



Multifractal Dimensions for Projections of Measures

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ABSTRACT: In this paper, we study the multifractal Hausdorff and packing dimensions of Borel probability measures and study their behaviors under orthogonal projections. In particular, we try through these results to improve the main result of M. Dai in [13] about the multifractal analysis of a measure of multifractal exact dimension.

Key Words: Multifractal analysis, Dimensions of measures, Projection.

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1. Introduction

The notion of dimensions is an important tool in the classification of subsets in \mathbb{R}^n . The Hausdorff and packing dimensions appear as some of the most common examples in the literature. The determination of set's dimensions is naturally connected to the auxiliary Borel measures supported by these sets. Moreover, the estimation of a set's dimension is naturally related to the dimension of a probability measure ν in \mathbb{R}^n . In this way, thinking particularly to sets of measure zero or one, leads to the respective definitions of the lower and upper Hausdorff dimensions of ν as follows

$$\underline{\dim}(\nu) = \inf \left\{ \dim(E); E \subseteq \mathbb{R}^n \text{ and } \nu(E) > 0 \right\}$$

and

$$\overline{\dim}(\nu) = \inf \left\{ \dim(E); E \subseteq \mathbb{R}^n \text{ and } \nu(E) = 1 \right\},$$

where $\dim(E)$ denotes the Hausdorff dimension of E (see [16]). If $\underline{\dim}(\nu) = \overline{\dim}(\nu)$, this common value is denoted by $\dim(\nu)$. In this case, we say that ν is unidimensional. Similarly, we define respectively the lower and upper packing dimensions of ν by

$$\underline{\text{Dim}}(\nu) = \inf \left\{ \text{Dim}(E); E \subseteq \mathbb{R}^n \text{ and } \nu(E) > 0 \right\}$$

and

$$\overline{\text{Dim}}(\nu) = \inf \left\{ \text{Dim}(E); E \subseteq \mathbb{R}^n \text{ and } \nu(E) = 1 \right\},$$

where $\text{Dim}(E)$ is the packing dimension of E (see [16]). Also, if the equality $\underline{\text{Dim}}(\nu) = \overline{\text{Dim}}(\nu)$ is satisfied, we denote by $\text{Dim}(\nu)$ their common value.

The lower and upper Hausdorff dimensions of ν were studied by H. Fan in [19,20]. They are related to the Hausdorff dimension of the support of ν . A similar approach, concerning the packing dimensions, was developed by Tamashiro in [45]. There are numerous works in which estimates of the dimension of a given measure are obtained [2,7,13,16,21,24,25,26,27,41]. When $\overline{\dim}(\nu)$ is small (resp. $\underline{\dim}(\nu)$ is large),

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it means that ν is *singular* (resp. *regular*) with respect to the Hausdorff measure. Similar definitions are used when concerned with the upper and lower packing dimensions.

Note that, in many works (see for example [16,24,25,26]), the quantities $\underline{\dim}(\nu)$, $\overline{\dim}(\nu)$, $\underline{\text{Dim}}(\nu)$ and $\overline{\text{Dim}}(\nu)$ are related to the asymptotic behavior of the function $\alpha_\nu(x, r) = \frac{\log \nu(B(x, r))}{\log r}$.

One of the main problems in multifractal analysis is to understand the multifractal spectrum, the Rényi dimensions and their relationship with each other. During the past 20 years, there has been enormous interest in computing the multifractal spectra of measures in the mathematical literature and within the last 15 years the multifractal spectra of various classes of measures in Euclidean space \mathbb{R}^n exhibiting some degree of self-similarity have been computed rigorously (see [16,33,38] and the references therein). In an attempt to develop a general theoretical framework for studying the multifractal structure of arbitrary measures, Olsen [33] and Pesin [37] suggested various ways of defining an auxiliary measure in very general settings. For more details and backgrounds on multifractal analysis and its applications, the readers may be referred also to the following essential references [1,2,8,5,10,11,12,14,15,22,30,31,33,34,35,36,39,42,43,46,47,48,49].

In this paper, we give a multifractal generalization of the results about Hausdorff and packing dimension of measures. We first estimate the multifractal Hausdorff and packing dimensions of a Borel probability measure. We try through these results to improve the main result of M. Dai in [13, Theorem A] about the multifractal analysis of a measure of exact multifractal dimension. We are especially based on the multifractal formalism developed by Olsen in [33]. Then, we investigate a relationship between the multifractal dimensions of a measure ν and its projections onto a lower dimensional linear subspace.

2. Preliminaries

We start by recalling the multifractal formalism introduced by Olsen in [33]. This formalism was motivated by Olsen's wish to provide a general mathematical setting for the ideas present in the physics literature on multifractals.

Let $E \subset \mathbb{R}^n$ and $\delta > 0$, we say that a collection of balls $(B(x_i, r_i))_i$ is a centered δ -packing of E if

$$\forall i, 0 < r_i < \delta, \quad x_i \in E, \quad \text{and} \quad B(x_i, r_i) \cap B(x_j, r_j) = \emptyset, \quad \forall i \neq j.$$

Similarly, we say that $(B(x_i, r_i))_i$ is a centered δ -covering of E if

$$\forall i, 0 < r_i < \delta, \quad x_i \in E, \quad \text{and} \quad E \subset \bigcup_i B(x_i, r_i).$$

Let μ be a Borel probability measure on \mathbb{R}^n . For $q, t \in \mathbb{R}$, $E \subseteq \mathbb{R}^n$ and $\delta > 0$, we define

$$\overline{\mathcal{P}}_{\mu, \delta}^{q, t}(E) = \sup \left\{ \sum_i \mu(B(x_i, r_i))^q (2r_i)^t \right\},$$

where the supremum is taken over all centered δ -packings of E . The generalized packing pre-measure is given by

$$\overline{\mathcal{P}}_\mu^{q, t}(E) = \inf_{\delta > 0} \overline{\mathcal{P}}_{\mu, \delta}^{q, t}(E).$$

In a similar way, we define

$$\overline{\mathcal{H}}_{\mu, \delta}^{q, t}(E) = \inf \left\{ \sum_i \mu(B(x_i, r_i))^q (2r_i)^t \right\},$$

where the infimum is taken over all centered δ -coverings of E . The generalized Hausdorff pre-measure is defined by

$$\overline{\mathcal{H}}_\mu^{q, t}(E) = \sup_{\delta > 0} \overline{\mathcal{H}}_{\mu, \delta}^{q, t}(E).$$

Especially, we have the conventions $0^q = \infty$ for $q \leq 0$ and $0^q = 0$ for $q > 0$.

