replacing the powers of $x^0, x^1, x^2, ..., x^n$ of x by the correlating polynomials $s_{0,q}(x), s_{1,q}(x), ..., s_{n,q}(x)$ and after summing up the terms of the obtained relation and denoting the resultant hybrid q-Sheffer polynomials by ${}_sA_{n,q}(x)$, the following generating function for the hybrid q-Sheffer polynomials ${}_sA_{n,q}(x)$ is obtained:

$$A_q(t)\mathcal{G}_q(t)e_q[xH_q(t)] = \sum_{n=0}^{\infty} {}_sA_{n,q}(x)\frac{t^n}{[n]_q!}. \tag{2.1}$$

Remark 2.1. *By taking $q \to 1$ in equation (2.1), the hybrid q-Sheffer polynomials ${}_sA_{n,q}(x)$ reduces to the hybrid Sheffer (Sheffer-Appell) polynomials ${}_sA_n(x)$ [14].*

Using equations (1.7) and (1.8) in the l.h.s. of generating function (2.1), simplifying and then equating the coefficients of identical powers of t in both sides of obtained relation, we get the following series expansion for the hybrid q-Sheffer polynomials ${}_sA_{n,q}(x)$:

$${}_sA_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q A_{k,q}\ s_{n-k,q}(x). \tag{2.2}$$

Certain members of the q-Appell family are given in Table 1. Since, corresponding to each member belonging to the q-Appell family, there exists a new special polynomial belonging to the hybrid q-Sheffer family. The generating function and series definition of the hybrid q-Sheffer members can be obtained by selecting appropriate generating function $A_q(t)$ in equation (2.1). These members along with their names, generating functions, series definitions and related numbers are given in Table 2.

Table 2: Certain members belonging to the hybrid q-Sheffer family

S. No.	Name of the q-special polynomials and related numbers	$A_q(t)$	Generating function	Series expansion
I.	q-Sheffer-Bernoulli polynomials and numbers	$\frac{t}{e_q(t)-1}$	$\left(\frac{t}{e_q(t)-1}\right)\mathcal{G}_q(t)e_q[xH_q(t)] = \sum\limits_{n=0}^{\infty} {}_sB_{n,q}(x)\frac{t^n}{[n]_q!}$ $\left(\frac{t}{e_q(t)-1}\right)\mathcal{G}_q(t) = \sum\limits_{n=0}^{\infty} {}_sB_{n,q}\frac{t^n}{[n]_q!}$, ${}_sB_{n,q} := {}_sB_{n,q}(0)$	${}_sB_{n,q}(x) = \sum\limits_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_{k,q}\ s_{n-k,q}(x)$
II.	q-Sheffer-Euler polynomials and numbers	$\frac{2}{e_q(t)+1}$	$\left(\frac{2}{e_q(t)+1}\right)\mathcal{G}_q(t)e_q[xH_q(t)] = \sum\limits_{n=0}^{\infty} {}_sE_{n,q}(x)\frac{t^n}{[n]_q!}$ $\left(\frac{2}{e_q(t)+1}\right)\mathcal{G}_q(t) = \sum\limits_{n=0}^{\infty} {}_sE_{n,q}\frac{t^n}{[n]_q!}$, ${}_sE_{n,q} := {}_sE_{n,q}(0)$	${}_sE_{n,q}(x) = \sum\limits_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q}\ s_{n-k,q}(x)$
III.	q-Sheffer-Genocchi polynomials and numbers	$\frac{2t}{e_q(t)+1}$	$\left(\frac{2t}{e_q(t)+1}\right)\mathcal{G}_q(t)e_q[xH_q(t)] = \sum\limits_{n=0}^{\infty} {}_sG_{n,q}(x)\frac{t^n}{[n]_q!}$ $\left(\frac{2t}{e_q(t)+1}\right)\mathcal{G}_q(t) = \sum\limits_{n=0}^{\infty} {}_sG_{n,q}\frac{t^n}{[n]_q!}$, ${}_sG_{n,q} := {}_sG_{n,q}(0)$	${}_sG_{n,q}(x) = \sum\limits_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q G_{k,q}\ s_{n-k,q}(x)$

By using a similar approach as in [13, p.359 (Theorem 7)] and with the help of equations (2.1) and (1.8), the following determinant definition for the hybrid q-Sheffer polynomials ${}_sA_{n,q}(x)$ is obtained:

Definition 2.2. *The hybrid q-Sheffer polynomials ${}_sA_{n,q}(x)$ of degree n are defined by*

$$\begin{aligned} {}_sA_{0,q}(x) &= \frac{1}{\beta_{0,q}}, \\ {}_sA_{n,q}(x) &= \frac{(-1)^n}{(\beta_{0,q})^{n+1}} \begin{vmatrix} 1 & s_{1,q}(x) & s_{2,q}(x) & \cdots & s_{n-1,q}(x) & s_{n,q}(x) \\ \beta_{o,q} & \beta_{1,q} & \beta_{2,q} & \cdots & \beta_{n-1,q} & \beta_{n,q} \\ 0 & \beta_{0,q} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \beta_{1,q} & \cdots & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \beta_{n-2,q} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q \beta_{n-1,q} \\ 0 & 0 & \beta_{0,q} & \cdots & \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q \beta_{n-3,q} & \begin{bmatrix} n \\ 2 \end{bmatrix}_q \beta_{n-2,q} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{0,q} & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q \beta_{1,q} \end{vmatrix}, \end{aligned} \tag{2.3}$$

