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A New Optimal Control Technique for Solution of HIV Infection Model

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ABSTRACT: In this paper, we introduce a new method, namely, the optimal control power series technique by using of the optimal control technique and power series technique. One can obtain numerical solutions of the HIV infection model of CD4$^+$T cells via this method. The obtained approximate solution is in good agreement with the experimental results and previous simulations by using of other methods.

Key Words: Optimal control technique, Power series technique, Numerical solution.

Contents

1. Introduction

In this paper, we consider a three dimensional model of the Human Immunodeficiency Virus (HIV) [1]

$$\begin{aligned}\frac{dT}{dt} &= q - \alpha\, T + r\, T\left(1 - \frac{T+I}{T_{\max}}\right) - k\, V\, T,\\ \frac{dI}{dt} &= k\, V\, T - \beta\, I,\\ \frac{dV}{dt} &= \mu\, \beta\, I - \gamma\, V,\end{aligned} \tag{1.1}$$

with the initial conditions: $T(0) = T_0, I(0) = I_0$ and $V(0) = V_0$. Here $T(t)$ represents the concentration of healthy CD4$^+$T cells at time t, $I(t)$ represents the concentration of infected CD4$^+$T cells at time t, and $V(t)$ represents the concentration of free HIV at time t [2,3]. α, β and γ denote natural turnover rates of uninfected T cells, infected T cells and virus particles, respectively, $\left(1 - \frac{T+I}{T_{\max}}\right)$ describes the logistic growth of the healthy CD4$^+$T cells, and proliferation of infected CD4$^+$T cells is neglected [1,3,4]. The term kVT describes the incidence of HIV infection of healthy CD4$^+$T cells, where $k > 0$ is the infection rate. Each infected CD4$^+$T cell is assumed to produce N virus particles during its lifetime, including any of its daughter cells [5]. The body is believed to produce CD4$^+$T cells from precursors in the bone marrow and thymus at a constant rate q. T cells multiply through mitosis with a rate r when T cells are stimulated by antigen or mitogen. $T_{\max}$ denotes the maximum CD4$^+$T cell concentration in the body [1,3,4,5].

In recent years, several methods have been proposed to solve numerically the HIV infection model of CD4$^+$T cells. For example, Ongun [6] presented a numerical solution for HIV CD4$^+$T cells by the Laplace Adomian decomposition method. New approximate solutions of (1) from Homotopy perturbation method are obtained in [7]. By using of the Bessel collocation method, new numerical solutions are obtained in [8]. Merdan et al. [9] by using of the variational iteration method are presented approximate solutions of HIV CD4$^+$T cells. Also, multistage variational iteration method is applied to compute the numerical

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solutions of (1) in [10]. Ghoreishi et al. [11] applied the homotopy analysis method for HIV CD4$^+$T cells. Yüzbaşı and Karaçayır [12] generalized the exponential polynomials reminiscent of the Galerkin method and they obtained the numerical solutions of (1). In [13], the differential transform method has been implemented for a dynamical model of HIV CD4$^+$T cells. Doğan introduced a new method called multi-step Laplace Adomian decomposition method to solution of the HIV infection model of CD4$^+$T cells [14]. Khalid et al. [15] developed the perturbation iteration algorithm and successfully applied it to solve a model for HIV infection of CD4$^+$T cells. For more references, see also [16,17,18,19,20]. In this paper, we obtain the approximate solutions of model (1) by developing the optimal control technique [21,22,23,24] and power series technique [25,26,27].
This paper is prepared as follows: In Section 2, the new numerical method to be used is presented. In Section 3, a dynamical model of HIV CD4$^+$T cells is solved by this method. In the last section, some conclusions are referred.

2. Method of solution

In this section, we introduce a new method based on optimal control technique and power series Technique, namely the optimal control power series Technique, for finding the numerical solution of dynamical epidemic models. We describe the optimal control power series technique in four steps. In summary, these steps are: Formulating this system as an optimal control problem, determining minimized function for system, determining optimality conditions, and solving algebraic systems by power series technique. To explain these fundamental steps in optimal control power series Technique, consider a nonlinear system as

$$\dot{x} = F(x(t)), \quad x(t_0) = x_0, \tag{2.1}$$

where $x \in \mathbf{R}^n$.

Step 1 We can formulate this system as an optimal control problem for optimization [21,22]. In the general form is

$$\dot{x} = F(x(t), u(t)), \quad x(t_0) = x_0, \tag{2.2}$$

where $u(t) \in \mathbf{R}^m$ are control vectors.