Olsen [33] introduced the following modifications on the generalized Hausdorff and packing measures,

$$\mathcal{H}_\mu^{q,t}(E) = \sup_{F \subseteq E} \overline{\mathcal{H}}_\mu^{q,t}(F) \quad \text{and} \quad \mathcal{P}_\mu^{q,t}(E) = \inf_{E \subseteq \bigcup_i E_i} \sum_i \overline{\mathcal{P}}_\mu^{q,t}(E_i).$$

The functions $\mathcal{H}_\mu^{q,t}$ and $\mathcal{P}_\mu^{q,t}$ are metric outer measures and thus measures on the family of Borel subsets of \mathbb{R}^n . An important feature of the Hausdorff and packing measures is that $\mathcal{P}_\mu^{q,t} \leq \overline{\mathcal{P}}_\mu^{q,t}$. Moreover, there exists an integer $\xi \in \mathbb{N}$, such that $\mathcal{H}_\mu^{q,t} \leq \xi \mathcal{P}_\mu^{q,t}$. The measure $\mathcal{H}_\mu^{q,t}$ is of course a multifractal generalization of the centered Hausdorff measure, whereas $\mathcal{P}_\mu^{q,t}$ is a multifractal generalization of the packing measure. In fact, it is easily seen that, for $t \geq 0$, one has

$$2^{-t} \mathcal{H}_\mu^{0,t} \leq \mathcal{H}^t \leq \mathcal{H}_\mu^{0,t} \quad \text{and} \quad \mathcal{P}_\mu^{0,t} = \mathcal{P}^t,$$

where \mathcal{H}^t and \mathcal{P}^t denote respectively the t -dimensional Hausdorff and t -dimensional packing measures.

The measures $\mathcal{H}_\mu^{q,t}$ and $\mathcal{P}_\mu^{q,t}$ and the pre-measure $\overline{\mathcal{P}}_\mu^{q,t}$ assign in a usual way a multifractal dimension to each subset E of \mathbb{R}^n . They are respectively denoted by $\dim_\mu^q(E)$, $\text{Dim}_\mu^q(E)$ and $\Delta_\mu^q(E)$ (see [33]) and satisfy

$$\begin{aligned} \dim_\mu^q(E) &= \inf \left\{ t \in \mathbb{R}; \quad \mathcal{H}_\mu^{q,t}(E) = 0 \right\} = \sup \left\{ t \in \mathbb{R}; \quad \mathcal{H}_\mu^{q,t}(E) = +\infty \right\}, \\ \text{Dim}_\mu^q(E) &= \inf \left\{ t \in \mathbb{R}; \quad \mathcal{P}_\mu^{q,t}(E) = 0 \right\} = \sup \left\{ t \in \mathbb{R}; \quad \mathcal{P}_\mu^{q,t}(E) = +\infty \right\}, \\ \Delta_\mu^q(E) &= \inf \left\{ t \in \mathbb{R}; \quad \overline{\mathcal{P}}_\mu^{q,t}(E) = 0 \right\} = \sup \left\{ t \in \mathbb{R}; \quad \overline{\mathcal{P}}_\mu^{q,t}(E) = +\infty \right\}. \end{aligned}$$

The number $\dim_\mu^q(E)$ is an obvious multifractal analogue of the Hausdorff dimension $\dim(E)$ of E whereas $\text{Dim}_\mu^q(E)$ and $\Delta_\mu^q(E)$ are obvious multifractal analogues of the packing dimension $\text{Dim}(E)$ and the pre-packing dimension $\Delta(E)$ of E respectively. In fact, it follows immediately from the definitions that

$$\dim(E) = \dim_\mu^0(E), \quad \text{Dim}(E) = \text{Dim}_\mu^0(E) \quad \text{and} \quad \Delta(E) = \Delta_\mu^0(E).$$

3. Multifractal Hausdorff and packing dimensions of measures

Now, we introduce the multifractal analogues of the Hausdorff and packing dimensions of a Borel probability measure.

Definition 3.1. *The lower and upper multifractal Hausdorff dimensions of a measure ν with respect to a measure μ are defined by*

$$\underline{\dim}_\mu^q(\nu) = \inf \left\{ \dim_\mu^q(E); \quad E \subseteq \mathbb{R}^n \text{ and } \nu(E) > 0 \right\}$$

and

$$\overline{\dim}_\mu^q(\nu) = \inf \left\{ \dim_\mu^q(E); \quad E \subseteq \mathbb{R}^n \text{ and } \nu(E) = 1 \right\}.$$

We denote by $\dim_\mu^q(\nu)$ their common value, if the equality $\underline{\dim}_\mu^q(\nu) = \overline{\dim}_\mu^q(\nu)$ is satisfied.

Definition 3.2. *The lower and upper multifractal packing dimensions of a measure ν with respect to a measure μ are defined by*

$$\underline{\text{Dim}}_\mu^q(\nu) = \inf \left\{ \text{Dim}_\mu^q(E); \quad E \subseteq \mathbb{R}^n \text{ and } \nu(E) > 0 \right\}$$

and

$$\overline{\text{Dim}}_\mu^q(\nu) = \inf \left\{ \text{Dim}_\mu^q(E); \quad E \subseteq \mathbb{R}^n \text{ and } \nu(E) = 1 \right\}.$$

When $\underline{\text{Dim}}_\mu^q(\nu) = \overline{\text{Dim}}_\mu^q(\nu)$, we denote by $\text{Dim}_\mu^q(\nu)$ their common value.

Definition 3.3. Let μ, ν be two Borel probability measures on \mathbb{R}^n .

1. We say that μ is absolutely continuous with respect to ν and write $\mu \ll \nu$ if, for any set $A \subset \mathbb{R}^n$, $\nu(A) = 0 \Rightarrow \mu(A) = 0$.
2. μ and ν are said to be mutually singular and we write $\mu \perp \nu$ if there exists a set $A \subset \mathbb{R}^n$, such that $\mu(A) = 0 = \nu(\mathbb{R}^n \setminus A)$.

The quantities $\underline{\dim}_\mu^q(\nu)$ and $\overline{\dim}_\mu^q(\nu)$ (resp. $\underline{\text{Dim}}_\mu^q(\nu)$ and $\overline{\text{Dim}}_\mu^q(\nu)$) allow to compare the measure ν with the generalized Hausdorff (resp. packing) measure. More precisely, we have the following result.

