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An Inverse Source Problem For a Two Terms Time-fractional Diffusion Equation

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abstract: In this paper, we consider an inverse problem for a linear heat equation involving two time-fractional derivatives, subject to a nonlocal boundary condition. We determine a source term independent of the space variable, and the temperature distribution with an over-determining function of integral type.

Key Words: Inverse problem, heat equation, fractional derivative, integral equations, bi-orthogonal system of functions, Fourier series.

Contents

1. Introduction

Recently, a great attention has been focused on the study of boundary value problems (BVP) for fractional differential equations. They appear as mathematical models in different branches in sciences as physics, chemistry, biology, geology, control theory, signal theory, nanoscience and so on. The reader can find many applications in the book [8] and references therein.

Many kinds of boundary problems, including inverse problems [4], were formulated for different type of PDEs of integer order or with fractional order differential operators. In [12], Luchko and Gorenflo studied a multi-terms fractional differential equation with Caputo fractional derivatives and using operational method, a solution of an initial boundary problem for that equation was obtained in explicit form containing multinomial Mittag-Leffler function, some properties of which were obtained by Zhiyuan Li et al [10]. Later, Yikan Liu [11] established strong maximum principle for fractional diffusion equations with multiple Caputo derivatives and investigated a related inverse problem. We note as well the work by Daftardar-Gejji and Bhalikar [2] where a multi-terms fractional diffusion-wave equation was considered and boundary-value problems for this equation were solved by the method of separation of variables. Kirane et al [9] considered a two dimensional inverse source problem for a time fractional diffusion equation and prove the well posedness of the inverse source problem using Fourier method. In [1] Aleroev et al studied a linear heat equation involving a Riemann-Liouville fractional derivative in time, with a nonlocal boundary condition.

In this paper we study the linear heat equation

$$ {}^{c}D_{0+}^{\alpha}u(x,t) + {}^{c}D_{0+}^{\beta}u(x,t) - \varrho u_{xx}(x,t) = F(x,t), \quad (x,t)\in\Omega_T, \tag{1.1}$$

with initial and nonlocal boundary conditions,

$$\begin{aligned} u(x,0) &= \varphi(x), \quad x\in(0,1), & (1.2)\\ u(0,t) &= u(1,t), \quad u_x(1,t)=0, \quad t\in(0,T], & (1.3)\end{aligned}$$

where $\Omega_T=(0,1)\times(0,T]$, ϱ is a positive constant, ${}^{c}D_{0+}^{\alpha}$ and ${}^{c}D_{0+}^{\beta}$ stand for the Caputo fractional derivatives of order α and β, respectively, with $0<\beta<\alpha<1$ and $\varphi(x)$ is the initial temperature.

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For (1.1)−(1.3) the direct problem is the determination of $u(x,t)$ in $\overline{\Omega}_T$ such that $u(.,t) \in C^2\left([0,1],\mathbb{R}\right)$ and

$${}^cD^{\alpha}_{0+}u(x,.),{}^cD^{\beta}_{0+}u(x,.) \in C\left((0,T],\mathbb{R}\right)$$

when the initial temperature $\varphi(x)$ and the source term $F(x,t)$ are given and continuous.

Letting the source term have the form $F(x,t)=a(t)f(x,t)$, the inverse problem consists of determining $a(t)$ and the temperature distribution $u(x,t)$, from the initial temperature $\varphi(x)$ and boundary conditions (1.3). This problem is not uniquely solvable.

To have the inverse problem uniquely solvable, we impose the over-determination condition

$$\int_0^1 xu(x,t)dx = g(t), \quad t \in [0,T], \tag{1.4}$$

where $g \in AC([0,T],\mathbb{R})$ (the space of absolutely continuous functions). The solvability of inverse problems with such condition has been considered earlier [5,6].

A solution of the inverse problem is a pair of functions $\{u(x,t),a(t)\}$ satisfying $u(.,t) \in C^2[(0,1),\mathbb{R}]$, ${}^cD^{\alpha}_{0+}u(x,.),{}^cD^{\beta}_{0+}u(x,.) \in C\left((0,T],\mathbb{R}\right)$ such that $a \in C\left((0,T],\mathbb{R}^+\right)$, satisfying the initial data and the condition (1.4).

When we want to solve the inverse problem (1.1) − (1.4) using separation of variables (Fourier's method), we have to consider the spectral problem

$$\begin{cases} X'' = -\mu X, & x \in (0,1), \\ X(0) = X(1), & X'(1) = 0. \end{cases} \tag{1.5}$$

The boundary-value problem (1.5) is non self-adjoint, it admits the following conjugate (adjoint) problem:

$$\begin{cases} Y'' = -\mu Y, & x \in (0,1), \\ Y(0) = 0, & Y'(0) = Y'(1). \end{cases} \tag{1.6}$$

Our approach to the solvability of the inverse problem (1.1) − (1.4) is based on the expansion of the solution $u(x,t)$ using a biorthogonal system of functions obtained from the eigenfunctions and associated eigenfunctions of the spectral problem (1.5) and its adjoint problem (1.6) (see Il'in [3] and Keldysh [7]).

The rest of the paper is organized as follows: in Section 2, for the sake of the convenience we recall some basic definitions and results needed in the sequel. In Section 3, we present our main results concerning the existence, uniqueness and continuous dependence of the solution of the inverse problem.

2. Preliminaries

In this section, we recall basic definitions and notations from fractional calculus. For a differentiable function $f:\mathbb{R}^+ \to \mathbb{R}$, the left sided Caputo fractional derivative of order $0<\alpha<1$ is defined by (see [8])

$${}^cD^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}}d\tau, \tag{2.1}$$

where $\Gamma(.)$ is the Euler Gamma function. The integral (2.1) can be written as a convolution

$${}^cD^{\alpha}_{0+}f(t) = \left(\gamma_{\alpha} * f'\right)(t),$$

where

$$\gamma_{\alpha}(t) := \begin{cases} \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, & t>0, \\ 0, & t \le 0. \end{cases}$$

In the discussion of single-term time fractional diffusion equations, it turns out that the expressions of solutions use the usual Mittag-Leffler function

$$E_{\zeta,\eta}(z) := \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\zeta k+\eta)}, \quad z \in \mathbb{C},\ \zeta>0,\ \eta \in \mathbb{R}. \tag{2.2}$$

The explicit solutions to two-terms case use a generalized form of (2.2) called the binomial Mittag-Leffler function of which several basic properties play remarkable roles especially for obtaining estimates for the stability.

The binomial Mittag-Leffler function is defined as (see [12])

$$E_{(\beta_1,\beta_2),\beta_0}(z_1,z_2) := \sum_{k=0}^{\infty}\sum_{i=0}^{k} \frac{k!}{i!\,(k-i)!}.\frac{z_1^i z_2^{k-i}}{\Gamma(\beta_0 + i\beta_1 + (k-i)\beta_2)}, \tag{2.3}$$

where we assume $0 < \beta_0 < 2$, $0 < \beta_1, \beta_2 < 1$, $z_1, z_2 \in \mathbb{C}$, i and k are non-negative integers.

