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Multiple Solutions for a Class of Bi-nonlocal Problems with Nonlinear Neumann Boundary Conditions

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ABSTRACT: In this paper, we are interested in a class of bi-nonlocal problems with nonlinear Neumann boundary conditions and sublinear terms at infinity. Using (S_+) mapping theory and variational methods, we establish the existence of at least two non-trivial weak solutions for the problem provided that the parameters are large enough. Our result complements and improves some previous ones for the superlinear case when the Ambrosetti-Rabinowitz type conditions are imposed on the nonlinearities.

Key Words: Bi-nonlocal problems, Nonlinear Neumann boundary conditions, Mountain pass theorem.

Contents

1. Introduction

In this paper, we are interested in a class of Kirchhoff type problems with nonlinear Neumann boundary conditions of the form

$$\begin{cases} M_1(L_1(u))\left(-\operatorname{div}(\ \varphi(x,\nabla u))+|u|^{p-2}u\right)=\lambda M_2(L_2(u))\ f(x,u), \quad x\in\Omega,\\ M_1(L_1(u))\ \varphi(x,\nabla u).\ \nu=\mu g(x,u), \quad x\in\partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N (N\geq 3)$, ν is the outward normal vector on the boundary $\partial\Omega$, $2\leq p<N$, λ,μ are parameters, $L_1(u)=\frac{1}{p}\int_\Omega (H(|\nabla u|^p)+|u|^p)dx$, $H(t)=\int_0^t h(s)ds$ for all $t\in\mathbb{R}$, $\varphi(x,v)=h(|v|^p)|v|^{p-2}v$ with increasing continuous functions h from $\mathbb{R}$ into $\mathbb{R}$, $L_2(u)=\int_\Omega F(x,u)dx$, where $F(x,u)=\int_0^u f(x,s)ds$ and $f:\Omega\times\mathbb{R}\to\mathbb{R}$, $g:\partial\Omega\times\mathbb{R}\to\mathbb{R}$ satisfy the Carathéodory condition. Moreover, $M_1:\mathbb{R}_0^+=[0,+\infty)\to\mathbb{R}$ and $M_2:\mathbb{R}_0^+\to\mathbb{R}$ are assumed to be continuous functions.

It should be noticed that if $h(t)\equiv 1$, problem (1.1) becomes a nonlocal Kirchhoff type equation with nonlinear boundary condition

$$\begin{cases} M_1\left(\frac{1}{p}\int_\Omega (|\nabla u|^p+|u|^p)\,dx\right)\left(-\operatorname{div}(|\nabla u|^{p-2}\nabla u)+|u|^{p-2}u\right)\\ \qquad\qquad = \lambda M_2(\int_\Omega F(x,u)\,dx)f(x,u), \quad x\in\Omega,\\ M_1\left(\frac{1}{p}\int_\Omega (|\nabla u|^p+|u|^p)\,dx\right)\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial\nu}\ =\mu g(x,u), \quad x\in\partial\Omega. \end{cases} \tag{1.2}$$

Since the first equation in (1.2) contains an integral over Ω, it is no longer a pointwise identity; therefore it is often called nonlocal problem. The interest of such problems comes from the fact that Kirchhoff type problems usually model several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density. Moreover, problem (1.2) is related to the stationary version of Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{p_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2}=0 \tag{1.3}$$

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presentend by Kirchhoff in 1883 (see [15]). This equation is an extension of the classical D'Alembert wave equation by considering the effects of the changes in the length of the string during the vibrations. The parameters in (1.3) have the following meanings: ρ denotes the mass density, p_0 denote the initial tension, h denotes the area of the cross-section, E denotes the Young modulus of the material and L denotes the lengh of the string.

Recently, Kirchhoff type problems have been studied by many authors and many important and interesting results are established, we refer to [2,6,7,8,9,10,11,13] for the problem with Dirichlet aboundary condition. In [11], Fan firstly considered a class of bi-nonlocal $p(x)$-Kirchhoff type problems with Dirichlet boundary conditions of the form

$$\begin{cases} -a\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = b(\int_\Omega F(x,u)\,dx)f(x,u), & x\in\Omega,\\ u=0, \quad x\in\partial\Omega, \end{cases} \tag{1.4}$$

where a, b are continuous functions. Under suitable conditions on a, b and the Ambrosetti-Rabinowitz type condition on the nonlinear term f, the author proved the existence of at least a non-trivial solution or the existence of infinitely many solutions for problem (1.4) by using variational methods. We notice that it follows from the Ambrosetti-Rabinowitz type condition that the nonlinear term is superlinear at infinity. In [9], Correa et al. considered problem (1.4) in the case when $b(t) = t^r$, $r > 0$ is a constant and proved the existence of infinitely many solutions for (1.4) by using Krasnoselskii's genus. In [12], Guo et al. developed the results of Fan [11] for the $p(x)$-Kirchhoff type problem with Neumann nonlinear boundary condition. Some further results on Kirchhoff type problems with Neumann nonlinear boundary condition can be found in [14,16,20,21], in which the authors studied the existence and multiplicity of solutions for the problem by using the Nehari manifold and fibering maps, Ekeland variational principle or the variational principles due to Bonanno et al. [3,4].