where $s_{n,q}(x)$ $(n = 1, 2, \cdots)$ are the q-Sheffer polynomials of degree n, $\beta_{0,q} \neq 0$ and

$$\begin{aligned} \beta_{0,q} &= \frac{1}{A_{0,q}}, \\ \beta_{n,q} &= -\frac{1}{A_{0,q}}\left(\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q A_{k,q}\ \beta_{n-k,q}\right),\ n = 1, 2, \cdots . \end{aligned} \tag{2.4}$$

Further, we find the determinant definitions of certain members belonging to the hybrid q-Sheffer family mentioned in Table 2 by taking $\left(\beta_{0,q} = 1,\ \beta_{i,q} = \frac{1}{[i+1]_q}\right)$;

$\left(\beta_{0,q}=1, \beta_{i,q}=\frac{1}{2}\right)$ and $\left(\beta_{0,q}=1, \beta_{i,q}=\frac{1}{2[i+1]_q}\right), i=1,2,\cdots n$ in determinant definition (2.3), respectively.

Definition 2.3. *The q-Sheffer-Bernoulli polynomials ${}_sB_{n,q}(x)$ of degree n are defined by*

$$
\begin{aligned}
&{}_sB_{0,q}(x)=1,\\
&{}_sB_{n,q}(x)=(-1)^n\begin{vmatrix}
1 & s_{1,q}(x) & s_{2,q}(x) & \cdots & s_{n-1,q}(x) & s_{n,q}(x)\\
1 & \frac{1}{[2]_q} & \frac{1}{[3]_q} & \cdots & \frac{1}{[n]_q} & \frac{1}{[n+1]_q}\\
0 & 1 & \begin{bmatrix}2\\1\end{bmatrix}_q\frac{1}{[2]_q} & \cdots & \begin{bmatrix}n-1\\1\end{bmatrix}_q\frac{1}{[n-1]_q} & \begin{bmatrix}n\\1\end{bmatrix}_q\frac{1}{[n]_q}\\
0 & 0 & 1 & \cdots & \begin{bmatrix}n-1\\2\end{bmatrix}_q\frac{1}{[n-2]_q} & \begin{bmatrix}n\\2\end{bmatrix}_q\frac{1}{[n-1]_q}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & 1 & \begin{bmatrix}n\\n-1\end{bmatrix}_q\frac{1}{[2]_q}
\end{vmatrix},
\end{aligned}
\tag{2.5}
$$

where $s_{n,q}(x),\ n=1,2,\cdots$ are the q-Sheffer polynomials of degree n.

Definition 2.4. *The q-Sheffer-Euler polynomials ${}_sE_{n,q}(x)$ of degree n are defined by*

$$
\begin{aligned}
&{}_sE_{0,q}(x)=1,\\
&{}_sE_{n,q}(x)=(-1)^n\begin{vmatrix}
1 & s_{1,q}(x) & s_{2,q}(x) & \cdots & s_{n-1,q}(x) & s_{n,q}(x)\\
1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2}\\
0 & 1 & \begin{bmatrix}2\\1\end{bmatrix}_q\frac{1}{2} & \cdots & \begin{bmatrix}n-1\\1\end{bmatrix}_q\frac{1}{2} & \begin{bmatrix}n\\1\end{bmatrix}_q\frac{1}{2}\\
0 & 0 & 1 & \cdots & \begin{bmatrix}n-1\\2\end{bmatrix}_q\frac{1}{2} & \begin{bmatrix}n\\2\end{bmatrix}_q\frac{1}{2}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & 1 & \begin{bmatrix}n\\n-1\end{bmatrix}_q\frac{1}{2}
\end{vmatrix},
\end{aligned}
\tag{2.6}
$$

where $s_{n,q}(x), n=1,2,\cdots$ are the q-Sheffer polynomials of degree n.

Definition 2.5. *The q-Sheffer-Genocchi polynomials ${}_sG_{n,q}(x)$ of degree n are defined by*

$$
\begin{aligned}
&{}_sG_{0,q}(x)=1,\\
&{}_sG_{n,q}(x)=(-1)^n\begin{vmatrix}
1 & s_{1,q}(x) & s_{2,q}(x) & \cdots & s_{n-1,q}(x) & s_{n,q}(x)\\
1 & \frac{1}{2[2]_q} & \frac{1}{2[3]_q} & \cdots & \frac{1}{2[n]_q} & \frac{1}{2[n+1]_q}\\
0 & 1 & \begin{bmatrix}2\\1\end{bmatrix}_q\frac{1}{2[2]_q} & \cdots & \begin{bmatrix}n-1\\1\end{bmatrix}_q\frac{1}{2[n-1]_q} & \begin{bmatrix}n\\1\end{bmatrix}_q\frac{1}{2[n]_q}\\
0 & 0 & 1 & \cdots & \begin{bmatrix}n-1\\2\end{bmatrix}_q\frac{1}{2[n-2]_q} & \begin{bmatrix}n\\2\end{bmatrix}_q\frac{1}{2[n-1]_q}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & 1 & \begin{bmatrix}n\\n-1\end{bmatrix}_q\frac{1}{2[2]_q}
\end{vmatrix},
\end{aligned}
\tag{2.7}
$$

where $s_{n,q}(x), n=1,2,\cdots$ are the q-Sheffer polynomials of degree n.