Theorem 3.4. Let μ, ν be two Borel probability measures on \mathbb{R}^n and $q \in \mathbb{R}$. We have,

1. $\underline{\dim}_\mu^q(\nu) = \sup \left\{ t \in \mathbb{R}; \nu \ll \mathcal{H}_\mu^{q,t} \right\}$ and $\overline{\dim}_\mu^q(\nu) = \inf \left\{ t \in \mathbb{R}; \nu \perp \mathcal{H}_\mu^{q,t} \right\}$.
2. $\underline{\text{Dim}}_\mu^q(\nu) = \sup \left\{ t \in \mathbb{R}; \nu \ll \mathcal{P}_\mu^{q,t} \right\}$ and $\overline{\text{Dim}}_\mu^q(\nu) = \inf \left\{ t \in \mathbb{R}; \nu \perp \mathcal{P}_\mu^{q,t} \right\}$.

Proof. 1) Let's prove that $\underline{\dim}_\mu^q(\nu) = \sup \left\{ t \in \mathbb{R}; \nu \ll \mathcal{H}_\mu^{q,t} \right\}$. Define

$$s = \sup \left\{ t \in \mathbb{R}; \nu \ll \mathcal{H}_\mu^{q,t} \right\}.$$

For any $t < s$ and $E \subseteq \mathbb{R}^n$, such that $\nu(E) > 0$, we have $\mathcal{H}_\mu^{q,t}(E) > 0$. It follows that $\dim_\mu^q(E) \geq t$ and then, $\underline{\dim}_\mu^q(\nu) \geq t$. We deduce that $\underline{\dim}_\mu^q(\nu) \geq s$.

On the other hand, for any $t > s$, there exists a set $E \subseteq \mathbb{R}^n$, such that $\nu(E) > 0$ and $\mathcal{H}_\mu^{q,t}(E) = 0$. Consequently, $\dim_\mu^q(E) \leq t$ and so, $\underline{\dim}_\mu^q(\nu) \leq t$. This leads to $\underline{\dim}_\mu^q(\nu) \leq s$.

Now, we prove that $\overline{\dim}_\mu^q(\nu) = \inf \left\{ t \in \mathbb{R}; \nu \perp \mathcal{H}_\mu^{q,t} \right\}$. For this, we define

$$s' = \inf \left\{ t \in \mathbb{R}; \nu \perp \mathcal{H}_\mu^{q,t} \right\}.$$

For $t > s'$, there exists a set $E \subseteq \mathbb{R}^n$, such that $\mathcal{H}_\mu^{q,t}(E) = 0 = \nu(\mathbb{R}^n \setminus E)$. Then, $\dim_\mu^q(E) \leq t$. Since $\nu(E) = 1$, then $\underline{\dim}_\mu^q(\nu) \leq t$ and $\underline{\dim}_\mu^q(\nu) \leq s'$.

Now, for $t < s'$, take $E \subseteq \mathbb{R}^n$, such that $\mathcal{H}_\mu^{q,t}(E) > 0$ and $\nu(E) = 1$. It can immediately be seen that $\dim_\mu^q(E) \geq t$. Then, $\underline{\dim}_\mu^q(\nu) \geq t$. It follows that $\underline{\dim}_\mu^q(\nu) \geq s'$. This ends the proof of assertion (1).

2) The proof of assertion (2) is given in [27, Theorem 2]. \square

Remark 3.5. When the upper multifractal Hausdorff (resp. packing) dimension of the measure is small, it means that the measure ν is “very singular” with respect to the generalized multifractal Hausdorff (resp. packing) measure. In the same way, when the lower multifractal (resp. packing) dimension of the measure is large, then the measure ν is “quite regular” with respect to the generalized multifractal Hausdorff (resp. packing) measure.

The quantities $\underline{\dim}_\mu^q(\nu)$, $\overline{\dim}_\mu^q(\nu)$, $\underline{\text{Dim}}_\mu^q(\nu)$ and $\overline{\text{Dim}}_\mu^q(\nu)$ are related to the asymptotic behavior of the function $\alpha_{\mu,\nu}^q(x, r)$, where

$$\alpha_{\mu,\nu}^q(x, r) = \frac{\log \nu(B(x, r)) - q \log \mu(B(x, r))}{\log r}.$$

Notice that the characterization of the lower and upper packing dimensions by the function $\alpha_{\mu,\nu}^q$ is proved by J. Li in [27, Theorem 3]. In the following theorem we prove similar results for the Hausdorff dimensions.

Theorem 3.6. *Let μ, ν be two Borel probability measures on \mathbb{R}^n and $q \in \mathbb{R}$. Let*

$$\underline{\alpha}_{\mu, \nu}^q(x) = \liminf_{r \rightarrow 0} \alpha_{\mu, \nu}^q(x, r) \quad \text{and} \quad \overline{\alpha}_{\mu, \nu}^q(x) = \limsup_{r \rightarrow 0} \alpha_{\mu, \nu}^q(x, r).$$

We have,

$$1. \underline{\dim}_{\mu}^q(\nu) = \text{ess inf } \underline{\alpha}_{\mu, \nu}^q(x) \quad \text{and} \quad \overline{\dim}_{\mu}^q(\nu) = \text{ess sup } \underline{\alpha}_{\mu, \nu}^q(x).$$

$$2. \underline{\text{Dim}}_{\mu}^q(\nu) = \text{ess inf } \overline{\alpha}_{\mu, \nu}^q(x) \quad \text{and} \quad \overline{\text{Dim}}_{\mu}^q(\nu) = \text{ess sup } \overline{\alpha}_{\mu, \nu}^q(x),$$

where the essential bounds being related to the measure ν .

Proof. We prove that $\underline{\dim}_{\mu}^q(\nu) = \text{ess inf } \underline{\alpha}_{\mu, \nu}^q(x)$.

Let $\alpha < \text{ess inf } \underline{\alpha}_{\mu, \nu}^q(x)$. For ν -almost every x , there exists $r_0 > 0$, such that $0 < r < r_0$ and

$$\nu(B(x, r)) < \mu(B(x, r))^q r^{\alpha}.$$

Denote by

$$F_n = \left\{ x; \nu(B(x, r)) < \mu(B(x, r))^q r^{\alpha}, \text{ for } 0 < r < \frac{1}{n} \right\}.$$

Let $F = \cup_n F_n$. It is clear that $\nu(F) = 1$. Take E be a Borel subset of \mathbb{R}^n satisfying $\nu(E) > 0$. We have $\nu(E \cap F) > 0$ and there exists an integer n , such that $\nu(E \cap F_n) > 0$.