Inspired by the papers mentioned above, in this note we study the existence of solutions for bi-nonlocal problem (1.1) with Neumann nonlinear boundary condition. More precisely, under the sublinear condition at infinity on the nonlinearities we obtain a multiplicity result by using the minimum principle combined with the mountain pass theorem. Our main result complements and improves some previous ones for the superlinear case when the Ambrosetti-Rabinowitz type conditions are imposed on the nonlinearities. It is worth mentioning that the nonlinear terms in problem (1.1) may change sign in Ω.

In order to state the main result of this paper, we need the following assumptions for f and g. Denote $F(x,t) = \int_0^t f(x,s)ds$ and $G(x,t) = \int_0^t g(x,s)ds$, then we assume that

(**F1**) $f : \Omega\times\mathbb{R}\to\mathbb{R}$ is a Caratheodory function such that

$$|f(x,t)| \le C_1(1+|t|^{p-1}), \quad \forall (x,t)\in\Omega\times\mathbb{R},\ 2\le p<N;$$

(**G1**) $g : \partial\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies the Caratheodory function such that

$$|g(x,t)| \le C_2(1+|t|^{p-1}), \quad \forall (x,t)\in\partial\Omega\times\mathbb{R}.$$

Let $\widetilde{C}$ be a fixed positive real number. We say that a C^1-function $\gamma : \mathbb{R}\to\mathbb{R}_0^+$ verifies the property (Γ) if and only if

$$\gamma(t) \le \widetilde{C}|t|^p, \quad \forall t\in\mathbb{R}. \tag{Γ}$$

Let K_i, $i = 1, 2, 3, 4$ be four functions satisfying property Γ. We introduce the following assumptions on the behavior of F and G at origin and at infinity:

(**F2**) It holds that

$$\limsup_{t\to 0}\frac{F(x,t)}{K_1(t)} \le 0$$

uniformly in $x\in\Omega$;

(**F3**) It holds that
$$\limsup_{t\to+\infty}\frac{F(x,t)}{K_2(t)}\le 0$$
uniformly in $x\in\Omega$.

(**G2**) It holds that
$$\limsup_{t\to 0}\frac{G(x,t)}{K_3(t)}\le 0$$
uniformly in $x\in\partial\Omega$;

(**G3**) It holds that
$$\limsup_{t\to+\infty}\frac{G(x,t)}{K_4(t)}\le 0$$
uniformly in $x\in\partial\Omega$.

We can see that there are many functions K_i satisfying the condition (Γ), for example $K_i(t)=\frac{\widetilde{C}}{2}|t|^p$, $\forall t\in\mathbb{R}$. Taking a C^1-function $w:\mathbb{R}\to\mathbb{R}$ such that $\lim_{|t|\to 0}w(t)=\lim_{|t|\to+\infty}w(t)=0$ we deduce that $F(x,t)=G(x,t)=\frac{\widetilde{C}}{2}w(t)|t|^p$ verify the conditions (**F1**)-(**F3**) and (**G1**)-(**G3**).

Regarding the functions h and M_i, $i=1,2$, we assume that

(**M1**) There are two positive constants m_0,m_1 such that
$$m_0\le M_1(t)\le m_1 \qquad \forall t\ge 0.$$

(**M2**) There are two positive constants m_0,m_1 such that
$$m_2\le M_2(t)\le m_3 \qquad \forall t\ge 0.$$

(**H1**) $h:[0,+\infty)\to\mathbb{R}$ is increasing continuous function and there exist α, $\beta>0$, such that
$$\alpha\le h(t)\le\beta$$
for all $t\ge 0$.

(**H2**) There is constants $\theta>0$, such that
$$\Big(h(|\xi|^p)|\xi|^{p-2}\xi-h(|\eta|^p)|\eta|^{p-2}\eta\Big)\cdot(\xi-\eta)\ge\theta|\xi-\eta|^p,$$
for all $\xi,\eta\in\mathbb{R}^N$.

It is noticed that the function $h(t)=1+\frac{t}{\sqrt{1+t^2}}$, $t\ge 0$ satisfies the conditions (**H1**)-(**H2**). In this case, (1.1) is called a capillarity system, see [17,18] for more details. For this reason, system (1.1) with the conditions (**H1**)-(**H2**) can be understood as a generalized capillarity system with nonlinear boundary conditions.