Since, the continuous q-Hermite polynomials $H_{n,q}(x;s)$ belong to the q-Sheffer family. Therefore, in view of equations (1.10) and (2.1), the following generating function for the continuous q-Hermite-Appell polynomials ${}_HA_{n,q}(x;s)$ is obtained:

$$
A_q(t)e_q\left(xt-\frac{st^2}{1+q}\right)=\sum_{n=0}^{\infty}{}_HA_{n,q}(x;s)\frac{t^n}{[n]_q!}.
\tag{2.8}
$$

Taking $s_{n,q}(x) = H_{n,q}(x;s)$ in equation (2.3), the following determinant definition for the continuous q-Hermite-Appell polynomials ${}_HA_{n,q}(x;s)$ is obtained.

Definition 2.6. *The continuous q-Hermite-Appell polynomials ${}_HA_{n,q}(x;s)$ of degree n are defined by*

$$\begin{aligned}
{}_HA_{0,q}(x;s) &= \tfrac{1}{\beta_{0,q}},\\
{}_HA_{n,q}(x;s) &= \tfrac{(-1)^n}{(\beta_{0,q})^{n+1}}\begin{vmatrix}
1 & H_{1,q}(x;s) & H_{2,q}(x;s) & \cdots & H_{n-1,q}(x;s) & H_{n,q}(x;s)\\
\beta_{o,q} & \beta_{1,q} & \beta_{2,q} & \cdots & \beta_{n-1,q} & \beta_{n,q}\\
0 & \beta_{0,q} & \begin{bmatrix}2\\1\end{bmatrix}_q\beta_{1,q} & \cdots & \begin{bmatrix}n-1\\1\end{bmatrix}_q\beta_{n-2,q} & \begin{bmatrix}n\\1\end{bmatrix}_q\beta_{n-1,q}\\
0 & 0 & \beta_{0,q} & \cdots & \begin{bmatrix}n-1\\2\end{bmatrix}_q\beta_{n-3,q} & \begin{bmatrix}n\\2\end{bmatrix}_q\beta_{n-2,q}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & \beta_{0,q} & \begin{bmatrix}n\\n-1\end{bmatrix}_q\beta_{1,q}
\end{vmatrix},
\end{aligned} \tag{2.9}$$

where $H_{n,q}(x;s), n = 1, 2, \cdots$ are the continuous q-Hermite polynomials of degree n, $\beta_{0,q} \neq 0$ and

$$\beta_{0,q} = \frac{1}{A_{0,q}},$$

$$\beta_{n,q} = -\frac{1}{A_{0,q}}\left(\sum_{k=1}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q A_{k,q}\ \beta_{n-k,q}\right),\ \ n = 1, 2, \cdots\ . \tag{2.10}$$

By suitably selecting $A_q(t)$ of the members belonging to the q-Appell family mentioned in Table 1, the generating functions and other results for the corresponding members belonging to the continuous q-Hermite-Appell family are obtained. These are given in Table 3.

Table 3: Certain members belonging to the continuous q-Hermite-Appell family

S. No.	Name of the q-special polynomials and related numbers	Generating functions	Series expansion
I.	Continuous q-Hermite-Bernoulli polynomials and numbers	$\left(\frac{t}{e_q(t)-1}\right) e_q\left(xt-\frac{st^2}{1+q}\right) = \sum_{n=0}^{\infty} {}_HB_{n,q}(x;s)\frac{t^n}{[n]_q!}$ $\left(\frac{t}{e_q(t)-1}\right) e_q\left(-\frac{st^2}{1+q}\right) = \sum_{n=0}^{\infty} {}_HB_{n,q}(s)\frac{t^n}{[n]_q!}$ ${}_HB_{n,q}(s) := {}_HB_{n,q}(0;s)$	${}_HB_{n,q}(x;s) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_{k,q}\ H_{n-k,q}(x;s)$
II.	Continuous q-Hermite-Euler polynomials and numbers	$\left(\frac{2}{e_q(t)+1}\right) e_q\left(xt-\frac{st^2}{1+q}\right) = \sum_{n=0}^{\infty} {}_HE_{n,q}(x;s)\frac{t^n}{[n]_q!}$ $\left(\frac{2}{e_q(t)+1}\right) e_q\left(-\frac{st^2}{1+q}\right) = \sum_{n=0}^{\infty} {}_HE_{n,q}(s)\frac{t^n}{[n]_q!}$ ${}_HE_{n,q}(s) := {}_HE_{n,q}(0;s)$	${}_HE_{n,q}(x;s) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q}\ H_{n-k,q}(x;s)$
III.	Continuous q-Hermite-Genocchi polynomials and numbers	$\left(\frac{2t}{e_q(t)+1}\right) e_q\left(xt-\frac{st^2}{1+q}\right) = \sum_{n=0}^{\infty} {}_HG_{n,q}(x;s)\frac{t^n}{[n]_q!}$ $\left(\frac{2t}{e_q(t)+1}\right) e_q\left(-\frac{st^2}{1+q}\right) = \sum_{n=0}^{\infty} {}_HG_{n,q}(s)\frac{t^n}{[n]_q!}$ ${}_HG_{n,q}(s) := {}_HG_{n,q}(0;s)$	${}_HG_{n,q}(x;s) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q G_{k,q}\ H_{n-k,q}(x;s)$