Let $\delta > 0$ and $(B(x_i, r_i))_i$ be a centered δ -covering of $E \cap F_n$. We have

$$\sum_i \nu(B(x_i, r_i)) \leq 2^{-\alpha} \sum_i \mu(B(x_i, r_i))^q (2r_i)^{\alpha},$$

so that

$$2^{\alpha} \nu(E \cap F_n) \leq \overline{\mathcal{H}}_{\mu, \delta}^{q, \alpha}(E \cap F_n).$$

Letting $\delta \rightarrow 0$ gives that

$$2^{\alpha} \nu(E \cap F_n) \leq \overline{\mathcal{H}}_{\mu}^{q, \alpha}(E \cap F_n) \leq \mathcal{H}_{\mu}^{q, \alpha}(E \cap F_n).$$

It follows that

$$\mathcal{H}_{\mu}^{q, \alpha}(E) \geq \mathcal{H}_{\mu}^{q, \alpha}(E \cap F_n) > 0 \Rightarrow \dim_{\mu}^q(E) \geq \alpha.$$

We have proved that

$$\underline{\dim}_{\mu}^q(\nu) \geq \text{ess inf } \underline{\alpha}_{\mu, \nu}^q(x).$$

On the other hand, if $\text{ess inf } \underline{\alpha}_{\mu, \nu}^q(x) = \alpha$. For $\varepsilon > 0$, let

$$E_{\varepsilon} = \left\{ x \in \text{supp } \nu; \underline{\alpha}_{\mu, \nu}^q(x) < \alpha + \varepsilon \right\}.$$

It is clear that $\nu(E_{\varepsilon}) > 0$. This means that $\underline{\dim}_{\mu}^q(\nu) \leq \dim_{\mu}^q(E_{\varepsilon})$. We will prove that

$$\dim_{\mu}^q(E_{\varepsilon}) \leq \alpha + \varepsilon, \quad \forall \varepsilon > 0.$$

Let $E \subset E_{\varepsilon}$ and $x \in E$. Then, for all $\delta > 0$ we can find $0 < r_x < \delta$, such that

$$\nu(B(x, r_x)) > \mu(B(x, r_x))^q r_x^{\alpha + \varepsilon}.$$

Take $\delta > 0$. The family $\left(B(x, r_x)\right)_{x \in E}$ is a centered δ -covering of E . Using Besicovitch's Covering Theorem (see [16,28]), we can construct ξ finite or countable sub-families $\left(B(x_{1j}, r_{1j})\right)_j, \dots, \left(B(x_{\xi j}, r_{\xi j})\right)_j$, such that each E satisfies

$$E \subseteq \bigcup_{i=1}^{\xi} \bigcup_j B(x_{ij}, r_{ij}) \quad \text{and} \quad \left(B(x_{ij}, r_{ij})\right)_j \text{ is a } \delta\text{-packing of } E.$$

We get

$$\sum_{i,j} \mu(B(x_{ij}, r_{ij}))^q (2r_{ij})^{\alpha+\varepsilon} \leq \xi 2^{\alpha+\varepsilon} \sum_j \nu(B(x_{ij}, r_{ij})) \leq \xi 2^{\alpha+\varepsilon} \nu(\mathbb{R}^n).$$

Consequently,

$$\overline{\mathcal{H}}_{\mu, \delta}^{q, \alpha+\varepsilon}(E) \leq \xi 2^{\alpha+\varepsilon} \nu(\mathbb{R}^n) \Rightarrow \overline{\mathcal{H}}_{\mu}^{q, \alpha+\varepsilon}(E) \leq \xi 2^{\alpha+\varepsilon} \nu(\mathbb{R}^n).$$

We obtain thus

$$\mathcal{H}_{\mu}^{q, \alpha+\varepsilon}(E_{\varepsilon}) \leq \xi 2^{\alpha+\varepsilon} \nu(\mathbb{R}^n) < \infty.$$

Therefore,

$$\dim_{\mu}^q(E) \leq \alpha + \varepsilon \quad \text{and} \quad \underline{\dim}_{\mu}^q(\nu) \leq \text{ess inf } \underline{\alpha}_{\mu, \nu}^q(x).$$

We prove in a similar way that $\overline{\dim}_{\mu}^q(\nu) = \text{ess sup } \underline{\alpha}_{\mu, \nu}^q(x)$. □

Corollary 3.7. *Let μ, ν be two Borel probability measures on \mathbb{R}^n and take $q, \alpha \in \mathbb{R}$. We have,*

1. $\underline{\dim}_{\mu}^q(\nu) \geq \alpha$ if and only if $\underline{\alpha}_{\mu, \nu}^q(x) \geq \alpha$ for ν -a.e. x .
2. $\overline{\dim}_{\mu}^q(\nu) \leq \alpha$ if and only if $\underline{\alpha}_{\mu, \nu}^q(x) \leq \alpha$ for ν -a.e. x .
3. $\underline{\text{Dim}}_{\mu}^q(\nu) \geq \alpha$ if and only if $\overline{\alpha}_{\mu, \nu}^q(x) \geq \alpha$ for ν -a.e. x .
4. $\overline{\text{Dim}}_{\mu}^q(\nu) \leq \alpha$ if and only if $\overline{\alpha}_{\mu, \nu}^q(x) \leq \alpha$ for ν -a.e. x .

Proof. Follows immediately from Theorem 3.6. □

Example 3.8.

We recall the definition of the deranged Cantor set (see [3,4,6,5]). Let $I_{\emptyset} = [0, 1]$. We obtain respectively the left and right sub-intervals $I_{\varepsilon,1}$ and $I_{\varepsilon,2}$ of I_{ε} by deleting the middle open sub-interval of I_{ε} inductively for each $\varepsilon \in \{1, 2\}^n$, where $n \in \mathbb{N}$. We consider the sequence

$$\mathcal{C}_n = \bigcup_{\varepsilon \in \{1,2\}^n} I_{\varepsilon}.$$

$\{\mathcal{C}_n\}_{n \in \mathbb{N}}$ is a decreasing sequence of closed sets.

For each $n \in \mathbb{N}$ and each $\varepsilon \in \{1, 2\}^n$, we put

$$|I_{\varepsilon,1}|/|I_{\varepsilon}| = c_{\varepsilon,1} \quad \text{and} \quad |I_{\varepsilon,2}|/|I_{\varepsilon}| = c_{\varepsilon,2},$$

where $|I|$ is the diameter of I . The set $\mathcal{C} = \bigcap_{n \geq 0} \mathcal{C}_n$ is called a deranged Cantor set.