Let $X=W^{1,p}(\Omega)$ be the usual Sobolev space equipped with the norm
$$\|u\|^p=\int_\Omega(|\nabla u|^p+|u|^p)dx$$
and $W_0^{1,p}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. For any $1\le p\le N$ and $1\le q\le p^*=\frac{Np}{N-p}$, we denote by $S_{q,\Omega}$ the best constant in the embedding $W^{1,p}(\Omega)\hookrightarrow L^q(\Omega)$ and for all $1\le q\le p_*=\frac{(N-1)p}{N-p}$, we also denote by $S_{q,\partial\Omega}$ the best constant in the embedding $W^{1,p}(\Omega)\hookrightarrow L^q(\partial\Omega)$, i.e.
$$S_{q,\partial\Omega}=\inf_{u\in W^{1,p}(\Omega)\setminus W_0^{1,p}(\Omega)}\frac{\int_\Omega(|\nabla u|^p+|u|^p)dx}{(\int_{\partial\Omega}|u|^q d\sigma)^{\frac{p}{q}}}.$$

Moreover, if $1 \leq q < p^*$, then the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact and if $1 \leq q < p_*$ then the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ is compact .

Let us define the functionals $L_1 : X \to \mathbb{R}$ by

$$L_1(u) = \frac{1}{p}\int_\Omega (H(|\nabla u|^p) + |u|^p)dx, \tag{1.5}$$

and $L_1' : X \to X^*$ by

$$\langle L_1'(u), v\rangle = \int_\Omega (h(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv)dx.$$

Let us define the mapping $J : X \to \mathbb{R}$ by

$$J(u) = \widehat{M_1}\Big(L_1(u)\Big),$$

where

$$\widehat{M_1}(t) = \int_0^t M_1(s)ds,$$

and $J' : X \to X^*$ by

$$\begin{aligned}\langle J'(u), v\rangle =& M_1\Big(\frac{1}{p}\int_\Omega (H(|\nabla u|^p + |u|^p)dx\Big)\\ &\times \int_\Omega (h(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv)dx\end{aligned}$$

for $u, v \in X$.

Let us define the functional $L_2 : X \to \mathbb{R}$ by

$$L_2(u) = \int_\Omega F(x, u)dx \tag{1.6}$$

and $L_2' : X \to X^*$ by

$$\langle L_2'(u), v\rangle = \int_\Omega f(x, u)vdx.$$

Let us define the mapping $I : X \to \mathbb{R}$ and $\psi : X \to \mathbb{R}$ by

$$I(u) = \widehat{M_2}\Big(L_2(u)\Big), \qquad \psi(u) = \int_{\partial\Omega} G(x, u)d\sigma \tag{1.7}$$

where

$$\widehat{M_2}(t) = \int_0^t M_2(s)ds,$$

and $I' : X \to X^*, \psi' : X \to X^*$ by

$$\langle I'(u), v\rangle = M_2\Big(\int_\Omega F(x, u)dx\Big)\int_\Omega f(x, u)vdx, \qquad \langle \psi'(u), v\rangle = \int_{\partial\Omega} g(x, u)vd\sigma$$

for any $u, v \in X$.

Definition 1.1. *We say that u is a weak solution of problem (1.1) if and only if*

$$\langle J'(u), v\rangle = \lambda\langle I'(u), v\rangle + \mu\langle \psi'(u), v\rangle$$

for any $v \in X$.

The main result of this paper is as follows.

Theorem 1.2. *Suppose that* **(F1)**-**(F3)**, **(G1)**-**(G3)**, **(M1)** − **(M2)** *and* **(H1)**-**(H2)** *are satisfied. Moreover, we assume that there exists t_0, such that $F(x, t_0) > 0$ for all $x \in \Omega$. Then, there exist $\lambda^*, \mu^* > 0$ such that problem (1.1) has at least two distinct, nonnegative, nontrivial weak solutions, provided that $\lambda \geq \lambda^*$ and $\mu \geq \mu^*$.*

2. Proofs of the main results

We will prove Theorem 1.2 by using critical point theory. Set $f(x,t) = g(x,t) = 0$ for $t < 0$. For all $\lambda, \mu \in \mathbb{R}$, we consider the functional $E_{\lambda,\mu} : X \to \mathbb{R}$ given by

$$E_{\lambda,\mu}(u) = J(u) - \lambda I(u) - \mu\psi(u). \tag{2.1}$$

By **(F1)**, **(G1)**, a simple computation implies that $E_{\lambda,\mu}$ is well-defined and of C^1 class in X. Thus, weak solutions of problem (1.1) correspond to the critical points of $E_{\lambda,\mu}$.

