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Generalization of Hartshorne's Connectedness Theorem

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ABSTRACT: In this paper, we use local cohomology theory to present some results about connectedness property of prime spectrum of modules. In particular, we generalize the Hartshorne's connectedness theorem.

Key Words: Connectedness, Local cohomology, Prime submodule, Zariski topology.

Contents

1. Introduction

In last decades, the connectedness of some varieties of the prime spectra of a commutative ring is investigated by many authors. *Falting's connectedness theorem* asserts that in an analytically irreducible local ring $(R, \mathfrak{m})$ of dimension n, if $\mathfrak{a} \subseteq \mathfrak{m}$ is an ideal generated by at most $n-2$ elements, then the punctured spectrum of $R/\mathfrak{a}$ is connected (see [10]). On the other hand, *Hartshorne's connectedness result* (see [12]), says that $\mathrm{Spec}(R) \setminus V(\mathfrak{a})$ is a connected subset of $\mathrm{Spec}(R)$ provided $\mathrm{grade}(\mathfrak{a}, R) > 1$. In [8] Divaani-Aazar and Schenzel proved a generalization of these results for finitely generated modules. The reader can refer to [7,9,19,20] for more references to the subject mentioned above. In this paper, we use local cohomology to prove some connectedness results for varieties of prime spectrum of certain modules. In particular, we generalize the Hartshorne's connectedness theorem.

Over the past several decades, the theory of prime modules and prime submodules (and its related topics such as Zariski topology on the prime spectrum of modules) is investigated by many algebraist (see [1,2,6,11,13,14]). The Zariski topology on the spectrum of prime ideals of a ring is one of the main tools in algebraic geometry. In the literature, there are many different generalizations of the Zariski topology of rings to modules via prime submodules (see [3,16,18]). Here, we use the Zariski topology on the prime spectrum of modules which is considered by C. P. Lu in [16]. It is shown by Lu that if M is finitely generated, then $\mathrm{Spec}(M)$, the set of all prime submodule of M when is equipped with the Zariski topology, is connected if and only if $\mathrm{Spec}(R/\mathrm{Ann}(M))$ is a connected space (see [16]). We find that, this is the only result on the connectedness of the prime spectra of modules in the previous literatures. Here, we are going to give some connectedness results for certain subspaces of prime spectrum of a module.

Throughout this paper, all rings are commutative with identity and all modules are unital. For a submodule N of an R-module M, $(N :_R M)$ denotes the ideal $\{r \in R \mid rM \subseteq N\}$ and *annihilator* of M, denoted by $\mathrm{Ann}_R(M)$, is the ideal $(\mathbf{0} :_R M)$. If there is no ambiguity, we will consider $(N : M)$ (resp. $\mathrm{Ann}(M)$) instead of $(N :_R M)$ (resp. $\mathrm{Ann}_R(M)$). A submodule N of an R-module M is said to be *prime* if $N \neq M$ and whenever $rm \in N$ (where $r \in R$ and $m \in M$), then $r \in (N : M)$ or $m \in N$. If N is prime, then $\mathfrak{p} = (N : M)$ is a prime ideal of R. In this case, N is said to be $\mathfrak{p}$*-prime* (see [14]). The set of all prime submodules of an R-module M is called the *prime spectrum* of M and denoted by $\mathrm{Spec}(M)$. Similarly, the collection of all $\mathfrak{p}$-prime submodules of an R-module M for any $\mathfrak{p} \in \mathrm{Spec}(R)$ is designated by $\mathrm{Spec}_{\mathfrak{p}}(M)$.

We remark that $\mathrm{Spec}(\mathbf{0}) = \emptyset$ and that $\mathrm{Spec}(M)$ may be empty for some nonzero R-module M (for example see [15,18]). Let B be a nonzero finitely generated R-module. Since every proper submodule of B is contained in a maximal submodule and since every maximal submodule is prime, $\mathrm{Spec}(B)$ is nonempty.

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Suppose that M is an R-module. For any submodule N of M, $V(N)$ is defined as $\{P \in \mathrm{Spec}(M) \mid (P : M) \supseteq (N : M)\}$. Set $\mathsf{Z}(M) = \{V(N) \mid N \leq M\}$. Then the elements of the set $\mathsf{Z}(M)$ satisfy the axioms for closed sets in a topological space $\mathrm{Spec}(M)$ (see [16]). The resulting topology due to $\mathsf{Z}(M)$ is called the *Zariski topology relative to* M. We recall that the *Zariski radical* of a submodule N of an R-module M, denoted by $\mathrm{rad}(N)$, is the intersection of all members of $V(N)$, that is, $\mathrm{rad}(N) = \cap_{P \in V(N)} P$ (see [17, Definitions 1.3]).