Concerning the relation between binomial Mittag-Leffler functions with different parameters, we have the following lemma.

Lemma 2.1. *(see lemma 3.1, [10]) Let $0 < \beta_0 < 2$, $0 < \beta_1, \beta_2 < 1$, $z_1, z_2 \in \mathbb{C}$ be fixed. Then*

$$\begin{aligned} \frac{1}{\Gamma(\beta_0)} &+ z_1 E_{(\beta_1,\beta_2),\beta_0+\beta_1}(z_1,z_2) + z_2 E_{(\beta_1,\beta_2),\beta_0+\beta_2}(z_1,z_2) \\ &= E_{(\beta_1,\beta_2),\beta_0}(z_1,z_2). \end{aligned} \tag{2.4}$$

For regularity of the solution to the two-terms time-fractional diffusion equation, the estimate

$$\left|E_{(\alpha_1,\alpha_1-\alpha_2),\beta}(z_1,z_2)\right| \le \frac{C}{1+|z_1|} \le C, \tag{2.5}$$

where $0 < \beta < 2$, $1 > \alpha_1 > \alpha_2 > 0$ and C a positive constant (see lemma 3.2, [10]) is useful.

In order to prove the convergence of the series corresponding to $D^{\alpha}_{0+}u(x,t)$, we need the following formula (see lemma 3.3, [10])

$$\begin{aligned} &\frac{d}{dt}\left(t^{\alpha_1} E_{(\alpha_1,\alpha_1-\alpha_2),1+\alpha_1}\left(-q_1 t^{\alpha_1}, -q_2 t^{\alpha_1-\alpha_2}\right)\right) \\ &= t^{\alpha_1-1} E_{(\alpha_1,\alpha_1-\alpha_2),\alpha_1}\left(-q_1 t^{\alpha_1}, -q_2 t^{\alpha_1-\alpha_2}\right), \quad t > 0, \end{aligned} \tag{2.6}$$

where q_1 and q_2 are positive constants, and the following lemma.

Lemma 2.2. *(see lemma 15.2 page 278 [13]) Let the fractional derivatives $D^{\alpha}_{a+}f_n$ exist for all $n \in \mathbb{N}$ and let the series $\sum_{n=0}^{\infty} f_n$ and $\sum_{n=0}^{\infty} D^{\alpha}_{a+}f_n$ uniformly converge on every sub-interval $[a+\varepsilon, b]$, $\varepsilon > 0$, then, the former series admits termwise fractional differentiation:*

$$\left({}^{c}D^{\alpha}_{a+}\sum_{n=0}^{\infty} f_n\right)(x) = \left(\sum_{n=0}^{\infty} {}^{c}D^{\alpha}_{a+} f_n\right)(x), \qquad \alpha > 0,\ a < x < b.$$

Let H be a Hilbert space with the scalar product $\langle .,. \rangle$. Two sets S_1 and S_2 of functions of H form a bi-orthogonal system of functions if a one-to-one correspondence can be established between them such that,

$$\langle f_i, g_j \rangle = \delta_{ij},$$

where $f_i \in S_1$, $g_j \in S_2$ and δ_{ij} is the Kronecker symbol.

Let $L^2(0,1)$ denote the classical Lebesgue space of measurable functions $\phi : (0,1) \to \mathbb{R}$ such that $|\phi(.)|^2$ is integrable, provided with the scalar product $\langle .,. \rangle$ defined by

$$\langle \phi, \xi \rangle = \int_0^1 \phi(x)\xi(x)dx, \qquad \text{for all } \phi, \xi \in L^2(0,1),$$

and the associated norm

$$\|\phi\|_{L^2} = \left(\int_0^1 \phi^2(x)dx\right)^{\frac{1}{2}}.$$

Let $C([0,T])$ denote the space of continuous functions $\omega:[0,T]\to\mathbb{R}$, equipped with the norm

$$\|\omega\| = \max_{0\le t\le T} |\omega(t)| .$$

A bi-orthogonal system of functions. The sets of functions

$$\left\{2, \{4\cos(2\pi nx)\}_{n=1}^{\infty}, \{4(1-x)\sin(2\pi nx)\}_{n=1}^{\infty}\right\} \tag{2.7}$$

and

$$\left\{x, \{x\cos(2\pi nx)\}_{n=1}^{\infty}, \{\sin(2\pi nx)\}_{n=1}^{\infty}\right\} \tag{2.8}$$

are obtained from the non-self-adjoint spectral problem (1.5) and its adjoint problem (1.6), respectively.

The set of functions (2.7) and (2.8) is complete in $L^2(0,1)$ and forms a Riesz basis in $L^2(0,1)$. Furthermore, the set of functions (2.7) $-$ (2.8) constitutes a bi-orthogonal system with the one to one correspondence

$$\begin{array}{ccccc} \Big\{ \underbrace{2}, & \underbrace{\{4\cos(2\pi nx)\}_{n=1}^{\infty}}, & \underbrace{\{4(1-x)\sin(2\pi nx)\}_{n=1}^{\infty}} & \Big\} \\ \downarrow & \downarrow & \downarrow & \\ \{ \ x, & \{x\cos(2\pi nx)\}_{n=1}^{\infty} \ , & \{\sin(2\pi nx)\}_{n=1}^{\infty} & \} . \end{array}$$

3. Mains Results

3.1. Existence and uniqueness of the solution of the inverse problem.

We have the following theorem.

Theorem 3.1. *Let the following conditions be satisfied*

$(H1)$ $\varphi\in C^4([0,1])$, $\varphi(1)=\varphi(0)$, $\varphi'(1)=0$, $\varphi''(0)=\varphi''(1)$, $\varphi'''(1)=0$,

$(H2)$ $f\in C^4(\overline{\Omega}_T,\mathbb{R})$, $f(1,t)=f(0,t)$, $f_x(1,t)=0$, $f_{xx}(1,t)=f_{xx}(0,t)$, $f_{xxx}(1,t)=0$, $\int_0^1 xf(x,t)dx\neq 0$, *and there exists a constant* $M_1>0$ *such that*

$$0<\frac{1}{M_1}\le\left|\int_0^1 xf(x,t)dx\right| . \tag{3.1}$$

$(H3)$ $g\in AC([0,T],\mathbb{R})$.

Then, for

$$T<\left(\frac{\alpha}{M_1C_1C}\right)^{\frac{1}{\alpha}}, \ \textit{where } C_1 \textit{ is defined in } (3.17),$$

the inverse problem (1.1) $-$ (1.4) *has a unique solution.*

Proof.
1-Existence of the solution of the inverse problem: We write the formal solution $u(x,t)$ for the linear system (1.1) $-$ (1.4) in the form

$$u(x,t)=2u_0(t)+\sum_{n=1}^{\infty}u_{2n-1}(t)4\cos(2\pi nx)+\sum_{n=1}^{\infty}u_{2n}(t)4(1-x)\sin(2\pi nx), \tag{3.2}$$

where $u_0(t)$, $u_{2n-1}(t)$, $u_{2n}(t)$ for $n\in\mathbb{N}$ are to be determined.