Let ν be a probability measure supported by the deranged Cantor set \mathcal{C} and μ be the Lebesgue measure on I_\emptyset . For $\varepsilon_1, \dots, \varepsilon_n \in \{1, 2\}$, we denote by $I_{\varepsilon_1, \dots, \varepsilon_n}$ the basic set of level n . For $x \in \mathcal{C}$, we denote by $I_n(x)$ the n -th level set containing x . We introduce the sequence of random variables X_n defined by

$$X_n(x) = -\log_3 \left(\frac{\nu(I_n(x))}{\nu(I_{n-1}(x))} \right).$$

We have

$$\frac{S_n(x)}{n} = \frac{X_1(x) + \dots + X_n(x)}{n} = \frac{\log(\nu(I_n(x)))}{\log |I_n(x)|}.$$

By Lemma 1 in [5], we have for all $x \in \mathcal{C}$,

$$\liminf_{n \rightarrow \infty} \frac{\log(\nu(I_n(x)))}{\log |I_n(x)|} = \liminf_{r \rightarrow 0} \frac{\log(\nu(B(x, r)))}{\log r}$$

and

$$\limsup_{n \rightarrow \infty} \frac{\log(\nu(I_n(x)))}{\log |I_n(x)|} = \limsup_{r \rightarrow 0} \frac{\log(\nu(B(x, r)))}{\log r}.$$

The quantities $\underline{\dim}_\mu^q(\nu)$ and $\overline{\dim}_\mu^q(\nu)$ are related to the asymptotic behavior of the sequence $\frac{S_n}{n}$. More precisely, we have the following two relations

$$\underline{\dim}_\mu^q(\nu) = \text{ess inf} \left\{ \liminf_{n \rightarrow \infty} \frac{S_n(x)}{n} - q \right\}$$

and

$$\overline{\dim}_\mu^q(\nu) = \text{ess sup} \left\{ \liminf_{n \rightarrow \infty} \frac{S_n(x)}{n} - q \right\}.$$

In the same way, we can also prove that

$$\underline{\text{Dim}}_\mu^q(\nu) = \text{ess inf} \left\{ \limsup_{n \rightarrow \infty} \frac{S_n(x)}{n} - q \right\}$$

and

$$\overline{\text{Dim}}_\mu^q(\nu) = \text{ess sup} \left\{ \limsup_{n \rightarrow \infty} \frac{S_n(x)}{n} - q \right\}.$$

We say that the measure ν is (q, μ) -unidimensional if $\overline{\dim}_\mu^q(\nu) = \underline{\dim}_\mu^q(\nu)$. We also say that ν has an exact multifractal packing dimension whenever $\overline{\text{Dim}}_\mu^q(\nu) = \underline{\text{Dim}}_\mu^q(\nu)$. In general, a Borel probability measure is not (q, μ) -unidimensional and $\overline{\text{Dim}}_\mu^q(\nu) \neq \underline{\text{Dim}}_\mu^q(\nu)$.

In the following, we are interested to the (q, μ) -unidimensionality and ergodicity of ν and to the calculus of its multifractal Hausdorff and packing dimensions. Our purpose in the following theorem is to prove the main Theorem of M. Dai [13, Theorem A] under less restrictive hypotheses.

Theorem 3.9. *The measure ν is (q, μ) -unidimensional with $\dim_\mu^q(\nu) = \alpha$ if and only if the following two conditions are satisfied.*

1. *There exists a set E of \mathbb{R}^n with $\dim_\mu^q(E) = \alpha$, such that $\nu(E) = 1$.*
2. *$\nu(E) = 0$, for every Borel set E satisfying $\dim_\mu^q(E) < \alpha$.*

Proof. We can deduce from Theorems 3.4 and 3.6 that ν is (q, μ) -unidimensional if and only if we have the following assertions.

1. ν is absolutely continuous with respect to $\mathcal{H}_\mu^{q, \alpha - \varepsilon}$, for all $\varepsilon > 0$.
2. ν and $\mathcal{H}_\mu^{q, \alpha + \varepsilon}$ are mutually singular, for all $\varepsilon > 0$.

Then, the proof of Theorem 3.9 becomes an easy consequence of the following lemma.

Lemma 3.10. [13] *The following conditions are equivalent.*

1. We have,
 - (a) there exists a set E of \mathbb{R}^n with $\dim_\mu^q(E) = \alpha$, such that $\nu(E) = 1$.
 - (b) $\nu(E) = 0$, for every Borel set E satisfying $\dim_\mu^q(E) < \alpha$.
2. We have,
 - (a) $\nu \ll \mathcal{H}_\mu^{q, \alpha - \varepsilon}$ for all $\varepsilon > 0$.
 - (b) $\nu \perp \mathcal{H}_\mu^{q, \alpha + \varepsilon}$ for all $\varepsilon > 0$.

□

Remark 3.11. *Theorem 3.9 improves Dai's result [13, Theorem A] (we need not to assume that μ is a doubling measure).*

The symmetrical results are true as well.

Theorem 3.12. *Let μ, ν be two Borel probability measures on \mathbb{R}^n and take $\alpha, q \in \mathbb{R}$. The following conditions are equivalent.*

1. $\overline{\text{Dim}}_\mu^q(\nu) = \underline{\text{Dim}}_\mu^q(\nu) = \alpha$.
2. We have,
 - (a) there exist a set $E \subset \mathbb{R}^n$ with $\text{Dim}_\mu^q(E) = \alpha$, such that $\nu(E) = 1$,
 - (b) if $E \subset \mathbb{R}^n$ satisfies $\text{Dim}_\mu^q(E) < \alpha$, then $\nu(E) = 0$.
3. We have,
 - (a) $\nu \ll \mathcal{P}_\mu^{q, \alpha - \varepsilon}$, for all $\varepsilon > 0$.
 - (b) $\nu \perp \mathcal{P}_\mu^{q, \alpha + \varepsilon}$, for all $\varepsilon > 0$.

Proof. We can deduce from Theorems 3.4 and 3.6 that the assertions (1) and (3) are equivalent. We only need to prove the equivalence of the assertions (2) and (3).