Remark 2.7. *Taking* $\left(\beta_{0,q}=1,\ \beta_{i,q}=\frac{1}{[i+1]_q}\right)$; $\left(\beta_{0,q}=1, \beta_{i,q}=\frac{1}{2}\right)$ *and* $\left(\beta_{0,q}=1, \beta_{i,q}=\frac{1}{2[i+1]_q}\right)$, $i=1,2,\cdots n$ *in determinant definition* (2.9) *of the continuous* q*-Hermite-Appell polynomials* ${}_HA_{n,q}(x;s)$*, we get the determinant definitions of the continuous* q*-Hermite-Bernoulli polynomials* ${}_HB_{n,q}(x;s)$*, continuous* q*-Hermite-Euler polynomials* ${}_HE_{n,q}(x;s)$ *and continuous* q*-Hermite-Genocchi polynomials* ${}_HG_{n,q}(x;s)$ *polynomials, respectively.*

In the next section, the graphs of the some members belonging to the hybrid q-Sheffer family are drawn. The zeros of these hybrid q-special polynomials are also explored using numerical computations.

3. Zeros of the hybrid q-Sheffer polynomials

In the recent past, the interest in solving mathematical problems with the help of computers has been increased rapidly. The numerical investigation is especially exciting as it enables us to understand the basic properties of q-numbers and q-special polynomials. It will lead to a new approach that employs numerical methods in the field of these hybrid q-special polynomials.

First, we plot the graph of the continuous q-Hermite polynomials $H_{n,q}(x;s)$, for $n=1,2,3,4$ and $q=\frac{1}{2},\ \frac{1}{5}(0<q<1);\ s=2$ and $x\to 2x$. The zeros of these polynomials are also explored and behaviour of these zeros is observed. This shows the four curves combined into one.

To show the graphical representation of the continuous q-Hermite polynomials $H_{n,q}(x;s)$, we require first few expressions of the $H_{n,q}(x;s)$. These expressions are given in Table 4.

Table 4: Expressions of first five $H_{n,1/2}(x;2)$ and $H_{n,1/5}(x;2)$

n	0	1	2	3	4
$H_{n,1/2}(x;2)$	1	$2x$	$4x^2-2$	$8x^3-20x$	$16x^4-34x^2+7$
$H_{n,1/5}(x;2)$	1	$2x$	$4x^2-2$	$8x^3-\frac{44}{5}x$	$16x^4-\frac{688}{25}x^2+\frac{124}{25}$

With the help of Matlab and by using the expressions for the $H_{n,1/2}(x;2)$ and $H_{n,1/5}(x;2)$ from Table 4 for $n=1,2,3,4$, Figure 1 is drawn.

Next, we investigate zeros of the continuous q-Hermite polynomials $H_{n,q}(x;s)$ for same values of n and q and $x\in\mathbb{C}$ with the help of Matlab as the manual computation of these zeros is too complicated. These are mentioned in Table 5 and are displayed in Figure 2.

Table 5: Real zeros of $H_{n,1/2}(x;2)$ and $H_{n,1/5}(x;2)$

Degree n	$H_{n,1/2}(x;2)$	$H_{n,1/5}(x;2)$
1	0	0
2	$0.707107, -0.707107$	$0.707107, -0.707107$
3	$0, 1.58114, -1.58114$	$0, 1.40881, -1.40881$
4	$0.480616, -0.480616, 1.37623, -1.37623$	$0.452284, 0.452284, 1.23103, -1.23103$

It is to be noted that the real zeros of the polynomials mentioned in Table 5 give the numerical results for the approximate solutions of the continuous q-Hermite polynomials $H_{n,1/2}(x;s)=0$ and $H_{n,1/5}(x;s)=0$, for $n=1,2,3,4$.

Next, to show the graphical representation of the continuous q-Hermite-Bernoulli polynomials ${}_HB_{n,q}(x;s)$ and continuous q-Hermite-Euler polynomials ${}_HE_{n,q}(x;s)$, respectively for $n=1,2,3,4$ and $q=\frac{1}{2},\ \frac{1}{5}$ $(0<q<1)$; $s=2$ and $x\to 2x$, we require first few expressions of the ${}_HB_{n,q}(x;s)$ and ${}_HE_{n,q}(x;s)$. These expressions are given in Table 6.