Let

$$f(x,t)=2f_0(t)+\sum_{n=1}^{\infty}f_{2n-1}(t)4\cos(2\pi nx)+\sum_{n=1}^{\infty}f_{2n}(t)4(1-x)\sin(2\pi nx),$$

where $f_0(t)$, $f_{2n-1}(t)$ and $f_{2n}(t)$ are given by

$$\begin{cases} f_0(t) &= \displaystyle\int_0^1 f(x,t)xdx, \\ f_{2n-1}(t) &= \displaystyle\int_0^1 f(x,t)x\cos(2\pi nx)\,dx, \\ f_{2n}(t) &= \displaystyle\int_0^1 f(x,t)\sin(2\pi nx)\,dx. \end{cases} \tag{3.3}$$

Using properties of the bi-orthogonal system we have

$$\begin{cases} u_0(t) &= \langle u(x,t), x\rangle, \\ u_{2n-1}(t) &= \langle u(x,t), x\cos(2\pi nx)\rangle, \\ u_{2n}(t) &= \langle u(x,t), \sin(2\pi nx)\rangle. \end{cases} \tag{3.4}$$

By virtue of (3.4), we have

$$\begin{aligned} D^{\alpha}_{0+}u_0(t) + D^{\beta}_{0+}u_0(t) &= \left\langle D^{\alpha}_{0+}u(x,t) + D^{\beta}_{0+}u(x,t), x\right\rangle, \\ D^{\alpha}_{0+}u_{2n}(t) + D^{\beta}_{0+}u_{2n}(t) &= \left\langle D^{\alpha}_{0+}u(x,t) + D^{\beta}_{0+}u(x,t), \sin(2\pi nx)\right\rangle, \\ D^{\alpha}_{0+}u_{2n-1}(t) + D^{\beta}_{0+}u_{2n-1}(t) &= \left\langle D^{\alpha}_{0+}u(x,t) + D^{\beta}_{0+}u(x,t), x\cos(2\pi nx)\right\rangle. \end{aligned}$$

Whereupon

$$D^{\alpha}_{0+}u_0(t) + D^{\beta}_{0+}u_0(t) = a(t)f_0(t), \tag{3.5}$$

$$D^{\alpha}_{0+}u_{2n}(t) + D^{\beta}_{0+}u_{2n}(t) + \lambda_n u_{2n}(t) = a(t)f_{2n}(t), \tag{3.6}$$

$$D^{\alpha}_{0+}u_{2n-1}(t) + D^{\beta}_{0+}u_{2n-1}(t) + (\lambda_n^2/\varrho)u_{2n-1}(t) + 2\lambda_n u_{2n}(t) = a(t)f_{2n-1}(t), \tag{3.7}$$

where $\lambda_n := 2\pi n\varrho$.

In order to get the unknown constants $u_0(0)$, $u_{2n-1}(0)$ and $u_{2n}(0)$, we use (3.4) and the initial temperature given in (1.2),

$$\begin{aligned} u_0(0) &= \langle u(x,0), x\rangle = \langle \varphi(x), x\rangle := \varphi_0, \\ u_{2n-1}(0) &= \langle u(x,0), x\cos(2\pi nx)\rangle = \langle \varphi(x), x\cos(2\pi nx)\rangle := \varphi_{2n-1}, \\ u_{2n}(0) &= \langle u(x,0), \sin(2\pi nx)\rangle = \langle \varphi(x), \sin(2\pi nx)\rangle := \varphi_{2n}, \end{aligned}$$

where φ_0, φ_{2n-1} and φ_{2n} are the coefficients of the series expansion in the basis (2.7) of the function $\varphi(x)$:

$$\varphi_0 = \int_0^1 \varphi(x)xdx; \quad \varphi_{2n-1} = \int_0^1 \varphi(x)x\cos(2\pi nx)\,dx; \quad \varphi_{2n} = \int_0^1 \varphi(x)\sin(2\pi nx)\,dx.$$

By using Theorem 4.1, [12] we obtain:

•The solution of the linear fractional differential equation (3.5) with the initial condition φ_0

$$u_0(t) = \left[\left(t^{\alpha-1}E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta})\right) * a(t)f_0(t)\right] + \varphi_0\left[1 - t^{\alpha-\beta}E_{\alpha-\beta,1+\alpha-\beta}(-t^{\alpha-\beta})\right]. \tag{3.8}$$

•The solution of the linear fractional differential equation (3.6) with the initial condition φ_{2n}

$$\begin{aligned} u_{2n}(t) = & \left[\left(t^{\alpha-1}E_{(\alpha-\beta,\alpha),\alpha}(-t^{\alpha-\beta}, -\lambda_n t^{\alpha})\right) * a(t)f_{2n}(t)\right] \\ & +\varphi_{2n}[1 - t^{\alpha-\beta}E_{(\alpha-\beta,\alpha),1+\alpha-\beta}(-t^{\alpha-\beta}, -\lambda_n t^{\alpha}) \\ & -\lambda_n t^{\alpha}E_{(\alpha-\beta,\alpha),1+\alpha}(-t^{\alpha-\beta}, -\lambda_n t^{\alpha})]. \end{aligned} \tag{3.9}$$

Using (2.4), we get

$$
\begin{aligned}
u_{2n}(t) &= \left[\left(t^{\alpha-1}E_{(\alpha-\beta,\alpha),\alpha}(-t^{\alpha-\beta},-\lambda_n t^{\alpha})\right) * a(t)f_{2n}(t)\right] \\
&\quad +\varphi_{2n}E_{(\alpha-\beta,\alpha),1}(-t^{\alpha-\beta},-\lambda_n t^{\alpha}).
\end{aligned} \tag{3.10}
$$

•The solution of the linear fractional differential equation (3.7) with the initial condition φ_{2n-1}

$$
\begin{aligned}
u_{2n-1}(t) &= \left[\left(t^{\alpha-1}E_{(\alpha-\beta,\alpha),\alpha}(-t^{\alpha-\beta},-\left(\lambda_n^2/\varrho\right)t^{\alpha})\right) * \left(a(t)f_{2n-1}(t)+2\lambda_n u_{2n}(t)\right)\right] \\
&\quad + \varphi_{2n-1}\left[1-t^{\alpha-\beta}E_{(\alpha-\beta,\alpha),1+\alpha-\beta}(-t^{\alpha-\beta},-\lambda_n t^{\alpha})\right. \\
&\quad \left. -\lambda_n t^{\alpha}E_{(\alpha-\beta,\alpha),1+\alpha}(-t^{\alpha-\beta},-\lambda_n t^{\alpha})\right].
\end{aligned}
$$