Step 2 The aim of the State-Dependent Riccati Equation (SDRE) control is to determine the sub-optimal controller for the system (2.2) such that the following coast functional is minimized [23]:

$$\begin{aligned} &J = \textstyle\int_{t_0}^{\infty} (x(t)^T\, Q\, x(t) + u(t)^T\, R\, u(t))dt, \\ &s.t. \quad \dot{x} = f(t, x(t)) + g(t, x(t))\, u(t), \end{aligned} \tag{2.3}$$

Where $Q \in \mathbf{R}^{n\times n}$ and $R \in \mathbf{R}^{m\times m}$ are state dependent weighting matrices which satisfy $Q \geq 0$ and $R > 0$ for all x.

Step 3 According to the Pontryagin's maximum principle [24], the optimality conditions for (2.3) are determined by the following nonlinear two-point boundary value problem (TPBVP):

$$\begin{aligned} &\dot{x} = f(t, x(t)) + g(t, x(t)) \left[-R^{-1}\, g^T(t, x(t))\, \lambda(t)\right], \\ &\dot{\lambda} = -\left(Q\, x(t) + (\tfrac{\partial f(t,x(t))}{\partial x})^T\, \lambda(t) + \sum_{i=1}^{n} \lambda_i \left[-R^{-1}\, g^T(t, x(t))\, \lambda(t)\right]^T \tfrac{\partial g_i(t,x(t))}{\partial x}\right) \end{aligned} \tag{2.4}$$

where $\lambda(t) \in \mathbf{R}^n$. On the other hand, the optimal control law is illustrated by $u(t)^* = -R^{-1}\, g^T(t, x(t))\lambda(t)$.

Step 4 The system (2.4) contains a nonlinear TPBVP that cannot be solved analytically. But a solution can be expressed in terms of a power series which takes the form

$$x(t) = \sum_{n=0}^{\infty} c_n\, (t - t_0)^n \qquad , \qquad \lambda(t) = \sum_{n=0}^{\infty} d_n\, (t - t_0)^n, \tag{2.5}$$

for some fixed t_0. Substituting the power series into the system (2.4) gives relationships among the coefficients $\{c_n\}$, which when solved gives a power series solution. This technique is called the power series method.

3. Numerical application

In this section, we will apply the optimal control power series technique to a dynamical model of HIV CD4$^+$T cells. Throughout this section, we set $q = 0.1, \alpha = 0.02, \beta = 0.3, r = 3, \gamma = 2.4, k = 0.0027, N = 10$ and $T_{\max} = 1500$. We have

$$\begin{aligned}
\frac{dT}{dt} &= 0.1 - 0.02\,T + 3\,T\left(1 - \frac{T+I}{1500}\right) - 0.0027\,V\,T,\\
\frac{dI}{dt} &= 0.0027\,V\,T - 0.3\,I,\\
\frac{dV}{dt} &= 3\,I - 2.4\,V,
\end{aligned} \tag{3.1}$$

given with the initial conditions $T(0) = 0.1, I(0) = 0, V(0) = 0$ and in the interval $0 \le t \le 0.9$. Applying all aforementioned terms, in order to minimizing $[u_T, u_I, 0]^T$, we define the minimize objective functional

$$\begin{aligned}
&\text{Min } J(u) = \textstyle\int_0^1 (I^2 - T^2 + u_T^2 + u_I^2)dt,\\
&\text{s.t.} \quad \dot{T} = 0.1 - 0.02\,T + 3\,T\left(1 - \tfrac{T+I}{1500}\right) - 0.0027\,V\,T + u_T,\\
&\qquad \dot{I} = 0.0027\,V\,T - 0.3\,I - u_I,\\
&\qquad \dot{V} = 3\,I - 2.4\,V.
\end{aligned}$$

Here, u_T and u_I are the control variables for $T(t)$ and $I(t)$, respectively. Our goal is to increase the number of the uninfected CD4$^+$T cells and minimizing the cost of treatment.
In this paper, we set $\omega_1 = \omega_2 = 1$. According to the Pontryagin's maximum principle, we could reach to the following co-state system:

$$\begin{aligned}
\dot{T} &= 0.1 - 2.98\,T - 0.002\,T^2 - 0.002\,T, I - 0.0027VT - \lambda_1,\\
\dot{I} &= 0.0027\,V\,T - 0.3\,I,\\
\dot{L} &= 3\,I - 2.4\,V,\\
\dot{\lambda_1} &= T - 2.98\lambda_1 + 0.004T\lambda_1 + 0.002I\lambda_1 + 0.0027V\lambda_1 - 0.0027V\lambda_2\\
\dot{\lambda_2} &= -I + 0.002T\lambda_1 + 0.3\lambda_2 - 3\lambda_3,\\
\dot{\lambda_3} &= 0.0027T\lambda_1 - 0.0027T\lambda_2 + 2.4\lambda_3,
\end{aligned} \tag{3.2}$$

and the optimal control law is given by

$$u^*(t) = \begin{bmatrix} u_1^*(t) \\ u_2^*(t) \\ u_3^*(t) \end{bmatrix} = \begin{bmatrix} -\lambda_1 \\ \lambda_2 \\ 0 \end{bmatrix}.$$