Table 6: Expressions of first five ${}_HB_{n,1/2}(x;2)$, ${}_HB_{n,1/5}(x;2)$, ${}_HE_{n,1/2}(x;2)$ and ${}_HE_{n,1/5}(x;2)$

Degree n	**0**	**1**	**2**	**3**	**4**
${}_HB_{n,1/2}(x;2)$	1	$2x-\frac{2}{3}$	$4x^2-2x-\frac{12}{7}$	$8x^3-\frac{14}{3}x^2-19x+\frac{211}{3}$	$16x^4-10x^3-\frac{63}{2}x^2+\frac{301}{12}x+\frac{44948}{7812}$
${}_HB_{n,1/5}(x;2)$	1	$2x-\frac{5}{6}$	$4x^2-2x-\frac{61}{31}$	$8x^3-\frac{124}{30}x^2-\frac{218}{25}x+\frac{7269}{3510}$	$16x^4-\frac{208}{25}x^3-\frac{17096}{625}x^2+\frac{134004}{14625}x+\frac{245974147}{272373750}$
${}_HE_{n,1/2}(x;2)$	1	$2x-\frac{1}{2}$	$4x^2-\frac{3}{2}x-\frac{17}{8}$	$8x^3-\frac{7}{2}x^2-\frac{327}{16}x+\frac{115}{64}$	$16x^4-\frac{15}{2}x^3-\frac{1123}{32}x^2+\frac{4845}{256}x+\frac{7791}{1024}$
${}_HE_{n,1/5}(x;2)$	1	$2x-\frac{1}{2}$	$4x^2-\frac{6}{5}x-\frac{11}{5}$	$8x^3-\frac{248}{125}x^2-\frac{5748}{625}x+\frac{58}{125}$	$16x^4-\frac{2496}{625}x^3-\frac{89224}{3125}x^2+\frac{332904}{78125}x+\frac{31899}{31250}$

With the help of Matlab and by using the expressions for the ${}_HB_{n,1/2}(x;2)$, ${}_HB_{n,1/5}(x;2)$, ${}_HE_{n,1/2}(x;2)$, and ${}_HE_{n,1/5}(x;2)$ from Table 6, for $n=1,2,3,4$, Figures 3 and 4 are drawn.

Next, we investigate the real and complex zeros of the polynomials ${}_HB_{n,1/2}(x;2)$, ${}_HB_{n,1/5}(x;2)$, ${}_HE_{n,1/2}(x;2)$, and ${}_HE_{n,1/5}(x;2)$ for same values of n and $x\in\mathbb{C}$. These zeros are computed with the help of Matlab and are given in Tables 7 and 8.

Table 7: Real zeros of ${}_HB_{n,1/2}(x;2)$, ${}_HB_{n,1/5}(x;2)$, ${}_HE_{n,1/2}(x;2)$ and ${}_HE_{n,1/5}(x;2)$

Degree n	${}_HB_{n,1/2}(x;2)$	${}_HB_{n,1/5}(x;2)$	${}_HE_{n,1/2}(x;2)$	${}_HE_{n,1/5}(x;2)$
1	0.333333	0.416667	0.25	0.25
2	$-0.450765, 0.950765$	$-0.494604, 0.994604$	$-0.5651, 0.9401$	$0.606637, 0.906637$
3	-2.23642	complex zero	complex zero	complex zero
4	$-1.41763, -0.188314$	complex zero	complex zero	complex zero

It is to be noted that the real zeros of the polynomials mentioned in Table 7 are giving the numerical results for the approximate solutions of the q-Hermite-Bernoulli polynomials ${}_HB_{n,1/2}(x;2)=0$, ${}_HB_{n,1/5}(x;2)=0$ and q-Hermite-Euler polynomials ${}_HE_{n,1/2}(x;2)=0$, ${}_HE_{n,1/5}(x;2)=0$, for $n=1,2,3,4$.

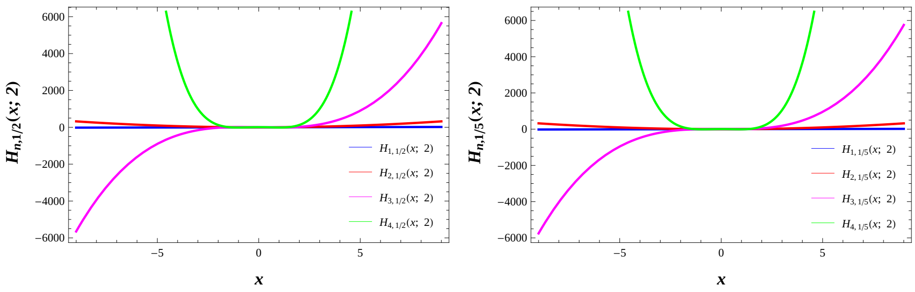

Figure 1: **(a) Curves of** $H_{n,1/2}(x;2)$**;** **(b) Curves of** $H_{n,1/5}(x;2)$**.**

Table 8: Complex zeros of ${}_HB_{n,1/2}(x;2)$, ${}_HB_{n,1/5}(x;2)$, ${}_HE_{n,1/2}(x;2)$ and ${}_HE_{n,1/5}(x;2)$