Lemma 2.1. *The functionals L_1, given by (1.5) is sequentially weakly lower semicontinuous.*

Proof. By Corollary III.8 in [5], it is enough to show that L_1 is sequentially lower semicontinuous. For this purpose, we fix $u \in X$ and $\epsilon > 0$. Since L_1 is convex, we deduce that for any $v \in X$ the following inequality holds true

$$L_1(v) \geq L_1(u) + \langle L_1'(u), v - u\rangle$$

or

$$\begin{aligned}
L_1(v) \geq L_1(u) &- \int_\Omega h(|\nabla u|^p)|\nabla u|^{p-2}|\nabla u||\nabla v - \nabla u|dx - \int_\Omega |u|^{p-2}|u||v-u|dx \\
&\geq L_1(u) - \beta\int_\Omega |\nabla u|^{p-1}|\nabla v - \nabla u|dx - \int_\Omega |u|^{p-1}|v-u|dx \\
&\geq L_1(u) - \beta\Big(\int_\Omega |\nabla u|^p dx\Big)^{\frac{p-1}{p}}\Big(\int_\Omega |\nabla v - \nabla u|^p dx\Big)^{\frac{1}{p}} \\
&\qquad - \Big(\int_\Omega |u|^p dx\Big)^{\frac{p-1}{p}}\Big(\int_\Omega |v-u|^p dx\Big)^{\frac{1}{p}} \\
&\geq L_1(u) - C\|v-u\| \\
&\geq L_1(u) - \epsilon
\end{aligned}$$

for all $v \in X$ with $\|v - u\| \leq \delta = \frac{\epsilon}{C}$, where C is a positive constant. The proof of Lemma 2.1 is complete. □

Lemma 2.2. *The functionals L_2 and ψ given by (1.6) and (1.7) are sequentially weakly continuous.*

Proof. Let $\{u_m\}$ be a sequence converging weakly to u in X. We will show that

$$(i)\ \lim_{m\to\infty}\int_\Omega F(x,u_m)dx = \int_\Omega F(x,u)dx, \quad (ii)\ \lim_{m\to\infty}\int_{\partial\Omega} G(x,u_m)d\sigma = \int_{\partial\Omega} G(x,u)d\sigma. \tag{2.2}$$

Indeed, by using **(F1)** we have

$$\begin{aligned}
\Big|\int_\Omega [F(x,u_m) - F(x,u)]dx\Big| &\leq \int_\Omega |f(x, u + \theta_m(u_m - u))||u_m - u|dx \\
&\leq C_1\int_\Omega \Big(1 + |u + \theta_m(u_m - u)|^{p-1}\Big)|u_m - u|dx \\
&\leq C_1\Big(\ |\Omega|_N^{\frac{p-1}{p}} + \|u + \theta_m(u_m - u)\|_{L^p(\Omega)}^{p-1}\ \Big) \\
&\quad \times \|u_m - u\|_{L^p(\Omega)}
\end{aligned} \tag{2.3}$$

where $0 \leq \theta_m(x) \leq 1$ for all $x \in \Omega$ and $|\Omega|_N$ denotes the Lebesgue measure of Ω in $\mathbb{R}^N$.

On the other hand, since $X \hookrightarrow L^p(\Omega)$ is compact, the sequence $\{u_m\}$ converges to u in the space $L^p(\Omega)$. Hence, it is easy to see that the sequences $\{\|u + \theta_m(u_m - u)\|_{L^p(\Omega)}\}$ is bounded. Thus, it follows from (2.3) that relation (2.2)$-(i)$ holds true. Similarly, since the embedding from X to $L^p(\partial\Omega)$ is compact, it follows that relation (2.2)$-(ii)$ hold true. □

Lemma 2.3. *The functional $E_{\lambda,\mu}$ given by (2.1) is sequentially weakly lower semicontinuous.*

Proof. We first prove that J is sequentially weakly lower semicontinuous in X. Let $\{u_m\}$ be a sequence that converges weakly in X. By the sequentially weakly lower semicontinuity of the functional L_1, we have

$$\liminf_{m\to\infty} L_1(u_m) \geq L_1(u).$$

Combining this with the continuity and monotonocity of the function $t \to \widehat{M_1}(t)$, we get

$$\liminf_{m\to\infty} J(u_m) = \liminf_{m\to\infty} \widehat{M_1}(L_1(u_m) \geq \widehat{M_1}\Big(\liminf_{m\to\infty} L(u_m)\Big) \geq \widehat{M_1}\Big(L_1(u)\Big) = J(u).$$

Thus, the functional J is sequentially weakly lower semicontinuous in X.