Let N be an R-module. For an ideal I of R we recall that the *i-th local cohomology module of N with respect to I* is defined as

$$H_I^i(N) = \varinjlim_{n \in \mathbb{N}} \mathrm{Ext}_R^i(R/I^n, N).$$

The reader can refer to [4] for the basic properties of local cohomology modules. Let H be an R-module. An element $a \in R$ is said to be H-regular if $ax \neq 0$ for all $0 \neq x \in H$. A sequence $a_1, \ldots, a_n$ of elements of R is an H-sequence (or an H-regular sequence) if the following two conditions hold: (1) a_1 is H-regular, a_2 is (H/a_1H)-regular, …, a_n is $(H/\sum_{i=1}^{n-1} a_iH)$-regular; (2) $H/\sum_{i=1}^{n} a_iH \neq 0$. Let R be a noetherian ring, M a finitely generated R-module and I an ideal such that $IM \neq M$. Then the common length of the maximal M-sequences in I is called the *grade* of I on M denoted by $\mathrm{grade}(I, M)$.

2. Main Results

Our first main result is the following statement which is a generalization of Hartshorne's connectedness result [12, Proposition 2.1].

Theorem 2.1. *Let R be a noetherian ring, M a finitely generated indecomposable R-module and N be a submodule of M such that* $\mathrm{grade}((N : M), M) > 1$. *Then* $\mathrm{Spec}(M) \setminus V(N)$ *is connected.*

Proof. Suppose that $\mathrm{Spec}(M) \setminus V(N)$ is disconnected. Then there are submodules N_1 and N_2 of M such that the following items hold:

1. $\mathrm{Spec}(M) \setminus V(N_1)$ and $\mathrm{Spec}(M) \setminus V(N_2)$ are disjoint and nonempty open subsets of $\mathrm{Spec}(M)$ and;
2. $\mathrm{Spec}(M) \setminus V(N) = [\mathrm{Spec}(M) \setminus V(N_1)] \cup [\mathrm{Spec}(M) \setminus V(N_2)]$.

The item (1) implies that

$$\begin{aligned} \emptyset &= [\mathrm{Spec}(M) \setminus V(N_1)] \cap [\mathrm{Spec}(M) \setminus V(N_2)] \\ &= \mathrm{Spec}(M) \setminus [V(N_1) \cup V(N_2)] \\ &= \mathrm{Spec}(M) \setminus V(N_1 \cap N_2). \end{aligned}$$

Since M is finitely generated, $V(N_1 \cap N_2) = \mathrm{Spec}(M)$. Thus, for every prime submodule P of M we have $(N_1 \cap N_2 : M) \subseteq (P : M)$. By [17, Proposition 2.3(5)], $(N_1 \cap N_2 : M) \subseteq \sqrt{\mathrm{Ann}(M)}$. Since R is noetherian, there exists an integer t such that

$$(N_1 \cap N_2 : M)^t \subseteq \left(\sqrt{\mathrm{Ann}(M)}\right)^t \subseteq \mathrm{Ann}(M). \tag{2.1}$$

From item (2), we deduce that $\mathrm{Spec}(M) \setminus V(N) = \mathrm{Spec}(M) \setminus [V(N_1) \cap V(N_2)]$. Thus,

$$\begin{aligned} V(N) &= V(N_1) \cap V(N_2) \\ &= V((N_1 : M)M + (N_2 : M)M) \\ &= V([(N_1 : M) + (N_2 : M)]M). \end{aligned}$$

Let $A := (N_1 : M) + (N_2 : M)$. Then $\mathrm{rad}(N) = \mathrm{rad}(AM)$. This implies that

$$(\mathrm{rad}(N) : M) = (\mathrm{rad}(AM) : M).$$

Since M is finitely generated, we infer that

$$\begin{array}{rcll} \sqrt{(N:M)} &=& (\mathrm{rad}(N):M) & \text{by [17, Proposition 2.3(5)]} \\ &=& (\mathrm{rad}(AM):M) & \\ &=& \sqrt{(AM:M)} & \text{by [17, Proposition 2.3(5)]} \\ &=& \sqrt{\mathrm{Ann}(M/AM)} & \\ &=& \sqrt{A+\mathrm{Ann}(M)}. & \end{array}$$