Degree n	${}_HB_{n,1/2}(x;2)$	${}_HB_{n,1/5}(x;2)$	${}_HE_{n,1/2}(x;2)$	${}_HE_{n,1/5}(x;2)$
1	Real zero	Real zero	Real zero	Real zero
2	Real zero	Real zero	Real zero	Real zero
3	$1.40988+1.39406i$, $1.40988-1.39406i$	$1.23121-3.94746\times10^{-17}i$, $-0.938558-1.11022\times10^{-16}i$, $0.224019+2.22045\times10^{-16}i$	$1.79268-2.96059\times10^{-16}i$, $0.0868844+6.66134\times10^{-16}i$, $-1.44206-3.88578\times10^{-16}i$	$1.18033-9.4739\times10^{-16}i$, $-0.982353-1.05971\times10^{-15}i$, $0.0500214+2.10942\times10^{-15}i$
4	$1.11547+0.320584i$, $1.11547-0.320584i$	$0.405393-2.10224\times10^{-16}i$, $1.42016+3.59413\times10^{-17}i$, $-1.22556-2.18879\times10^{-17}i$, $-0.0799936+1.9617\times10^{-16}i$	$0.994036-2.24714\times10^{-16}i$, $-1.45021-1.89008\times10^{-17}i$, $1.19985+1.7986\times10^{-16}i$, $-0.274924+6.37552\times10^{-17}i$	$0.278764+1.08318\times10^{-16}i$, $1.37916-1.77985\times10^{-17}i$, $-1.27853+1.51446\times10^{-17}i$, $-0.129791-1.05665\times10^{-16}i$

The real and complex zeros mentioned in Tables 7 and 8 are displayed in Figures 5 and 6.

Remark 3.1. *In view of the values given in Tables 7 and 8, the following important relation between real and complex zeros of the hybrid q-Sheffer polynomials is observed:*

$$Number\ of(real\ zeros\ of\ {}_sA_{n,q}(x)) = n - Number\ of(complex\ zeros\ of\ {}_sA_{n,q}(x)),$$

where n is the degree of the polynomial.

Remark 3.2. *In view of the values given in Tables 5, 7 and 8, we observe that the hybrid q-Sheffer polynomials ${}_sA_{n,q}(x)$ have both real and complex zeros, while the ordinary q-Sheffer polynomials $s_{n,q}(x)$ possess only real zeros. This is the important point which distinguishes the hybrid q-Sheffer polynomials from the ordinary q-Sheffer polynomials.*

The figures drawn here provides an unbounded capability to create visual mathematical investigations of the behavior of the hybrid q-Sheffer polynomials ${}_sA_{n,q}(x)$ and also the roots of these polynomials. Moreover, it is possible to construct new mathematical ideas and analyze them in ways that are generally not possible by manual computations.

In the next section, we establish an important result related to the orthogonality of the hybrid q-Sheffer polynomials.

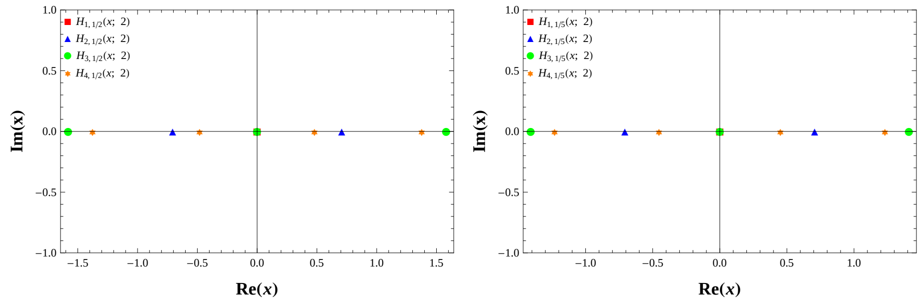

Figure 2: **(a) Zeros of $H_{n,1/2}(x;2)$; (b) Zeros of $H_{n,1/5}(x;2)$.**

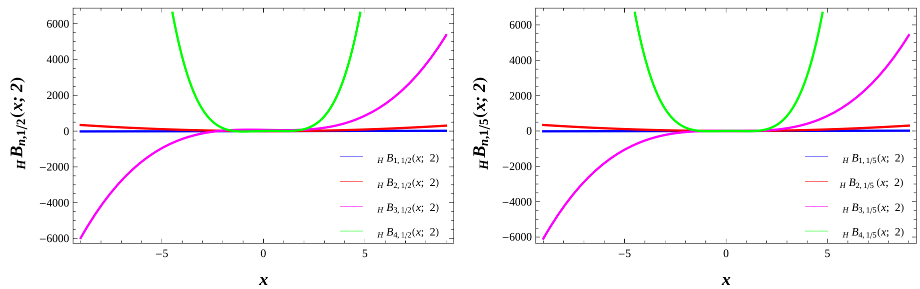

Figure 3: **(a) Curves of ${}_HB_{n,1/2}(x;2)$; (b) Curves of ${}_HB_{n,1/5}(x;2)$.**