Next, we prove that the functional I given by (1.7) is sequentially weakly continuous in X. This follows that $E_{\lambda,\mu}$ is sequentially weakly lower semicontinuous in X. Let $\{u_m\}$ be a sequence that converges weakly in X. By the sequentially weakly continuity of the functional L_2, we have

$$\lim_{m\to\infty} L_2(u_m) = L_2(u).$$

Combining this with the continuity of the function $t \to \widehat{M_2}(t)$, we get

$$\lim_{m\to\infty} I(u_m) = \lim_{m\to\infty} \widehat{M_2}(L_2(u_m) = \widehat{M_2}\Big(\lim_{m\to\infty} L_2(u_m)\Big) = \widehat{M_2}\Big(L_2(u)\Big) = I(u).$$

Thus, the functional I is sequentially weakly continuous in X. □

Lemma 2.4. *The functional $E_{\lambda,\mu}$ is coercive and bounded from below.*

Proof. Consider $\widetilde{C}$ as in (Γ). By $(\mathbf{F1}), (\mathbf{F3})$ and $(\mathbf{G1}), (\mathbf{G3})$, there exists $C_\lambda = C(\lambda) > 0$ and $C'_\mu = C(\mu) > 0$ such that

$$\lambda F(x,t) \leq \frac{m_0 \min\{1,\alpha\} S_{p,\Omega}}{4\widetilde{C} m_2 p} K_2(t) + C_\lambda, \quad \text{for a.e. } x \in \Omega, \quad t \in \mathbb{R}$$

and

$$\mu G(x,t) \leq \frac{m_0 \min\{1,\alpha\} S_{p,\partial\Omega}}{4\widetilde{C} p} K_4(t) + C'_\mu, \quad \text{for a.e. } x \in \partial\Omega, \quad t \in \mathbb{R}.$$

Hence, using $(\mathbf{M1})$ and $(\mathbf{H1})$

$$\begin{aligned} E_{\lambda,\mu}(u) &\geq \frac{m_0}{p} \int_\Omega (\alpha|\nabla u|^p + |u|^p) dx - m_2 \int_\Omega \Big(\frac{m_0 \min\{1,\alpha\} S_{p,\Omega}}{4\widetilde{C} m_2 p} K_2(u) + C_\lambda\Big) dx \\ &\quad - \int_{\partial\Omega} \Big(\frac{m_0 \min\{1,\alpha\} S_{p,\partial\Omega}}{4\widetilde{C} p} K_4(u) + C'_\mu\Big) d\sigma \\ &\geq \frac{m_0 \min\{1,\alpha\}}{p} \int_\Omega (|\nabla u|^p + |u|^p) dx \\ &\quad - \frac{m_0 \min\{1,\alpha\} S_{p,\Omega}}{4p} \int_\Omega |u|^p dx - C_\lambda |\Omega|_N \\ &\quad - \frac{m_0 \min\{1,\alpha\} S_{p,\partial\Omega}}{4p} \int_{\partial\Omega} |u|^p d\sigma - C'_\mu |\partial\Omega|_{N-1} \\ &\geq \frac{m_0 \min\{1,\alpha\}}{p} \int_\Omega (|\nabla u|^p + |u|^p) dx - \frac{m_0 \min\{1,\alpha\}}{4p} \int_\Omega (|\nabla u|^p + |u|^p) dx \\ &\quad - \frac{m_0 \min\{1,\alpha\}}{4p} \int_\Omega (|\nabla u|^p + |u|^p) dx - C_\lambda |\Omega|_N - C'_\mu |\partial\Omega|_{N-1} \\ &= \frac{m_0 \min\{1,\alpha\}}{2p} \int_\Omega (|\nabla u|^p + |u|^p) dx - C_\lambda |\Omega|_N - C'_\mu |\partial\Omega|_{N-1}. \end{aligned} \tag{2.4}$$

Since $\partial\Omega$ is bounded, the functional $E_{\lambda,\mu}$ is coercive and bounded from below and coercive on X. □

Lemma 2.5. *If $u \in X$ is a weak solution of problem (1.1) then $u \geq 0$ in Ω.*

Proof. Observe that if u is a weak solution of (1.1), denoting by u_- the negative part of u, i.e. $u_-(x) = \min\{u(x), 0\}$, we have

$$\begin{aligned}
0 &= \langle E'_{\lambda,\mu}(u), u_-\rangle \\
&= M_1\left(\frac{1}{p}\int_\Omega (H(|\nabla u|^p + |u|^p))dx\right)\int_\Omega (h(|\nabla u|^p)|\nabla u|^{p-2}\nabla u.\nabla u_- + |u|^{p-2}uu_-)dx \\
&\quad - \lambda M_2\left(\int_\Omega F(x,u_-)dx\right)\int_\Omega f(x,u)u_- dx - \mu\int_{\partial\Omega} g(x,u)u_- d\sigma \\
&\geq m_0\alpha\int_\Omega (|\nabla u_-|^p + |u_-|^p) = m_0\alpha\|u_-\|_X^p dx.
\end{aligned} \tag{2.5}$$