Since $\mathrm{Ann}(M) \subseteq A$, we have

$$\sqrt{(N:M)} = \sqrt{A} = \sqrt{(N_1:M)+(N_2:M)}. \tag{2.2}$$

Now, we consider the Mayer-Vietoris sequence

$$\begin{aligned} 0 &\to H^0_{(N_1:M)+(N_2:M)}(M) \to H^0_{(N_1:M)}(M) \oplus H^0_{(N_2:M)}(M) \\ &\to H^0_{(N_1:M)\cap(N_2:M)}(M) \to H^1_{(N_1:M)+(N_2:M)}(M) \to \cdots . \end{aligned} \tag{2.3}$$

By (2.2) and [4, Remark 1.2.3], we have $H^i_{(N_1:M)+(N_2:M)}(M) = H^i_{(N:M)}(M)$ for each i. Since $\mathrm{grade}((N:M),M) > 1$, $H^i_{(N:M)}(M) = 0$, for $i = 0,1$. So, in the light of (2.3), we obtain that

$$H^0_{(N_1:M)}(M) \oplus H^0_{(N_2:M)}(M) \cong H^0_{(N_1:M)\cap(N_2:M)}(M). \tag{2.4}$$

By (2.1), $H^0_{(N_1:M)\cap(N_2:M)}(M)$ is equal to M. Since M is indecomposable, it follows from (2.4) that either $M = H^0_{(N_1:M)}(M)$ and $H^0_{(N_2:M)}(M) = 0$ or $M = H^0_{(N_2:M)}(M)$ and $H^0_{(N_1:M)}(M) = 0$. Thus, there are integers a, b such that $(N_1:M)^a \subseteq \mathrm{Ann}(M)$ or $(N_2:M)^b \subseteq \mathrm{Ann}(M)$. Therefore, $V(N_1) = \mathrm{Spec}(M)$ or $V(N_2) = \mathrm{Spec}(M)$, contrary to item (1). □

Theorem 2.1 yields the following interesting consequences.

Corollary 2.2. *Let R be a noetherian ring and M be a finitely generated indecomposable R-module. Suppose that I is an ideal of R such that $\sqrt{I} \supseteq \mathrm{Ann}(M)$ and $\mathrm{grade}(I,M) > 1$. Then $\mathrm{Spec}(M) \setminus V(IM)$ is connected.*

Proof. By [14, p. 65, Proposition 8], $(\sqrt{I}M : M) = \sqrt{I}$. Hence, by [5, Proposition 1.2.10(b)],

$$\mathrm{grade}((\sqrt{I}M:M),M) = \mathrm{grade}(\sqrt{I},M) = \mathrm{grade}(I,M) > 1.$$

Therefore, $\mathrm{Spec}(M) \setminus V(\sqrt{I}M)$ is connected by Theorem 2.1. It is easy to see that $V(\sqrt{I}M) = V(IM)$. Consequently, $\mathrm{Spec}(M) \setminus V(IM)$ is connected. □

Corollary 2.3. *Let R be a noetherian ring and M be a finitely generated indecomposable R-module. Suppose that $\mathfrak{p} \in \mathrm{Supp}(M)$ and $\mathrm{grade}(\mathfrak{p},M) > 1$. Then $\mathrm{Spec}(M) \setminus V(\mathfrak{p}M)$ is connected.*

Proof. Use Corollary 2.2. □

Corollary 2.4. ***(Hartshorne's connectedness result)*** *Let $(R,\mathfrak{m})$ be a noetherian local ring and $\mathrm{depth}(R) > 1$. Then $\mathrm{Spec}(R) \setminus V(\mathfrak{m})$ is connected.*

Proof. Use Corollary 2.3. □

Before bringing the final main result of this paper, recall that for any topological space Z and $y \in Z$, Z_y is the set of all points $y' \in Z$ whose closure contains y (see [12]).

Lemma 2.5. *Let T be a connected topological space and Y a closed subspace such that for each $y \in Y$, $T_y \setminus \{y\}$ is nonempty and connected. Then $T \setminus Y$ is connected.*

Proof. See [12, Lemma 1.2]. □

We are now ready to state and prove the second main result which is an application of Theorem 2.1.