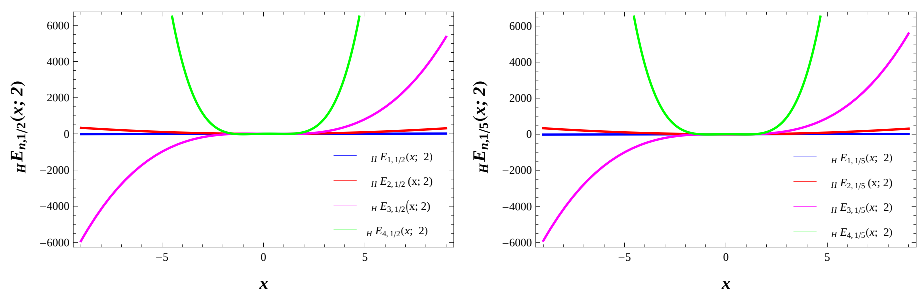

Figure 4: **(a) Curves of ${}_HE_{n,1/2}(x;2)$; (b) Curves of ${}_HE_{n,1/5}(x;2)$.**

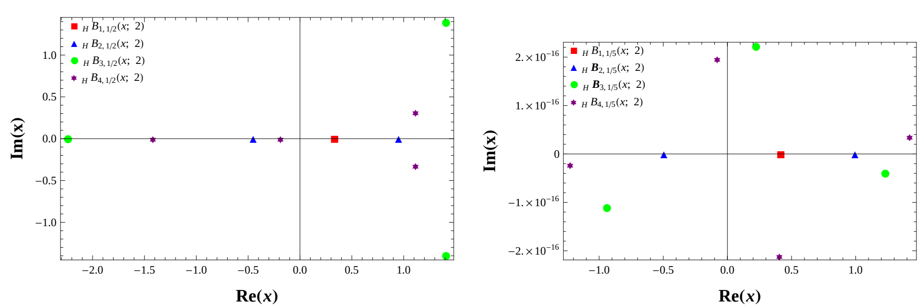

Figure 5: **(a) Zeros of ${}_HB_{n,1/2}(x;2)$; (b) Zeros of ${}_HB_{n,1/5}(x;2)$.**

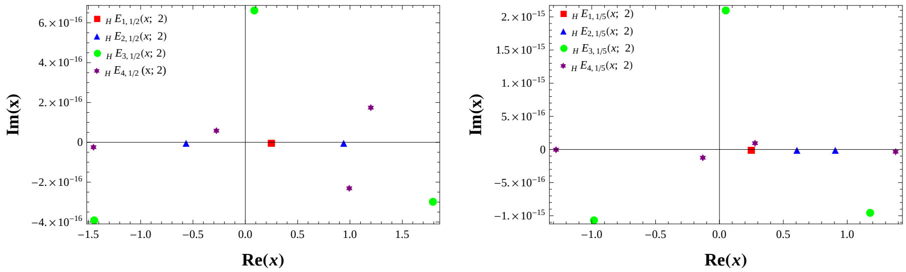

Figure 6: **(a) Zeroes of $_HE_{n,1/2}(x;2)$; (b) Zeroes of $_HE_{n,1/5}(x;2)$.**

4. Concluding remarks

The orthogonal polynomials in general and the classical orthogonal polynomials in particular have been the objects of extensive works. The classical orthogonal polynomials constitute a very important and interesting set of special functions and more specifically of orthogonal polynomials. They are very interesting mathematical objects that have attracted the attention of mathematicians. They are connected with numerous problems of applied mathematics, theoretical physics, chemistry, approximation theory and several other mathematical branches.

We recall a well-known fact about orthogonal polynomials that is: A monic polynomial $P_n(x)$ is orthogonal [1], if and only if, it satisfies the three term recurrence relation

$$P_{n+1}(x) = (x - t_n)P_n(x) - \lambda_n\ P_{n-1}(x), \quad n \geq 1, \tag{4.1}$$

where $P_0(x) = 1,\ P_1(x) = x - t_0$ and $\lambda_n \neq 0$ for all $n \geq 1$.

In order to establish the orthogonality of the hybrid q-Sheffer polynomials $_sA_{n,q}(x)$, we derive the three term recurrence relation for the hybrid q-Sheffer polynomials $_sA_{n,q}(x)$ by proving the following result.

Theorem 4.1. *Let $_sA_{n,q}(x)$ be the hybrid q-Sheffer polynomials for the pair $(A_q(t)\mathcal{G}_q(t), H_q(t))$. If $A_q(t), \mathcal{G}_q(t) \in \mathcal{E}_q(0)$ and $H_q(t) \in \mathcal{E}_q(1)$ satisfy the following:*

$$\begin{aligned}
&\frac{A_q'(t)}{A_q(t)} = -\frac{\alpha_0 + \alpha_1 t}{1 + \beta_1 t + \beta_2 t^2}, \qquad \frac{\mathcal{G}_q'(t)}{\mathcal{G}_q(t)} = -\frac{(\gamma_0 + \gamma_1 t)A_q(t)}{(1 + \beta_1 t + \beta_2 t^2)A_q(qt)},\\
&H_q'(t) = \frac{\mathcal{G}_q(t)A_q(t)}{(1 + \beta_1 t + \beta_2 t^2)\mathcal{G}_q(qt)A_q(qt)},
\end{aligned} \tag{4.2}$$

where $\alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2, \gamma_0$ and γ_1 are any functions of q.