It is easy to see that if $u \in X$ then $u_- \in X$, so from (2.5) we have $u \geq 0$ in Ω. □

Lemma 2.1 - 2.4 imply by applying the minimum principle in [19] that $E_{\lambda,\mu}$ has a global minimizer u_1 and by lemma 2.5 , u_1 is a non-negative solution of problem (1.1) . The following lemma shows that the solution u_1 is not trivial provided that λ and μ are large enough.

Lemma 2.6. *There exists a constant $\lambda^*, \mu^* > 0$ such that for all $\lambda \geq \lambda^*$ and $\mu \geq \mu^*$ we have $\inf_{u\in X} E_{\lambda,\mu}(u) < 0$, hence $u_1 \not\equiv 0$, i.e., the solution u_1 is not trivial.*

Proof. Indeed, let Ω' be a sufficiently large compact subset of Ω and a function $u_0 \in C_0^\infty(\Omega)$, such that $u_0(x) = t_0$ on $\Omega', 0 \leq u_0(x) \leq t_0$ on $\Omega \setminus \Omega'$. Then we have

$$\begin{aligned}
\int_\Omega F(x,u_0)dx &= \int_{\Omega'} F(x,u_0)dx + \int_{\Omega\setminus\Omega'} F(x,u_0)dx \\
&\geq \int_{\Omega'} F(x,t_0)dx - C_1\int_{\Omega\setminus\Omega'} (1 + |u_0|^p)dx \\
&\geq \int_{\Omega'} F(x,t_0)dx - C_1(1 + |t_0|^p)|\Omega\setminus\Omega'|_N > 0,
\end{aligned}$$

provided that $|\Omega\setminus\Omega'| > 0$ is small enough. So, we deduce that

$$\begin{aligned}
E_{\lambda,\mu}(u_0) &\leq \frac{m_1}{p}\int_\Omega (\beta|\nabla u_0|^p + |u_0|^p)dx \\
&\quad - \lambda\widehat{M_2}\Big(\int_{\Omega'} F(x,t_0)dx - C_1(1 + |t_0|^p)|\Omega\setminus\Omega'|_N\Big) - \mu\int_{\partial\Omega} G(x,u_0)d\sigma \\
&\leq \frac{m_1}{p}\int_\Omega (\beta|\nabla u_0|^p + |u_0|^p)dx \\
&\quad - \lambda m_3\Big(\int_{\Omega'} F(x,t_0)dx - C_1(1 + |t_0|^p)|\Omega\setminus\Omega'|_N\Big) \\
&\quad - \mu C_2(1 + |t_0|^p)|\partial\Omega|_{N-1} \\
&< 0
\end{aligned}$$

for all $\lambda \geq \lambda^*$ and $\mu \geq \mu^*$ large enough. This completes the proof. □

Our idea is to obtain the second weak solution $u_2 \in X$ by applying the mountain pass theorem in [1]. To this purpose, we first show that for all $\lambda \geq \lambda^*$ and $\mu \geq \mu^*$, the functional $E_{\lambda,\mu}$ has the geometry of the mountain pass theorem.

Lemma 2.7. *There exist a constant $\rho \in (0, \|u_1\|_X)$ and a constant $r > 0$ such that $E_{\lambda,\mu}(u) \geq r$ for all $u \in X$ with $\|u\|_X = \rho$.*

Proof. By **(F1)**, **(F2)** and **(G1)**, **(G2)**, we have

$$\lambda F(x,t) \leq \frac{m_0 \min\{1,\alpha\} S_{p,\Omega}}{4\widetilde{C} m_2 p} K_1(t) + C_\lambda |t|^\alpha, \quad \forall (x,t) \in \Omega \times \mathbb{R} \tag{2.6}$$

and

$$\mu G(x,t) \leq \frac{m_0 \min\{1,\alpha\} S_{p,\partial\Omega}}{4\widetilde{C} p} K_3(t) + C'_\mu |t|^\alpha, \quad \forall (x,t) \in \partial\Omega \times \mathbb{R}, \tag{2.7}$$