Using (2.4) and (3.10), we get

$$
\begin{aligned}
u_{2n-1}(t) &= \left[\left(t^{\alpha-1}E_{(\alpha-\beta,\alpha),\alpha}(-t^{\alpha-\beta},-\left(\lambda_n^2/\varrho\right)t^{\alpha})\right) * \left(a(t)f_{2n-1}(t)\right)\right] \\
&\quad + \left[2\lambda_n h(t) * a(t)f_{2n}(t)\right] \\
&\quad + \left[2\lambda_n\varphi_{2n}t^{\alpha-1}E_{(\alpha-\beta,\alpha),\alpha}(-t^{\alpha-\beta},-\left(\lambda_n^2/\varrho\right)t^{\alpha}) * E_{(\alpha-\beta,\alpha),1}(-t^{\alpha-\beta},-\lambda_n t^{\alpha})\right] \\
&\quad +\varphi_{2n-1}E_{(\alpha-\beta,\alpha),1}(-t^{\alpha-\beta},-\left(\lambda_n^2/\varrho\right)t^{\alpha}),
\end{aligned} \tag{3.11}
$$

where

$$
h(t) = t^{\alpha-1}E_{(\alpha-\beta,\alpha),\alpha}(-t^{\alpha-\beta},-\left(\lambda_n^2/\varrho\right)t^{\alpha}) * t^{\alpha-1}E_{(\alpha-\beta,\alpha),\alpha}(-t^{\alpha-\beta},-\lambda_n t^{\alpha}).
$$

Now, we get the expression of the term $a(t)$. By multiplying both sides of the equation by x and integrating the resulting equation between 0 and 1, we obtain

$$
\int_0^1 x\left[{}^cD_{0+}^{\alpha}u(x,t)+{}^cD_{0+}^{\beta}u(x,t)\right]dx = \int_0^1 x\left[\varrho u_{xx}(x,t)+a(t)f(x,t)\right]dx.
$$

Using (1.4), we have

$$
\begin{aligned}
a(t) &= \left(\int_0^1 xf(x,t)dx\right)^{-1}\left[{}^cD_{0+}^{\alpha}g(t)+{}^cD_{0+}^{\beta}g(t)+\varrho\int_0^1 xu_{xx}(x,t)dx\right] \\
&= \left(\int_0^1 xf(x,t)dx\right)^{-1}\left[{}^cD_{0+}^{\alpha}g(t)+{}^cD_{0+}^{\beta}g(t)+\varrho u_x(0,t)\right],
\end{aligned} \tag{3.12}
$$

where

$$
\begin{aligned}
u_x(0,t) &= \sum_{n=1}^{\infty} 8\pi n u_{2n}(t) \\
&= \sum_{n=1}^{\infty} 8\pi n[(t^{\alpha-1}E_{(\alpha-\beta,\alpha),\alpha}(-t^{\alpha-\beta},-\lambda_n t^{\alpha}) * a(t)f_{2n}(t)) \\
&\quad +\varphi_{2n}E_{(\alpha-\beta,\alpha),1}(-t^{\alpha-\beta},-\lambda_n t^{\alpha})].
\end{aligned} \tag{3.13}
$$

Let $\hat{\mathbf{A}}(a(t)) := a(t)$, where the operator $\hat{\mathbf{A}} : C([0,T]) \to C([0,T])$ is defined for $a \in C([0,T])$ by

$$
\hat{\mathbf{A}}(a(t)) = \left(\int_0^1 xf(x,t)dx\right)^{-1}\left[{}^cD_{0+}^{\alpha}g(t)+{}^cD_{0+}^{\beta}g(t)+\varrho u_x(0,t)\right].
$$

By (3.3) and (3.13), we obtain the Volterra integral equation

$$
\hat{\mathbf{A}}(a(t)) = \mathcal{L}(t) + (f_0(t))^{-1}\int_0^t K(t,\tau)a(\tau)d\tau, \tag{3.14}
$$

where,

$$\mathcal{L}(t) := (f_0(t))^{-1} \left[{}^c D_{0+}^{\alpha} g(t) + {}^c D_{0+}^{\beta} g(t) + \sum_{n=1}^{\infty} 4\lambda_n \varphi_{2n} E_{(\alpha-\beta,\alpha),1}(-t^{\alpha-\beta}, -\lambda_n t^{\alpha}) \right], \tag{3.15}$$

and

$$K(t,\tau) := \sum_{n=1}^{\infty} 4\lambda_n f_{2n}(\tau) \left[(t-\tau)^{\alpha-1} E_{(\alpha-\beta,\alpha),\alpha}(-(t-\tau)^{\alpha-\beta}, -\lambda_n (t-\tau)^{\alpha}) \right]. \tag{3.16}$$

Before we proceed further, notice that under the assumptions $(H2)$, the series $\sum_{n=1}^{\infty} 4\lambda_n f_{2n}(t)$ is uniformly convergent by the Weierstrass M-test because it is bounded from above by a uniformly convergent numerical series

$$\sum_{n=1}^{\infty} 4\lambda_n f_{2n}(t) = \sum_{n=1}^{\infty} \frac{\varrho}{2\pi^3 n^3} f_{2n}^{(4)}(t) \leq \sum_{n=1}^{\infty} \frac{\varrho}{2\pi^3 n^3} \left| f_{2n}^{(4)}(t) \right|,$$

where $f_{2n}^{(4)}(t)$ is the coefficient of the Fourier sine series of the function $f^{(4)}(.,t)$. Furthermore, by Bessel's inequality, we have

$$\sum_{n=1}^{\infty} \left[f_{2n}^{(4)}(t) \right]^2 \leq c \left\| f^{(4)} \right\|_{L^2(0,1)}^2,$$

where c is a constant independent of t and n. Thus, we have

$$\sum_{n=1}^{\infty} 4\lambda_n f_{2n}(t) \leq C_1, \tag{3.17}$$

where C_1 is a constant independent of t and n.

We select $T < \left(\frac{\alpha}{M_1 C_1 C} \right)^{\frac{1}{\alpha}}$, where M_1 is from assumption $(H2)$ of Theorem 3.1 and C is from the inequality (2.5).

We shall show that $\hat{\mathbf{A}} : C([0,T]) \to C([0,T])$ and the mapping $\hat{\mathbf{A}}$ is a contraction.

For $a \in C([0,T])$, using (2.5) and assumptions $(H1)$ and $(H2)$, the series in the expression of $u_x(0,t)$ (see (3.13)) is uniformly convergent on $[0,T]$ and represents a continuous function. The term ${}^c D_{0+}^{\alpha} g(t) + {}^c D_{0+}^{\beta} g(t)$ is continuous being the difference of two continuous functions (see equation (3.12)). Hence $\hat{\mathbf{A}}(a) \in C([0,T])$.