By using the power series technique, the approximate solutions of system (8) are:

$$\begin{aligned}
T(t) &= \textstyle\sum_{n=0}^{\infty} c_{1n}\, t^n = T(0) + c_{11}\, t + c_{12}\, t^2 + c_{13}\, t^3 + \cdots \\
I(t) &= \textstyle\sum_{n=0}^{\infty} c_{2n}\, t^n = I(0) + c_{21}\, t + c_{22}\, t^2 + c_{23}\, t^3 + \cdots \\
V(t) &= \textstyle\sum_{n=0}^{\infty} c_{3n}\, t^n = V(0) + c_{31}\, t + c_{32}\, t^2 + c_{33}\, t^3 + \cdots \\
\lambda_1(t) &= \textstyle\sum_{n=0}^{\infty} c_{4n}\, t^n = \lambda_1(0) + c_{41}\, t + c_{42}\, t^2 + c_{43}\, t^3 + \cdots \\
\lambda_2(t) &= \textstyle\sum_{n=0}^{\infty} c_{5n}\, t^n = \lambda_2(0) + c_{51}\, t + c_{52}\, t^2 + c_{53}\, t^3 + \cdots \\
\lambda_3(t) &= \textstyle\sum_{n=0}^{\infty} c_{6n}\, t^n = \lambda_3(0) + c_{61}\, t + c_{62}\, t^2 + c_{63}\, t^3 + \cdots .
\end{aligned} \tag{3.3}$$

Substituting approximate solutions (9) into system (8) and equating the terms with identical powers of t, we can obtain the following approximate solutions:

$$\begin{aligned}
T &= 0.1 + 0.3977800000\, t + 0.5922148450\, t^2 + 0.5875974940\, t^3 \\
&\quad + 0.4370467812\, t^4 + 0.2598795654\, t^5 + 0.1284691841\, t^6 \\
&\quad + 0.05433087940\, t^7 + 0.01990146661\, t^8 + 0.006438780022\, t^9, \\
I &= 0.1225420020000\, t + 0.9993696680\, t^2 - 2.120992477\, t^3 \\
&\quad + 3.799090900\, t^4 - 4.597792247\, t^5 + 4.911909967\, t^6 \\
&\quad - 4.262423197\, t^7 + 3.505583168\, t^8 - 2.275237974\, t^9, \\
V &= 0.1 - 0.2400000000\, t + 0.1880213003\, t^2 - 0.05048007343\, t^3 \\
&\quad - 0.1601232001\, t^4 + 0.2882108934\, t^5 - 0.3384965683\, t^6 \\
&\quad + 0.3108457456\, t^7 - 0.2444359104\, t^8 + 0.1690433931\, t^9, \\
\lambda_1 &= -0.00001349360991\, t^3 + 0.00004676554452\, t^4 - 0.00008592448460\, t^5 \\
&\quad + 0.0001201155034\, t^6 - 0.0001276990959\, t^7 + 0.00009925660561\, t^8 \\
&\quad - 0.00004150041898\, t^9, \\
\lambda_2 &= 0.1499289990\, t^2 - 0.3181303228\, t^3 + 0.5533934190\, t^4 \\
&\quad - 0.6896240302\, t^5 + 0.7218014052\, t^6 - 0.6393289643\, t^7 \\
&\quad + 0.4958400408\, t^8 - 0.3412705856\, t^9, \\
\lambda_3 &= -0.1\, t - 0.06267376677\, t^3 - 0.02498424169\, t^4 + 0.02003220400\, t^5 \\
&\quad - 0.04002226730\, t^6 + 0.03463473240\, t^7 - 0.02846529848\, t^8 \\
&\quad + 0.07043474712\, t^9.
\end{aligned}$$

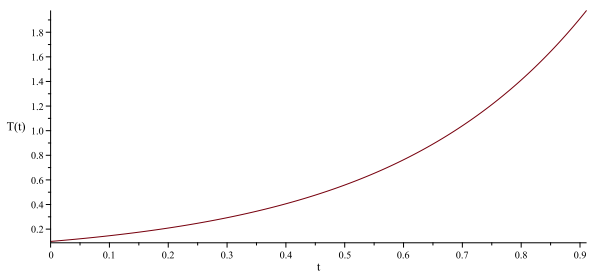

Figure 1: Graphic of the approximate solutions of T(t)