where $p < \alpha < p^*$. Hence, using the continuous embeddings, we get

$$\begin{aligned}
E_{\lambda,\mu}(u) &\geq \frac{m_0 \min\{1,\alpha\}}{p} \int_\Omega (|\nabla u|^p + |u|^p) dx - \lambda \widehat{M_2}\Big(\int_\Omega F(x,u) dx\Big) \\
&\quad - \mu \int_{\partial\Omega} G(x,u) d\sigma \\
&\geq \frac{m_0 \min\{1,\alpha\}}{p} \int_\Omega (|\nabla u|^p + |u|^p) dx \\
&\quad - m_2 \int_\Omega \left(\frac{m_0 \min\{1,\alpha\} S_{p,\Omega}}{4\widetilde{C} m_2 p} K_1(u) + C_\lambda |u|^\alpha \right) dx \\
&\quad - \int_{\partial\Omega} \left(\frac{m_0 \min\{1,\alpha\} S_{p,\partial\Omega}}{4\widetilde{C} p} K_3(u) + C'_\mu |u|^\alpha \right) d\sigma \\
&\geq \frac{m_0 \min\{1,\alpha\}}{p} \int_\Omega (|\nabla u|^p + |u|^p) dx - \frac{m_0 \min\{1,\alpha\} S_{p,\Omega}}{4p} \int_\Omega |u|^p dx \\
&\quad - C_\lambda \int_\Omega |u|^\alpha dx - \frac{m_0 \min\{1,\alpha\} S_{p,\partial\Omega}}{4p} \int_{\partial\Omega} |u|^p d\sigma - C'_\mu \int_{\partial\Omega} |u|^\alpha d\sigma \\
&\geq \frac{m_0 \min\{1,\alpha\}}{p} \|u\|_X^p - \frac{m_0 \min\{1,\alpha\}}{4p} \|u\|_X^p - \overline{C_\lambda} \|u\|_X^\alpha \\
&\quad - \frac{m_0 \min\{1,\alpha\}}{4p} \|u\|_X^p - \overline{C'_\mu} \|u\|_X^\alpha \\
&= \Big(\frac{m_0 \min\{1,\alpha\}}{2p} - \overline{C_\lambda} \|u\|_X^{\alpha-p} - \overline{C'_\mu} \|u\|_X^{\alpha-p} \Big) \|u\|_X^p,
\end{aligned}$$

where $\overline{C_\lambda}$ and $\overline{C'_\mu}$ are positive constants. Since $p < \alpha < p^*$, there are positive constants $\rho < \|u_1\|_X$ and r such that $E_{\lambda,\mu}(u) \geq r$ for all $u \in X$ with $\|u\|_X = \rho$. □

Lemma 2.8. *The mappings I' and ψ' are sequentially weakly-strongly continuous, namely, $u_n \rightharpoonup u$ in X implies $I'(u_n) \to I'(u)$ and $\psi'(u_n) \to \psi'(u)$ in X^*.*

Proof. Let $\{u_n\}$ be a sequence converging weakly to u in X. Since the embeddings $X \hookrightarrow L^p(\Omega)$ is compact, from **(F1)**, we can see that the Nemytskii operator $N_f : X \to X^*$ defined by

$$\langle N_f(u), v \rangle = \int_\Omega f(x,u(x)) v(x) dx$$

is sequentially weakly-strongly continuous (see [11]). By the sequentially weakly continuity of the functional L_2 combining with the continuity of the function M_2, we get $I'(u_n) \to I'(u)$ in X^*. Similarly, since the embedding $X \hookrightarrow L^p(\partial\Omega)$ is compact, we can see that the functional ψ' is sequentially weakly-strongly continuous (see [12]). □

Lemma 2.9. *The mapping L'_1 is of type (S_+), i.e.*

$$u_n \rightharpoonup u \text{ in } X \quad \text{and} \quad \limsup_{n\to\infty} \langle L'_1(u_n), u_n - u \rangle \leq 0, \quad n \to \infty$$

implies $u_n \to u$ in X.

Proof. Define the mappings $K, G : X \to X^*$ respectively by

$$\langle K(u), v\rangle = \int_\Omega |u|^{p-2}uvdx, \quad \forall u, v \in X,$$

$$\langle G(u), v\rangle = \frac{1}{p}\int_\Omega h(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\nabla vdx, \quad \forall u, v \in X.$$

Since the embedding $X \hookrightarrow L^p(\Omega)$ is compact, we can see that K is sequentially weakly-strongly continuous.