As

$$\begin{aligned}
\left| \hat{\mathbf{A}}(a)(t) - \hat{\mathbf{A}}(b)(t) \right| &\leq \left| (f_0(t))^{-1} \right| \int_0^t |K(t,\tau)| \, |a(\tau) - b(\tau)| \, d\tau \\
&\leq M_1 C_1 C \frac{T^{\alpha}}{\alpha} \max_{0 \leq t \leq T} |a(t) - b(t)|, \\
\left\| \hat{\mathbf{A}}(a) - \hat{\mathbf{A}}(b) \right\| &= \max_{0 \leq t \leq T} \left| \hat{\mathbf{A}}(a(t)) - \hat{\mathbf{A}}(b(t)) \right| \qquad (3.18) \\
&\leq M_1 C_1 C \frac{T^{\alpha}}{\alpha} \|a - b\|. \qquad (3.19)
\end{aligned}$$

Since $T < \left(\frac{\alpha}{M_1 C_1 C} \right)^{\frac{1}{\alpha}}$,

$$M_1 C_1 C \frac{T^{\alpha}}{\alpha} < 1,$$

thus the mapping $\hat{\mathbf{A}}$ is a contraction for $t \in [0,T]$. This ensures the unique determination of $a \in C([0,T])$ by the Banach fixed point theorem.

Under assumptions $(H1)-(H3)$ and following [1], we shall show that the series solution given by (3.2) and the series corresponding to $u_{xx}(x,t)$ are uniformly convergent and represent continuous functions on Ω_T . Also, we shall show that the series corresponding to $u(x,t)$ is $\alpha-$differentiable and $\beta-$differentiable.

For $g \in AC([0,T])$ the term ${}^cD_{0+}^{\alpha}g(t) +^c D_{0+}^{\beta}g(t)$ is continuous being the difference of continuous functions (see equation (3.12)). Furthermore, for any $\varepsilon > 0$ the term ${}^cD_{0+}^{\alpha}g(t)+ {}^cD_{0+}^{\beta}g(t)$ is bounded on the interval $(\varepsilon, T]$. In the estimates below, we will use this fact.

From (3.15) and (3.14), we obtain

$$\|\mathcal{L}\| \leq M_4, \qquad \|a\| \leq \frac{M_4}{1 - M_1C_1C\frac{T^{\alpha}}{\alpha}} := N,$$

where $M_4 = M_1(M_2 + CM_3)$, M_2 is a bound of ${}^cD_{0+}^{\alpha}g(t) +^c D_{0+}^{\beta}g(t)$,

$$\sum_{n=1}^{\infty} 4\lambda_n\varphi_{2n} \leq M_3$$

and both M_2 and M_3 are positive constants independent of t and n.

Due to the assumptions $(H1)$ and $(H2)$ of Theorem 3.1, we have

$$\begin{aligned}
\varphi_{2n} &= \frac{1}{16\pi^4n^4}\int_0^1 \varphi^{(4)}(x)\sin(2\pi nx)\,dx = \frac{1}{16\pi^4n^4}\varphi_{2n}^{(4)},\\
\varphi_{2n-1} &= \frac{1}{16\pi^4n^4}\int_0^1 \varphi^{(4)}(x)\,x\cos(2\pi nx)\,dx = \frac{1}{16\pi^4n^4}\varphi_{2n-1}^{(4)},\\
f_{2n}(t) &= \frac{1}{16\pi^4n^4}\int_0^1 f_x^{(4)}(x,t)\sin(2\pi nx)\,dx = \frac{1}{16\pi^4n^4}f_{2n}^{(4)}(t),\\
f_{2n-1}(t) &= \frac{1}{16\pi^4n^4}\int_0^1 f_x^{(4)}(x,t)\,x\cos(2\pi nx)\,dx = \frac{1}{16\pi^4n^4}f_{2n-1}^{(4)}(t),
\end{aligned}$$

where $\varphi_{2n-1}^{(4)}$, $\varphi_{2n}^{(4)}$ and $f_{2n-1}^{(4)}(t)$, $f_{2n}^{(4)}(t)$ are the coefficients of the Fourier cosine and the Fourier sine series of the functions $\varphi^{(4)}(x)$ and $f_x^{(4)}(x,t)$, respectively. These functions are bounded by virtue of Bessel's inequality.

Then the sum of the series (3.2) is bounded above by

$$\begin{aligned}
&2\left(CN\|f_0\|\frac{T^{\alpha}}{\alpha} + 2|\varphi_0|\right) + \sum_{n=1}^{\infty}\frac{C}{16\pi^4n^4}\left(N\left\|f_{2n}^{(4)}\right\|\frac{T^{\alpha}}{\alpha} + \left|\varphi_{2n}^{(4)}\right|\right)\\
&+\sum_{n=1}^{\infty}\frac{C}{16\pi^4n^4}\left(\lambda_nCN\frac{T^{2\alpha}}{\alpha}\text{ß}(\alpha,\alpha)\left\|f_{2n}^{(4)}\right\|\right.\\
&\qquad\left.+\frac{T^{\alpha}}{\alpha}\left(N\left\|f_{2n-1}^{(4)}\right\| + 2\lambda_nC\left|\varphi_{2n}^{(4)}\right|\right) + \left|\varphi_{2n-1}^{(4)}\right|\right),
\end{aligned}$$

where ß(α,α) is the Euler beta function.

By the Weierstrass M-test the series (3.2) is uniformly convergent. Similarly, we can show that the series corresponding to $u_{xx}(x,t)$ is uniformly convergent and represents a continuous function.

Let us show that the series (3.2) corresponding to $u(x,t)$

$$2u_0(t) + \sum_{n=1}^{\infty} u_{2n-1}(t)4\cos(2\pi nx) + \sum_{n=1}^{\infty} u_{2n}(t)4(1-x)\sin(2\pi nx),$$

is $\alpha-$differentiable and $\beta-$differentiable.

We need to show that the series

$$2{}^cD_{0+}^{\alpha}u_0(t) + \sum_{n=1}^{\infty}\left({}^cD_{0+}^{\alpha}u_{2n-1}(t)\right)4\cos(2\pi nx) + \sum_{n=1}^{\infty}\left({}^cD_{0+}^{\alpha}u_{2n}(t)\right)4(1-x)\sin(2\pi nx), \tag{3.20}$$

and

$$2{}^{c}D_{0+}^{\beta}u_0(t)+\sum_{n=1}^{\infty}\left({}^{c}D_{0+}^{\beta}u_{2n-1}(t)\right)4\cos(2\pi nx)+\sum_{n=1}^{\infty}\left({}^{c}D_{0+}^{\beta}u_{2n}(t)\right)4(1-x)\sin(2\pi nx) \tag{3.21}$$

are uniformly convergent.