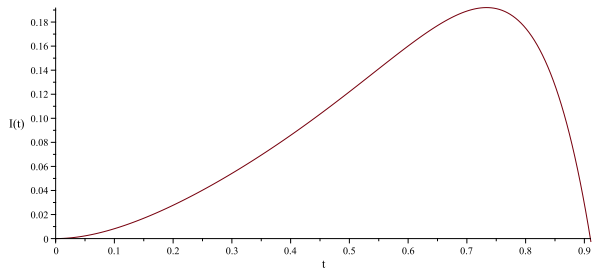

Figure 2: Graphic of the approximate solutions of I(t)

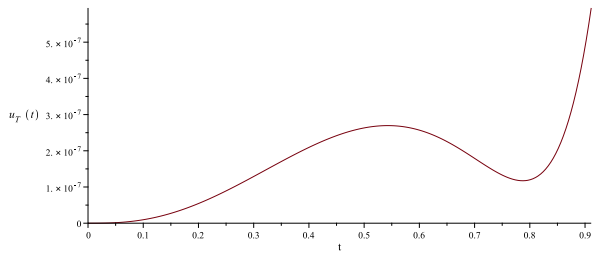

Figure 3: The control variable $u_T(t)$.

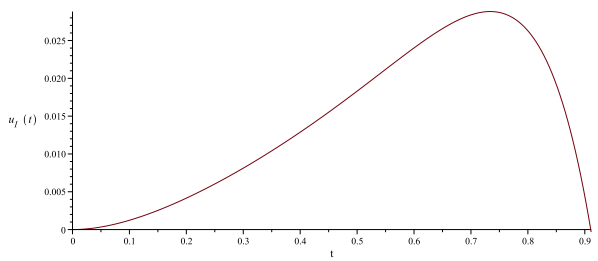

Figure 4: The control variable $u_I(t)$.

4. Conclusions and Discussions

In this paper, we obtained numerical solutions of a model for HIV infection of CD4$^+$T cells by presented method. We plotted the solutions of $T(t)$ and $I(t)$ for $0 \leq t \leq 0.9$, as shown in Figs. 1-2. Figure 1 shows that applying optimal control the number of concentration of healthy CD4$^+$T cells (T(t)) increases gradually. In Figure 2, after introducing control variable $u_I(t)$, the density of the concentration of infected CD4$^+$T cells (I(t)) declines towards zero.

The obtained numerical solutions are in very good coincidence with the other numerical solutions [7-16]. In Tables 1-2, the obtained values of the approximate solutions of a model for HIV infection of CD4$^+$T cells at several values of t are compared with those of Laplace Adomian decomposition method with Pade approximation [6] and Bessel collocation method [8]. In Table 1, it found that the obtained solutions by using of our method are in a good agreement with the approximate solutions in [6] and [8] at $0 \leq t \leq 0.6$. Also, it can be concluded that our present results at $0.6 \leq t \leq 0.9$, even better than the results obtained by the Laplace Adomian decomposition method with Pade approximation [6] and the results obtained by the Bessel collocation method [8]. The results in Table 2 show that, in [6] and [8] the density of infected cells $I(t)$ increase at $0 \leq t \leq 0.9$. But in our present method, the intensity of infected cells $I(t)$

decrease with the passage of time after applying control variable $u_I(t)$. Moreover, in Figs. 3-4, we see that control variables $u_T(t)$ and $u_I(t)$ are in agreement with $0 \leq u_T(t), u_I(t) \leq 1$. It can be concluded that our proposed method is powerful mathematical tool for solving a wide variety of other epidemic models.

Table 1: Numerical comparison for $T(t)$.

t	$LADM - Pade$	*Bessel coll.*	*Present method*
0	0.1	0.1	0.1
0.2	0.2088072731	0.2038616561	0.2087367817
0.4	0.4061052625	0.3803309335	0.4059521352
0.6	0.7611467713	0.6954623767	0.7635498323
0.9	1.5245154522	1.4521254658	1.977214845

Table 2: Numerical comparison for $I(t)$.

t	$LADM - Pade$	*Bessel coll.*	*Present method*
0	0	0	0
0.2	0.0000060327	0.0000062478	0.02772720781
0.4	0.0000131591	0.0000129355	0.08586776364
0.6	0.0000212683	0.0000203526	0.16088182680
0.9	0.0000385462	0.0000325452	0.00000600060

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