Let $\{u_n\}$ be a sequence such that converges weakly to u in X and

$$\limsup_{n\to\infty}\langle G(u_n), u_n - u\rangle = \limsup_{n\to\infty}\int_\Omega h(|\nabla u_n|^p)|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n - u)dx \le 0.$$

Since $\{u_n\}$ converges weakly to u, we have

$$\lim_{n\to\infty}\int_\Omega h(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\nabla(u_n - u)dx = 0.$$

From (**H2**) we have

$$\begin{aligned} 0 &\ge \limsup_{n\to\infty}\Bigg[\int_\Omega h(|\nabla u_n|^p)|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n - u)dx \\ &\qquad\qquad - \int_\Omega h(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\nabla(u_n - u)dx\Bigg] \\ &= \limsup_{n\to\infty}\left[\int_\Omega (h(|\nabla u_n|^p)|\nabla u_n|^{p-2}\nabla u_n - h(|\nabla u|^p)|\nabla u|^{p-2}\nabla u)(\nabla u_n - \nabla u)dx\right] \\ &\ge \limsup_{n\to\infty}\theta\int_\Omega |\nabla u_n - \nabla u|^p dx. \end{aligned}$$

So $\{u_n\} \to u$ in X. This shows that the functional G is of type (S_+). Moreover, since K is sequentially weakly-strongly continuous, the mapping $L_1' = K - G$ is of type (S_+). This completes the proof of Lemma 2.9. □

Lemma 2.10. *The mappings J' and $E'_{\lambda,\mu} : X \to X^*$ are of type (S_+).*

Proof. Suppose that $\{u_n\} \subset X$ is a sequence that converges weakly to u in X and

$$\limsup_{n\to\infty}\langle J'(u_n), u_n - u\rangle = \limsup_{n\to\infty}\ M_1(L_1(u_n))\ \langle L_1'(u_n), u_n - u\rangle \le 0.$$

From (**M1**) we have

$$\limsup_{n\to\infty}\ \langle L_1'(u_n), u_n - u\rangle \le 0.$$

Since L_1' is of type (S_+), we have $u_n \to u$ in X. This shows that the mapping $J' : X \to X^*$ is of type (S_+). Moreover, since I' and ψ' are sequentially weakly-strongly continuous, this implies that $E'_{\lambda,\mu} : X \to X^*$ is of type (S_+). □

Lemma 2.11. *The functional $E_{\lambda,\mu}$ satisfies the Palais-Smale condition in X, i.e. a sequence $\{u_n\}$ such that $E_{\lambda,\mu}(u_n) \to c$ and $E'_{\lambda,\mu}(u_n) \to 0$, has a strongly convergent subsequence.*

Proof. By Lemma 2.4, we deduce that $E_{\lambda,\mu}$ is coercive on X. Let $\{u_n\} \subset X$ be a Palais-Smale sequence for the functional $E_{\lambda,\mu}$ in X, i.e.

$$E_{\lambda,\mu}(u_m) \to c, \quad E'_{\lambda,\mu}(u_m) \to 0 \text{ in } X^{-1} \text{ as } m \to \infty, \tag{2.8}$$

where X^{-1} is the dual space of X.

Since $E_{\lambda,\mu}$ is coercive on X, relation (2.8) implies that the sequence $\{u_m\}$ is bounded in X. Since X is reflexive , we can take a subsequence of $\{u_n\}$ denoted still by $\{u_n\}$, such that it converges weakly to u in X. The condition $E'_{\lambda,\mu}(u_n) \to 0$ implies that $\langle E'_{\lambda,\mu}(u_n), u_n - u\rangle \to 0$. Since $E'_{\lambda,\mu} : X \to X^*$ is of type (S_+), we have $u_n \to u \in X$. This completes the proof. □

Proof of Theorem 1.2. By Lemmas 2.1-2.6, system (1.1) admits a non-negative, non-trivial weak solution u_1 as the global minimizer of $E_{\lambda,\mu}$. Setting

$$\overline{c} := \inf_{\chi\in\mathbf{\Gamma}} \max_{u\in\chi([0,1])} E_{\lambda,\mu}(u), \tag{2.9}$$

where $\mathbf{\Gamma} := \{\chi \in C([0, 1], X) : \chi(0) = 0, \chi(1) = u_1\}$.

Lemmas 2.7- 2.11 show that all assumptions of the mountain pass theorem in [1] are satisfied, $E_{\lambda,\mu}(u_1) < 0$ and $\|u_1\|_X > \rho$. Then, $\overline{c}$ is a critical value of $E_{\lambda,\mu}$, i.e. there exists $u_2 \in X$ such that $E_{\lambda,\mu}(u_2)(\varphi) = 0$ for all $\varphi \in X$ or u_2 is a weak solution of (1.1). Moreover, u_2 is not trivial and $u_2 \not\equiv u_1$ since $E_{\lambda,\mu}(u_2) = \overline{c} > 0 > E_{\lambda,\mu}(u_1)$. Theorem 1.2 is proved. □

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