For this, we first calculate $u_0'(t), u_{2n}'(t)$ and $u_{2n-1}'(t)$, then, we estimate ${}^{c}D_{0+}^{\alpha}u_0(t)$, ${}^{c}D_{0+}^{\alpha}u_{2n-1}(t)$ and ${}^{c}D_{0+}^{\alpha}u_{2n}(t)$ on $[\varepsilon, T]$ for all $\varepsilon > 0$. We have

$$\begin{aligned}
u_0'(t) &= \frac{d}{dt}\left[\left[\left(t^{\alpha-1}E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta})\right)*a(t)f_0(t)\right]\right.\\
&\quad\left.+\varphi_0\left[1-t^{\alpha-\beta}E_{\alpha-\beta,1+\alpha-\beta}(-t^{\alpha-\beta})\right]\right]\\
&= \left[\left(t^{\alpha-2}E_{\alpha-\beta,\alpha-1}(-t^{\alpha-\beta})\right)*a(t)f_0(t)\right]-\varphi_0 t^{\alpha-\beta-1}E_{\alpha-\beta,\alpha-\beta}(-t^{\alpha-\beta}).
\end{aligned}$$

Using the fact that

$$xE_{(\alpha-\beta,\alpha),\alpha}(x,y)\le\frac{|x|}{1+|x|}C\le C,$$

we have

$$\begin{aligned}
\left|{}^{c}D_{0+}^{\alpha}u_0(t)\right| &\le \frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{|u_0'(\tau)|}{(t-\tau)^{\alpha}}d\tau\\
&\le \frac{C}{\Gamma(1-\alpha)}\int_0^t(t-\tau)^{-\alpha}\left(\frac{\tau^{\alpha-1}}{1-\alpha}\|a\|\|f_0\|+\|\varphi_0\|\tau^{-1}\right)d\tau\\
&\le \frac{C\|a\|\|f_0\|}{(1-\alpha)\Gamma(1-\alpha)}\int_0^t(t-\tau)^{-\alpha}\tau^{\alpha-1}d\tau\\
&\quad+\frac{C\|\varphi_0\|}{\Gamma(1-\alpha)}\int_0^t(t-\tau)^{-\alpha}\tau^{-1}d\tau.
\end{aligned}$$

Therefore,

$$\left|{}^{c}D_{0+}^{\alpha}u_0(t)\right|\le\frac{C}{\Gamma(1-\alpha)}\left[\frac{ß(1-\alpha,\alpha)}{1-\alpha}\|a\|\|f_0\|+\|\varphi_0\|t^{-\alpha}\right]. \tag{3.22}$$

Also

$$\begin{aligned}
u_{2n}'(t) &= \frac{d}{dt}\left[\left[\left(t^{\alpha-1}E_{(\alpha-\beta,\alpha),\alpha}(-t^{\alpha-\beta},-\lambda_n t^{\alpha})\right)*a(t)f_{2n}(t)\right]\right.\\
&\quad+\varphi_{2n}\left[1-t^{\alpha-\beta}E_{(\alpha-\beta,\alpha),1+\alpha-\beta}(-t^{\alpha-\beta},-\lambda_n t^{\alpha})\right.\\
&\quad\left.\left.-\lambda_n t^{\alpha}E_{(\alpha-\beta,\alpha),1+\alpha}(-t^{\alpha-\beta},-\lambda_n t^{\alpha})\right]\right]\\
&= \left[\left(t^{\alpha-2}E_{(\alpha-\beta,\alpha),\alpha-1}(-t^{\alpha-\beta},-\lambda_n t^{\alpha})\right)*a(t)f_{2n}(t)\right]\\
&\quad-\varphi_{2n}\left[t^{\alpha-\beta-1}E_{(\alpha-\beta,\alpha),\alpha-\beta}(-t^{\alpha-\beta},-\lambda_n t^{\alpha})\right.\\
&\quad\left.+\lambda_n t^{\alpha-1}E_{(\alpha-\beta,\alpha),\alpha}(-t^{\alpha-\beta},-\lambda_n t^{\alpha})\right],
\end{aligned}$$

and

$$\begin{aligned}
\left|{}^{c}D_{0+}^{\alpha}u_{2n}(t)\right| &\le \frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{|u_{2n}'(\tau)|}{(t-\tau)^{\alpha}}d\tau\\
&\le \frac{C\|a\|\|f_{2n}\|}{\Gamma(1-\alpha)}\left(\int_0^t(t-\tau)^{-\alpha}\frac{\tau^{\alpha-1}}{1-\alpha}d\tau\right)\\
&\quad+\frac{C\|\varphi_{2n}\|}{\Gamma(1-\alpha)}\left(\int_0^t(t-\tau)^{-\alpha}\tau^{-1}d\tau+\lambda_n\int_0^t(t-\tau)^{-\alpha}\tau^{\alpha-1}d\tau\right).
\end{aligned}$$

By applying the change of variable $s=\frac{\tau}{t}$ in the calculation of the integrals

$$\int_0^t (t-\tau)^{-\alpha}\,\tau^{\alpha-1} \qquad \text{and} \qquad \int_0^t (t-\tau)^{-\alpha}\,\tau^{-1}d\tau,$$

we find

$$\begin{aligned}\left|{}^cD^{\alpha}_{0^+}u_{2n}(t)\right| &\leq \frac{C\left\|a\right\|\left\|f^{(4)}_{2n}\right\|}{16\pi^4n^4\Gamma(1-\alpha)}\frac{\text{ß}\left(1-\alpha,\alpha\right)}{1-\alpha}\\ &\quad+\frac{C\left\|\varphi^{(4)}_{2n}\right\|}{16\pi^4n^4\Gamma(1-\alpha)}\left(\lambda_n\text{ß}\left(1-\alpha,\alpha\right)+t^{-\alpha}\right). \qquad (3.23)\end{aligned}$$