Assume that the measure ν satisfies the hypothesis (a) and (b) of (2). Let $E \subset \mathbb{R}^n$ and suppose that $\mathcal{P}_\mu^{q, \alpha - \varepsilon}(E) = 0$, for all $\varepsilon > 0$. Then, we have that $\text{Dim}_\mu^q(E) \leq \alpha - \varepsilon < \alpha$. By condition (b) of (2), we obtain $\nu(E) = 0$. Thus,

$$\nu \ll \mathcal{P}_\mu^{q, \alpha - \varepsilon}, \quad \text{for all } \varepsilon > 0.$$

Thanks to condition (a) of (2), there exists a set $E \subset \mathbb{R}^n$ of multifractal packing dimension α , such that $\nu(E) = 1$ and $\text{Dim}_\mu^q(E) = \alpha < \alpha + \varepsilon$, for all $\varepsilon > 0$. Then, $\mathcal{P}_\mu^{q, \alpha + \varepsilon}(E) = 0$. Thus,

$$\nu \perp \mathcal{P}_\mu^{q, \alpha + \varepsilon}, \quad \text{for all } \varepsilon > 0.$$

Now, assume that ν satisfies conditions (a) and (b) of (3). This means that $\nu \ll \mathcal{P}_\mu^{q, \alpha - \varepsilon}$, for all $\varepsilon > 0$. Taking a Borel set E with $\text{Dim}_\mu^q(E) = \beta < \alpha$ and $\varepsilon = \frac{\alpha - \beta}{2}$, we get $\mathcal{P}_\mu^{q, (\alpha + \beta)/2}(E) = 0$. Then, $\nu(E) = 0$.

Since $\nu \perp \mathcal{P}_\mu^{q, \alpha + \epsilon}$, for all $\epsilon > 0$, there exists a set F_ϵ with $\mathcal{P}_\mu^{q, \alpha + \epsilon}(F_\epsilon) = 0$ and $\nu(F_\epsilon) = 1$. Hence, $\text{Dim}_\mu^q(F_\epsilon) \leq \alpha + \epsilon$. Choose a sequence $(\epsilon_k)_k$ such that $\epsilon_k \rightarrow 0$ as $k \rightarrow +\infty$ and consider the set $F = \bigcap_{k \geq 1} F_{\epsilon_k}$. It is clear that $\nu(F) = 1$ and

$$\text{Dim}_\mu^q(F) \leq \liminf_{k \rightarrow \infty} \text{Dim}_\mu^q(F_{\epsilon_k}) \leq \liminf_{k \rightarrow \infty} (\alpha + \epsilon_k) = \alpha.$$

If $\text{Dim}_\mu^q(F) = \alpha$, then the condition (a) of (2) is satisfied for $E = F$.

If $\text{Dim}_\mu^q(F) < \alpha$, then putting $E = F \cup G$, for some Borel set G of multifractal packing dimension α , we obtain

$$\nu(E) = 1 \quad \text{and} \quad \text{Dim}_\mu^q(E) = \max \left\{ \text{Dim}_\mu^q(F), \text{Dim}_\mu^q(G) \right\} = \alpha.$$

□

Proposition 3.13. *Let μ be the Lebesgue measure on \mathbb{R}^n , ν be a compactly supported Borel probability measure on \mathbb{R}^n and $T : \text{supp } \nu \rightarrow \text{supp } \nu$ a K -lipschitz function. Suppose that ν is T -invariant and ergodic on $\text{supp } \nu$. Then,*

$$\overline{\text{dim}}_\mu^q(\nu) = \underline{\text{dim}}_\mu^q(\nu) \quad \text{and} \quad \overline{\text{Dim}}_\mu^q(\nu) = \underline{\text{Dim}}_\mu^q(\nu).$$

Proof. T is a K -lipschitz function, then $T(B(x, r)) \subseteq B(T(x), Kr)$. Since ν is T -invariant, then we can deduce that

$$\nu(B(x, r)) \leq \nu(T^{-1}(T(B(x, r)))) \leq \nu(T^{-1}(B(T(x), Kr))) = \nu(B(T(x), Kr)).$$

It follows that,

$$\begin{aligned} \frac{\log \nu(B(x, r)) - q \log \mu(B(x, r))}{\log r} &= \frac{\log \nu(B(x, r))}{\log r} - q \\ &\geq \frac{\log \nu(B(T(x), Kr))}{\log(Kr)} \times \frac{\log(Kr)}{\log r} - q, \end{aligned}$$

which proves that

$$\underline{\alpha}_{\mu, \nu}^q(x) \geq \underline{\alpha}_{\mu, \nu}^q(T(x)) \quad \text{and} \quad \overline{\alpha}_{\mu, \nu}^q(x) \geq \overline{\alpha}_{\mu, \nu}^q(T(x)).$$

Since ν is ergodic, then the functions

$$\underline{\alpha}_{\mu, \nu}^q(x) - \underline{\alpha}_{\mu, \nu}^q(T(x))$$

and

$$\overline{\alpha}_{\mu, \nu}^q(x) - \overline{\alpha}_{\mu, \nu}^q(T(x))$$

are positive and satisfies

$$\int \left(\underline{\alpha}_{\mu, \nu}^q(x) - \underline{\alpha}_{\mu, \nu}^q(T(x)) \right) d\nu(x) = 0 \quad \text{and} \quad \int \left(\overline{\alpha}_{\mu, \nu}^q(x) - \overline{\alpha}_{\mu, \nu}^q(T(x)) \right) d\nu(x) = 0.$$

We can conclude that,

$$\underline{\alpha}_{\mu, \nu}^q(x) = \underline{\alpha}_{\mu, \nu}^q(T(x)) \quad \text{and} \quad \overline{\alpha}_{\mu, \nu}^q(x) = \overline{\alpha}_{\mu, \nu}^q(T(x)) \quad \text{for } \nu\text{-a.e. } x$$

and that the functions $\underline{\alpha}_{\mu, \nu}^q, \overline{\alpha}_{\mu, \nu}^q$ are T -invariant. On the other hand, the measure ν is ergodic and

$$-q \leq \underline{\alpha}_{\mu, \nu}^q \leq \overline{\alpha}_{\mu, \nu}^q \leq n - q.$$

It follows that $\underline{\alpha}_{\mu, \nu}^q$ ($\overline{\alpha}_{\mu, \nu}^q$) is ν -almost every where constant, which says that

$$\overline{\text{dim}}_\mu^q(\nu) = \underline{\text{dim}}_\mu^q(\nu) \quad \text{and} \quad \overline{\text{Dim}}_\mu^q(\nu) = \underline{\text{Dim}}_\mu^q(\nu).$$

□

Remark 3.14. *In the case where $\overline{\alpha}_{\mu,\nu}^q(x) = \underline{\alpha}_{\mu,\nu}^q(x) = \alpha$ for ν -almost all x , we have $\dim_{\mu}^q(\nu) = \text{Dim}_{\mu}^q(\nu) = \alpha$. The results developed by Heurteaux in [24,25,26] and Fan et al. in [19,20,21] are obtained as a special case of the multifractal Theorems when q equals 0.*

Example 3.15.