Theorem 2.6. *Let R be a noetherian ring and M be an R-module such that $X := \mathrm{Spec}(M)$ is a connected topological space. Suppose that Y is a closed subset of X such that for each $P \in Y$, $M_{\mathfrak{p}}$, where $\mathfrak{p} := (P : M)$, is a cyclic indecomposable $R_{\mathfrak{p}}$-module with $\mathrm{depth}(M_{\mathfrak{p}}) > 1$. Then $\mathrm{Spec}(M) \setminus Y$ is connected.*

Proof. By Lemma 2.5, it is enough to show that for each $P \in Y$, $X_P \setminus \{P\}$ is nonempty and connected. Note that by [16, Proposition 5.2(1)], X_P can be described as following:

$$X_P = \{Q \in X \mid P \in V(Q)\} = \{Q \in X \mid (Q : M) \subseteq (P : M)\}.$$

By [17, Theorem 3.7(1)], this set (as a subspace of X) is homeomorphic to $\mathrm{Spec}(M_{\mathfrak{p}})$. More precisely, $P \in X_P$ is corresponded to $P^e \in \mathrm{Spec}(M_{\mathfrak{p}})$ (the extension of P with respect to the natural map $M \to M_{\mathfrak{p}}$). Moreover,

$$(P :_R M)_{\mathfrak{p}} = (P^e :_{R_{\mathfrak{p}}} M_{\mathfrak{p}}) = \mathfrak{p}R_{\mathfrak{p}} \in \mathrm{Max}(R_{\mathfrak{p}}).$$

Since, $M_{\mathfrak{p}}$ is a cyclic $R_{\mathfrak{p}}$-module, $|\mathrm{Spec}_{\mathfrak{q}}(M_{\mathfrak{p}})| \leq 1$ for all $\mathfrak{q} \in \mathrm{Spec}(R_{\mathfrak{p}})$ by [18, Theorem 3.5]. Therefore, $P^e = \mathfrak{p}M_{\mathfrak{p}}$ is the unique maximal submodule of $M_{\mathfrak{p}}$. Hence, $X_P \setminus \{P\}$ is homeomorphic to

$$\mathrm{Spec}(M_{\mathfrak{p}}) \setminus \{\mathfrak{p}M_{\mathfrak{p}}\} = \mathrm{Spec}(M_{\mathfrak{p}}) \setminus V(\mathfrak{p}M_{\mathfrak{p}}).$$

Since $1 < \mathrm{depth}(M_{\mathfrak{p}}) \leq \dim(M_{\mathfrak{p}})$, there are distinct prime ideals $\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_2$ in the $\mathrm{Supp}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ such that $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2$. Since $M_{\mathfrak{p}}$ is cyclic, according to [16], there are prime submodules P_0, P_1, P_2 of $M_{\mathfrak{p}}$ such that $\mathfrak{p}_i = (P_i :_{R_{\mathfrak{p}}} M_{\mathfrak{p}})$ for $i = 0, 1, 2$. These prime submodules are distinct and $P_2 = \mathfrak{p}M_{\mathfrak{p}}$ by [18, Theorem 3.5]. Therefore, $\mathrm{Spec}(M_{\mathfrak{p}}) \setminus \{\mathfrak{p}M_{\mathfrak{p}}\}$ is nonempty. Since

$$\mathrm{grade}((\mathfrak{p}M_{\mathfrak{p}} : M_{\mathfrak{p}}), M_{\mathfrak{p}}) = \mathrm{grade}(\mathfrak{p}R_{\mathfrak{p}}, M_{\mathfrak{p}}) = \mathrm{depth}(M_{\mathfrak{p}}) > 1,$$

$\mathrm{Spec}(M_{\mathfrak{p}}) \setminus \{\mathfrak{p}M_{\mathfrak{p}}\}$ is connected by Theorem 2.1. This completes the proof. □

Recently, a sheaf on the prime spectra of modules is introduced in [13]. Let N be an R-module. Then the *sheaf associated to N relative to M* is denoted by $\mathcal{A}(N, M)$. For exact definition and the results on this sheaf see [13]. By [13, Proposition 3.2], for each $P \in \mathrm{Spec}(M)$, the stalk $\mathcal{A}(N, M)_P = \varinjlim_{P \in U} \mathcal{A}(N, M)(U)$ of the sheaf $\mathcal{A}(N, M)$ is isomorphic to $N_{\mathfrak{p}}$, where $\mathfrak{p} := (P : M)$.

Corollary 2.7. *Let R be a noetherian ring and M be an R-module such that $\mathrm{Spec}(M)$ is a connected topological space. Suppose that Y is a closed subset of $\mathrm{Spec}(M)$ such that for each $P \in Y$, the stalk of the sheaf $\mathcal{A}(M, M)$ at P, is a cyclic indecomposable $R_{\mathfrak{p}}$-module with $\mathrm{depth}(M_{\mathfrak{p}}) > 1$, where $\mathfrak{p} := (P : M)$. Then $\mathrm{Spec}(M) \setminus Y$ is connected.*

Proof. Use Theorem 2.6. □

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