Also

$$\begin{aligned}u'_{2n-1}(t)=&\frac{d}{dt}[\left[\left(t^{\alpha-1}E_{(\alpha-\beta,\alpha),\alpha}(-t^{\alpha-\beta},-\left(\lambda_n^2/\varrho\right)t^{\alpha}\right)\right)*\left(a(t)f_{2n-1}(t)\right)\right]\\ &+[2\lambda_n h(t)*a(t)f_{2n}(t)]\\ &+\left[2\lambda_n\varphi_{2n}t^{\alpha-1}E_{(\alpha-\beta,\alpha),\alpha}(-t^{\alpha-\beta},-\left(\lambda_n^2/\varrho\right)t^{\alpha})*E_{(\alpha-\beta,\alpha),1}(-t^{\alpha-\beta},-\lambda_nt^{\alpha})\right]\\ &+\varphi_{2n-1}[1-t^{\alpha-\beta}E_{(\alpha-\beta,\alpha),1+\alpha-\beta}(-t^{\alpha-\beta},-(\lambda_n^2/\varrho)t^{\alpha})\\ &-(\lambda_n^2/\varrho)t^{\alpha}E_{(\alpha-\beta,\alpha),1+\alpha}(-t^{\alpha-\beta},-(\lambda_n^2/\varrho)t^{\alpha})]]\\ =&\left[\left(t^{\alpha-2}E_{(\alpha-\beta,\alpha),\alpha-1}(-t^{\alpha-\beta},-\left(\lambda_n^2/\varrho\right)t^{\alpha}\right)\right)*\left(a(t)f_{2n-1}(t)\right)\right]\\ &+2\lambda_n h'(t)*a(t)f_{2n}(t)\\ &+\left[2\lambda_n\varphi_{2n}t^{\alpha-2}E_{(\alpha-\beta,\alpha),\alpha-1}(-t^{\alpha-\beta},-\left(\lambda_n^2/\varrho\right)t^{\alpha})*E_{(\alpha-\beta,\alpha),1}(-t^{\alpha-\beta},-\lambda_nt^{\alpha})\right]\\ &-\varphi_{2n-1}[t^{\alpha-\beta-1}E_{(\alpha-\beta,\alpha),\alpha-\beta}(-t^{\alpha-\beta},-\left(\lambda_n^2/\varrho\right)t^{\alpha})\\ &+\left(\lambda_n^2/\varrho\right)t^{\alpha-1}E_{(\alpha-\beta,\alpha),\alpha}(-t^{\alpha-\beta},-\left(\lambda_n^2/\varrho\right)t^{\alpha})],\end{aligned}$$

where

$$h'(t)=t^{\alpha-2}E_{(\alpha-\beta,\alpha),\alpha-1}(-t^{\alpha-\beta},-\left(\lambda_n^2/\varrho\right)t^{\alpha})*t^{\alpha-1}E_{(\alpha-\beta,\alpha),\alpha}(-t^{\alpha-\beta},-\lambda_nt^{\alpha}).$$

We have

$$|h'(t)|\leq C^2\int_0^t (t-\tau)^{\alpha-2}\,\tau^{\alpha-1}d\tau.$$

Therefore,

$$\begin{aligned}\left|u'_{2n-1}(t)\right| &\leq 2\lambda_n\left\|a\right\|\left\|f_{2n}\right\|\int_0^t\left|h'(\tau)\right|d\tau+C\left\|a\right\|\left\|f_{1n}\right\|\frac{t^{\alpha-1}}{1-\alpha}\\ &\quad+2\lambda_nC^2\left\|\varphi_{2n}\right\|\frac{t^{\alpha-1}}{1-\alpha}\\ &\quad+\left\|\varphi_{1n}\right\|\left(Ct^{-1}+\left(\lambda_n^2/\varrho\right)Ct^{\alpha-1}\right)\\ &\leq \frac{1}{16\pi^4n^4}[2\lambda_n\left\|a\right\|\left\|f^{(4)}_{2n}\right\|C^2\int_0^t\int_0^s (s-\tau)^{\alpha-2}\,\tau^{\alpha-1}d\tau ds\\ &\quad+C\left\|a\right\|\left\|f^{(4)}_{1n}\right\|\frac{t^{\alpha-1}}{1-\alpha}\\ &\quad+2\lambda_nC^2\left\|\varphi^{(4)}_{2n}\right\|\frac{t^{\alpha-1}}{1-\alpha}+C\left\|\varphi^{(4)}_{1n}\right\|\left(t^{-1}+\left(\lambda_n^2/\varrho\right)t^{\alpha-1}\right)].\end{aligned}$$

As

$$\begin{aligned}\int_0^t\int_0^s (s-\tau)^{\alpha-2}\tau^{\alpha-1}d\tau ds &= \int_0^t \tau^{\alpha-1}\left(\int_\tau^t (s-\tau)^{\alpha-2}ds\right)d\tau \\ &= \frac{1}{\alpha-1}\int_0^t \tau^{\alpha-1}(t-\tau)^{\alpha-1}d\tau \\ &= \frac{t^{2\alpha-1}}{\alpha-1}ß(\alpha,\alpha),\end{aligned}$$

we get the estimate

$$\begin{aligned}\left|u'_{2n-1}(t)\right| \leq\ & \frac{1}{16\pi^4 n^4}[2\lambda_n \|a\|\left\|f_{2n}^{(4)}\right\| C^2\frac{t^{2\alpha-1}}{1-\alpha}ß(\alpha,\alpha)+C\|a\|\left\|f_{1n}^{(4)}\right\|\frac{t^{\alpha-1}}{1-\alpha} \\ &+2\lambda_n C^2\left\|\varphi_{2n}^{(4)}\right\|\frac{t^{\alpha-1}}{1-\alpha}+C\left\|\varphi_{1n}^{(4)}\right\|\left(t^{-1}+\left(\lambda_n^2/\varrho\right)t^{\alpha-1}\right)].\end{aligned}$$

Similarly

$$\begin{aligned}\left|{}^cD_{0+}^\alpha u_{2n-1}(t)\right| \leq & \frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\left|u'_{2n-1}(\tau)\right|}{(t-\tau)^\alpha}d\tau \\ \leq & \frac{1}{16\pi^4n^4\Gamma(1-\alpha)}[2\lambda_n\|a\|\left\|f_{2n}^{(4)}\right\|C^2\frac{ß(\alpha,\alpha)}{1-\alpha}\int_0^t(t-\tau)^{-\alpha}\tau^{2\alpha-1}d\tau \\ &+\left(\|a\|\left\|f_{1n}^{(4)}\right\|+2\lambda_nC\left\|\varphi_{2n}^{(4)}\right\|\right)\frac{C}{1-\alpha}\int_0^t(t-\tau)^{-\alpha}\tau^{\alpha-1}d\tau \\ &+C\left\|\varphi_{1n}^{(4)}\right\|\int_0^t(t-\tau)^{-\alpha}\left(\tau^{-1}+\left(\lambda_n^2/\varrho\right)\tau^{\alpha-1}\right)d\tau].\end{aligned}$$

By applying the change of variable $s=\frac{\tau}{t}$ in the integrals

$$\int_0^t(t-\tau)^{-\alpha}\tau^{2\alpha-1}d\tau,\ \int_0^t(t-\tau)^{-\alpha}\tau^{\alpha-1}d\tau$$

and

$$\int_0^t(t-\tau)^{-\alpha}\left(\tau^{-1}+\left(\lambda_n^2/\varrho\right)\tau^{\alpha-1}\right)d\tau,$$

we estimate

$$\begin{aligned}\left|{}^cD_{0+}^\alpha u_{2n-1}(t)\right| \leq\ & \frac{1}{16\pi^4n^4\Gamma(1-\alpha)}[2\lambda_n\|a\|\left\|f_{2n}^{(4)}\right\|\frac{C^2t^\alpha}{1-\alpha}ß(\alpha,\alpha)ß(1-\alpha,2\alpha) \\ &+\left(\|a\|\left\|f_{1n}^{(4)}\right\|+2\lambda_nC\left\|\varphi_{2n}^{(4)}\right\|\right)\frac{C}{1-\alpha}ß(1-\alpha,\alpha) \\ &+C\left\|\varphi_{1n}^{(4)}\right\|\left(t^\alpha+\left(\lambda_n^2/\varrho\right)ß(1-\alpha,\alpha)\right)] \\ \leq\ & \frac{1}{16\pi^4n^4\Gamma(1-\alpha)}[2\lambda_n\|a\|\left\|f_{2n}^{(4)}\right\|\frac{C^2T^\alpha}{1-\alpha}ß(\alpha,\alpha)ß(1-\alpha,2\alpha) \\ &+\left(\|a\|\left\|f_{1n}^{(4)}\right\|+2\lambda_nC\left\|\varphi_{2n}^{(4)}\right\|\right)\frac{C}{1-\alpha}ß(1-\alpha,\alpha) \\ &+C\left\|\varphi_{1n}^{(4)}\right\|\left(T^\alpha+\left(\lambda_n^2/\varrho\right)ß(1-\alpha,\alpha)\right)].\end{aligned}\tag{3.24}$$