We say that the probability measure μ is a quasi-Bernoulli measure on the Cantor set $\mathcal{C} = \{0, 1\}^{\mathbb{N}^*}$ if we can find $C \geq 1$ such that

$$\forall x, y \in \mathcal{F} \quad C^{-1}\mu(x)\mu(y) \leq \mu(xy) \leq C\mu(x)\mu(y),$$

where \mathcal{F} is the set of words written with the alphabet $\{0, 1\}$. Let \mathcal{F}_n be the set of words of length n , and take $x = x_1x_2\dots \in \mathcal{C}$, let $I_n(x)$ be the unique cylinder \mathcal{F}_n that contains x . Let us introduce the function τ_{μ} defined for $p \in \mathbb{R}$, by

$$\tau_{\mu}(p) = \lim_{n \rightarrow \infty} \tau_{\mu}(p, n) \quad \text{with} \quad \tau_{\mu}(p, n) = \frac{1}{n \log 2} \log \left(\sum_{x \in \mathcal{F}_n} \mu(x)^p \right).$$

Let μ and ν be two probability measures on \mathcal{C} such that, $\nu \ll \mu$ and μ is a quasi-Bernoulli measure. Then, $\tau'_{\mu}(1)$ exists and we have

$$\lim_{n \rightarrow \infty} \frac{\log \mu(B(x, 2^{-n}))}{\log(2^{-n})} = \lim_{n \rightarrow \infty} \frac{\log_2 \mu(I_n(x))}{-n} = -\tau'_{\mu}(1) \text{ for } \mu\text{-a.e. } x \in \mathcal{C}, \quad (3.1)$$

and

$$\lim_{n \rightarrow \infty} \frac{\log_2 \nu(I_n(x))}{-n} = -\tau'_{\nu}(1) = -\tau'_{\mu}(1) \quad \text{for } \nu\text{-a.e. } x \in \mathcal{C}. \quad (3.2)$$

For more details about (3.1) and (3.2), the reader can see [23,25]. We have immediately from (3.1) and (3.2) that the measure ν is (q, μ) -unidimensional and

$$\dim_{\mu}^q(\nu) = \text{Dim}_{\mu}^q(\nu) = (q-1)\tau'_{\mu}(1) = (q-1)\tau'_{\nu}(1).$$

4. Projections results

In this section, we show that the multifractal Hausdorff and packing dimensions of a measure ν are preserved under almost every orthogonal projection. Casually, we briefly recall some basic definitions and facts which will be repeatedly used in subsequent developments. Let m be an integer with $0 < m < n$ and $G_{n,m}$ the Grassmannian manifold of all m -dimensional linear subspaces of \mathbb{R}^n . Denote by $\gamma_{n,m}$ the invariant Haar measure on $G_{n,m}$, such that $\gamma_{n,m}(G_{n,m}) = 1$. For $V \in G_{n,m}$, define the projection map $\pi_V : \mathbb{R}^n \rightarrow V$ as the usual orthogonal projection onto V . Then, the set $\{\pi_V, V \in G_{n,m}\}$ is compact in the space of all linear maps from \mathbb{R}^n to \mathbb{R}^m and the identification of V with π_V induces a compact topology for $G_{n,m}$. Also, for a Borel probability measure μ with compact support on \mathbb{R}^n , denoted by $\text{supp } \mu$, and for $V \in G_{n,m}$, define the projection μ_V of μ onto V , by

$$\mu_V(A) = \mu(\pi_V^{-1}(A)) \quad \forall A \subseteq V.$$

Since μ is compactly supported and $\text{supp } \mu_V = \pi_V(\text{supp } \mu)$ for all $V \in G_{n,m}$, then for any continuous function $f : V \rightarrow \mathbb{R}$, we have

$$\int_V f d\mu_V = \int f(\pi_V(x)) d\mu(x)$$

whenever these integrals exist. For more details, see for example [17,18,28,29,40,44]. The convolution is defined, for $1 \leq m < n$ and $r > 0$, by

$$\begin{aligned} \overline{\phi}_r^m : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ x &\longmapsto \gamma_{n,m} \left\{ V \in G_{n,m}; |\pi_V(x)| \leq r \right\}. \end{aligned}$$

Moreover, define

$$\begin{aligned} \phi_r^m : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ x &\longmapsto \min \left\{ 1, r^m |x|^{-m} \right\}. \end{aligned}$$

We have that $\phi_r^m(x)$ is equivalent to $\overline{\phi}_r^m(x)$ and write $\phi_r^m(x) \asymp \overline{\phi}_r^m(x)$.

For a probability measure μ and for $V \in G_{n,m}$, we have

$$\mu * \phi_r^m(x) \asymp \mu * \overline{\phi}_r^m(x) = \int \mu_V(B(x_V, r)) dV$$

and

$$\mu * \phi_r^m(x) = \int \min \left\{ 1, r^m |x - y|^{-m} \right\} d\mu(y).$$

So, integrating by parts and converting into spherical coordinates (see [18]), we obtain

$$\mu * \phi_r^m(x) = mr^m \int_r^{+\infty} u^{-m-1} \mu(B(x, u)) du.$$

We present the tools, as well as the intermediate results, which will be used in the proofs of our main results. The following straightforward estimates concern the behaviour of the convolution $\mu * \phi_r^m(x)$ as $r \rightarrow 0$.

Lemma 4.1. [18] *Let $1 \leq m \leq n$ and μ be a compactly supported Borel probability measure on \mathbb{R}^n . For all $x \in \mathbb{R}^n$, we have*

$$cr^m \leq \mu * \phi_r^m(x)$$

for all sufficiently small r , where $c > 0$ independent of r .

Definition 4.2. *Let $E \subseteq \mathbb{R}^n$ and $0 < s < +\infty$. We say that E is s -Ahlfors regular if it is closed and if there exists a Borel measure μ on \mathbb{R}^n and a constant $1 \leq C_E < +\infty$, such that $\mu(E) > 0$ and*

$$C_E^{-1} r^s \leq \mu(B(x, r)) \leq C_E r^s, \quad \text{for all } x \in E \quad \text{and} \quad 0 < r \leq 1.$$

Lemma 4.3. *Let $0 < m \leq n$.*

1. *Let μ be a compactly supported Borel probability measure on \mathbb{R}^n . Then, for all $x \in \mathbb{R}^n$ and $r > 0$,*

$$\mu(B(x, r)) \leq \mu * \phi_r^m(x).$$

2. *Suppose that μ is a compactly supported Borel probability measure on \mathbb{R}^n with support contained in an s -Ahlfors regular set for some $0 < s \leq m$. Then, for all $\varepsilon > 0$ and μ -almost all x there is $c > 0$ such that*

$$c r^{-\varepsilon} \mu(B(x, r)) \geq \mu * \phi_r^m(x).$$

for sufficiently small r .