Then the hybrid q-Sheffer polynomials $_sA_{n,q}(x)$ satisfy the following three term recurrence relation:

$$_sA_{n+1,q}(x) = (x - t_{n,q})\,{}_sA_{n,q}(x) - \lambda_{n,q}\,{}_sA_{n-1,q}(x), \ \ n \geq 1, \tag{4.3}$$

where $t_{n,q}$ and $\lambda_{n,q}$ are determined by

$$t_{n,q} = \alpha_0 + \gamma_0 + [n]_q\beta_1 \quad \text{and} \quad \lambda_{n,q} = [n]_q(\alpha_1 + \gamma_1 + [n-1]_q\beta_2) \tag{4.4}$$

and $\alpha_1 + \gamma_1 \neq -[n-1]_q\beta_2$, for all $n \geq 1$.

Proof. Differentiating both side of generating function (2.1) w.r.t. t by using formula (1.6), we find

$$\begin{aligned}
&A_q(qt)\mathcal{G}_q(qt)xH_q'(t)e_q[xH_q(t)] + \left(A_q(qt)\mathcal{G}_q'(t) + A_q'(t)\mathcal{G}_q(t)\right)e_q[xH_q(t)]\\
&= \sum_{n=0}^{\infty} {}_sA_{n+1,q}(x)\frac{t^n}{[n]_q!}.
\end{aligned} \tag{4.5}$$

Using generating function (2.1) in the l.h.s. of equation (4.5), it follows that

$$\begin{aligned}&\left(\frac{A_q(qt)\mathcal{G}_q(qt)xH'_q(t)}{A_q(t)\mathcal{G}_q(t)}+\frac{A_q(qt)\mathcal{G}'_q(t}{A_q(t)\mathcal{G}_q(t)}+\frac{A'_q(t)}{A_q(t)}\right)\sum_{n=0}^{\infty}{}_sA_{n,q}(x)\frac{t^n}{[n]_q!}\\&=\sum_{n=0}^{\infty}{}_sA_{n+1,q}(x)\frac{t^n}{[n]_q!}.\end{aligned}\tag{4.6}$$

Using relations (4.2) in equation (4.6), it gives

$$(x-(\alpha_0+\gamma_0)-(\alpha_1+\gamma_1)t)\sum_{n=0}^{\infty}{}_sA_{n,q}(x)\frac{t^n}{[n]_q!}=(1+\beta_1t+\beta_2t^2)\sum_{n=0}^{\infty}{}_sA_{n+1,q}(x)\frac{t^n}{[n]_q!}.\tag{4.7}$$

Taking $n=0$, letting ${}_sA_{0,q}(x)=1$ and equating the coefficients of terms not containing t, we find

$${}_sA_{1,q}(x)=x-(\alpha_0+\gamma_0).\tag{4.8}$$

Thus, for $n\geq 1$, equation (4.7) can be written as:

$$\begin{aligned}&\sum_{n=1}^{\infty}\Big(x-(\alpha_0+\gamma_0)\ {}_sA_{n,q}(x)-[n]_q(\alpha_1+\gamma_1)\ {}_sA_{n-1,q}(x)\Big)\frac{t^n}{[n]_q!}\\&=\sum_{n=1}^{\infty}\Big({}_sA_{n+1,q}(x)+[n]_q\beta_1\ {}_sA_{n,q}(x)+[n]_q[n-1]_q\ {}_sA_{n-1,q}(x)\beta_2\Big)\frac{t^n}{[n]_q!}.\end{aligned}\tag{4.9}$$

Now equating the coefficients of t^n on both sides of the above equation, we find for $n\geq 1$

$${}_sA_{n+1,q}(x)=\Big(x-(\alpha_0+\gamma_0+[n]_q\ \beta_1)\Big){}_sA_{n,q}(x)-[n]_q\Big(\alpha_1+\gamma_1+[n-1]_q\ \beta_2\Big){}_sA_{n-1,q}(x),\tag{4.10}$$

which in view of relations (4.4), yields assertion (4.3). $\square$

Remark 4.2. *Taking $\gamma_1=s$ and $\gamma_0=\beta_1=\beta_2=0$ in relation (4.3), we deduce the following consequence of Theorem 4.1.*

Corollary 4.3. *For the continuous q-Hermite Appell polynomials ${}_HA_{n,q}(x;s)$, the following three-term recurrence relation holds true:*

$${}_HA_{n+1,q}(x;s)=(x-\alpha_0){}_HA_{n,q}(x;s)-[n]_q(\alpha_1+s){}_HA_{n-1,q}(x;s),\tag{4.11}$$

where ${}_HA_{0,q}(x;s)=1,\ {}_HA_{1,q}(x;s)=x-\alpha_0$.

The notions of d-dimensional orthogonality for polynomials [15], vectorial orthogonality [11] or simultaneous orthogonality are the generalizations of ordinary orthogonality for polynomials. Such polynomials are characterized by the fact that they satisfy a $d+1$-order recurrence relationship, that is a relation between $d+2$ consecutive polynomials [11]. All these new notions of d-orthogonality for polynomials and equivalently, $1/d$-orthogonality [7] appear as particular cases of the general notion of biorthogonality studied in [8]. Recently, these concepts related to orthogonality have been the subject of numerous investigations and applications. The hybrid q-Sheffer polynomials considered in this article may be studied using these concepts. To study the numbers related to these hybrid q-Sheffer polynomials from combinatorial point of view may also be taken as further related research problem.

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