Proof. The proof of assertion (1) is exactly the same as that given in [18]. The assertion (2) is nothing but Lemma 5.8 of [32]. \square

We use the properties of $\mu * \phi_r^m(x)$ to have a relationship between the kernels and the projected measures.

Lemma 4.4. [18] *Let $1 \leq m \leq n$ and μ be a compactly supported Borel probability measure on \mathbb{R}^n . We have,*

1. Let $\varepsilon > 0$. For all $V \in G_{n,m}$, for μ -almost all x and for sufficiently small r ,

$$r^\varepsilon \mu * \phi_r^m(x) \leq \mu_V(B(x_V, r)).$$

2. Let $\varepsilon > 0$. For $\gamma_{n,m}$ -almost all $V \in G_{n,m}$, for all $x \in \mathbb{R}^n$ and for sufficiently small r ,

$$r^{-\varepsilon} \mu * \phi_r^m(x) \geq \mu_V(B(x_V, r)).$$

Throughout this section, we consider a compactly supported Borel probability measure μ on \mathbb{R}^n with support contained in an s -Ahlfors regular set for some $0 \leq s \leq m < n$ and ν be a compactly supported Borel probability measure on \mathbb{R}^n such that $\text{supp } \nu \subseteq \text{supp } \mu$ and $\nu \ll \mu$.

We introduce the function $\underline{\alpha}_{\mu,\nu}^{q,m}$ and $\overline{\alpha}_{\mu,\nu}^{q,m}$, by

$$\underline{\alpha}_{\mu,\nu}^{q,m}(x) = \liminf_{r \rightarrow 0} \frac{\log \nu * \phi_r^m(x) - q \log \mu * \phi_r^m(x)}{\log r},$$

and

$$\overline{\alpha}_{\mu,\nu}^{q,m}(x) = \limsup_{r \rightarrow 0} \frac{\log \nu * \phi_r^m(x) - q \log \mu * \phi_r^m(x)}{\log r}.$$

Proposition 4.5. *Let $q \in \mathbb{R}$. We have that for ν -almost all x*

1. *If $q > 0$, then*

$$\underline{\alpha}_{\mu,\nu}^{q,m}(x) = \underline{\alpha}_{\mu,\nu}^q(x).$$

2. *If $q \leq 0$ and $\underline{\alpha}_{\mu,\nu}^q(x) \leq m(1 - q)$, then*

$$\underline{\alpha}_{\mu,\nu}^{q,m}(x) = \underline{\alpha}_{\mu,\nu}^q(x).$$

Proof. 1. We will prove that for ν -almost all x , we have $\underline{\alpha}_{\mu,\nu}^{q,m}(x) \leq \underline{\alpha}_{\mu,\nu}^q(x)$. The proof of the other inequality is similar.

By using Lemma 4.3, we have

$$\log \nu(B(x, r)) \leq \log \nu * \phi_r^m(x).$$

Since ν is absolutely continuous with respect to μ and $q > 0$, we have that for ν -almost all x

$$-q(\log c - \varepsilon \log r + \log \mu(B(x, r))) \leq -q \log \mu * \phi_r^m(x).$$

So, for ν -almost all x ,

$$\frac{\log \nu(B(x, r)) - q(\log c - \varepsilon \log r + \log \mu(B(x, r)))}{\log r} \geq \frac{\log \nu * \phi_r^m(x) - q \log \mu * \phi_r^m(x)}{\log r}$$

Finally, Letting $\varepsilon \rightarrow 0$, we get $\underline{\alpha}_{\mu,\nu}^q(x) \geq \underline{\alpha}_{\mu,\nu}^{q,m}(x)$.

2. The inequality $\underline{\alpha}_{\mu,\nu}^{q,m}(x) \leq m(1 - q)$ follows immediately from Lemma 4.1.

By using Lemma 4.3, we have

$$\log \nu(B(x, r)) \leq \log \nu * \phi_r^m(x).$$

Since $q \leq 0$, then

$$-q \log \mu(B(x, r)) \leq -q \log \mu * \phi_r^m(x).$$

It follows that $\underline{\alpha}_{\mu,\nu}^{q,m}(x) \leq \underline{\alpha}_{\mu,\nu}^q(x)$. The proof for other inequality is similar to that given for assertion (1). □

The following proposition is a consequence of Lemma 4.4.

Proposition 4.6. *Let $q \in \mathbb{R}$. For $\gamma_{n,m}$ -almost all $V \in G_{n,m}$ and ν -almost all x , we have*

$$\underline{\alpha}_{\mu_V, \nu_V}^q(x_V) = \underline{\alpha}_{\mu, \nu}^{q,m}(x)$$

and

$$\overline{\alpha}_{\mu_V, \nu_V}^q(x_V) = \overline{\alpha}_{\mu, \nu}^{q,m}(x).$$

The following theorem presents general relations between the multifractal Hausdorff and the multifractal packing dimension of a measure ν and that of its orthogonal projections.

Theorem 4.7. *Let $q \in \mathbb{R}$.*

1. *For $\gamma_{n,m}$ -almost all $V \in G_{n,m}$, we have*

$$\underline{\text{Dim}}_{\mu_V}^q(\nu_V) = \text{ess inf } \overline{\alpha}_{\mu, \nu}^{q,m}(x) \quad \text{and} \quad \overline{\text{Dim}}_{\mu_V}^q(\nu_V) = \text{ess sup } \overline{\alpha}_{\mu, \nu}^{q,m}(x)$$

where the essential bounds being related to the measure ν .

2. *For $\gamma_{n,m}$ -almost all $V \in G_{n,m}$, we have*

(a) *for $q > 0$,*

$$\underline{\text{dim}}_{\mu_V}^q(\nu_V) = \underline{\text{dim}}_{\mu}^q(\nu) \quad \text{and} \quad \overline{\text{dim}}_{\mu_V}^q(\nu_V) = \overline{\text{dim}}_{\mu}^q(\nu).$$

(b) *for $q \leq 0$ and $\overline{\text{dim}}_{\mu}^q(\nu) \leq m(1 - q)$,*

$$\underline{\text{dim}}_{\mu_V}^q(\nu_V) = \underline{\text{dim}}_{\mu}^q(\nu) \quad \text{and} \quad \overline{\text{dim}}_{\mu_V}^q(\nu_V) = \overline{\text{dim}}_{\mu}^q(\nu).$$

Proof. Follows immediately from Propositions 4.5 and 4.6 and Corollary 3.7. □

Remark 4.8. *Due to Proposition 5.10 in [32], the result is optimal. If in addition, $q = 0$, then the results of Falconer and O'Neil hold (see [18]).*

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