Consequently, by (3.22), (3.23) and (3.24) the series (3.20) and (3.21) are uniformly convergent by the Weierstrass M-test. Hence the series (3.2) corresponding to $u(x,t)$ is α-differentiable and alike β-

differentiable with respect to the time variable and the relations

$$\begin{aligned} {}^cD_{0+}^{\alpha}u(x,t) &= 2{}^cD_{0+}^{\alpha}u_0(t)+\sum_{n=1}^{\infty}\left({}^cD_{0+}^{\alpha}u_{2n-1}(t)\right)4\cos\left(2\pi nx\right) \\ &\quad+\sum_{n=1}^{\infty}\left({}^cD_{0+}^{\alpha}u_{2n}(t)\right)4\left(1-x\right)\sin\left(2\pi nx\right), \end{aligned}$$

and

$$\begin{aligned} {}^cD_{0+}^{\beta}u(x,t) &= 2{}^cD_{0+}^{\beta}u_0(t)+\sum_{n=1}^{\infty}({}^cD_{0+}^{\beta}u_{2n-1}(t))4\cos\left(2\pi nx\right) \\ &\quad+\sum_{n=1}^{\infty}({}^cD_{0+}^{\beta}u_{2n}(t))4\left(1-x\right)\sin\left(2\pi nx\right) \end{aligned}$$

hold true. Similarly, we can show that the series corresponding to $u_{xx}(x,t)$ is uniformly convergent and represents a continuous function.

2-Uniqueness of the solution. Let $\{u(x,t),a(t)\}$ and $\{v(x,t),b(t)\}$ be two solution sets of the inverse problem then

$$\begin{aligned} u(x,t)-v(x,t) &= 2\left(u_0(t)-v_0(t)\right)+\sum_{n=1}^{\infty}\left(u_{2n-1}(t)-v_{2n-1}(t)\right)4\cos\left(2\pi nx\right) \\ &\quad+\sum_{n=1}^{\infty}\left(u_{2n}(t)-v_{2n}(t)\right)4\left(1-x\right)\sin\left(2\pi nx\right) \\ &= \left[\left(t^{\alpha-1}E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta})\right)*\left(a(t)-b(t)\right)f_0(t)\right] \\ &\quad+\left[\left(t^{\alpha-1}E_{(\alpha-\beta,\alpha),\alpha}(-t^{\alpha-\beta},-\lambda_nt^{\alpha})\right)*\left(a(t)-b(t)\right)f_{2n}(t)\right] \\ &\quad+\left[\left(t^{\alpha-1}E_{(\alpha-\beta,\alpha),\alpha}(-t^{\alpha-\beta},-\left(\lambda_n^2/\varrho\right)t^{\alpha})\right)*\left(a(t)-b(t)\right)f_{2n-1}(t)\right] \\ &\quad+[2\lambda_nt^{\alpha-1}E_{(\alpha-\beta,\alpha),\alpha}(-t^{\alpha-\beta},-\left(\lambda_n^2/\varrho\right)t^{\alpha}) \\ &\quad*t^{\alpha-1}E_{(\alpha-\beta,\alpha),\alpha}(-t^{\alpha-\beta},-\lambda_nt^{\alpha})*\left(a(t)-b(t)\right)f_{2n}(t)], \end{aligned} \tag{3.25}$$

and

$$a(t)-b(t)=\left(f_0(t)\right)^{-1}\int_0^tK(t,\tau)\left(a(\tau)-b(\tau)\right)d\tau.$$

Due to the estimate (3.19), we have $a=b$ and by substituting $a=b$ in (3.25), we obtain $u=v$. □

3.2. Continuous dependence of the solution on the data.

Let $\mathcal{F}$ be the set of triplets $\{\varphi,f,g\}$ where the functions φ,f,g satisfy the assumptions of Theorem 3.1 and

$$\|\varphi\|_{C^4([0,1])}\leq M_5,\qquad \|f\|_{C^4(\Omega_T)}\leq M_6,\qquad \|g\|_{AC([0,T])}\leq M_7.$$

For $\psi\in\mathcal{F}$, we define the norm

$$\|\psi\|_{\mathcal{F}}:=\|\varphi\|_{C^4([0,T])}+\|f\|_{C^4(\Omega_T)}+\|g\|_{AC([0,T])}.$$

Now, we present the result on the stability of the solution of the inverse problem.

Theorem 3.2. *The solution $\{u(x,t),a(t)\}$ of the inverse problem (1.1) – (1.4), under the assumptions of Theorem 3.1, depends continuously upon the data for $T<\left(\frac{\alpha}{M_1C_1C}\right)^{\frac{1}{\alpha}}$.*

Proof. Let $\{u(x,t),a(t)\}$, $\{\tilde{u}(x,t),\tilde{a}(t)\}$ be two solution sets of the inverse problem (1.1) − (1.4), corresponding to the data $\psi=\{\varphi,f,g\}$, $\tilde{\psi}=\left\{\tilde{\varphi},\tilde{f},\tilde{g}\right\}$, respectively.

From (3.15), we have

$$\begin{aligned}
\mathcal{L}(t)-\tilde{\mathcal{L}}(t) &= (f_0(t))^{-1}\left[{}^cD^{\alpha}_{0+}g(t)+{}^cD^{\beta}_{0+}g(t)+\sum_{n=1}^{\infty}4\lambda_n\varphi_{2n}E_{(\alpha-\beta,\alpha),1}(-t^{\alpha-\beta},-\lambda_nt^{\alpha})\right]\\
&\quad-\left(\tilde{f}_0(t)\right)^{-1}\left[{}^cD^{\alpha}_{0+}\tilde{g}(t)+{}^cD^{\beta}_{0+}\tilde{g}(t)+\sum_{n=1}^{\infty}4\lambda_n\tilde{\varphi}_{2n}E_{(\alpha-\beta,\alpha),1}(-t^{\alpha-\beta},-\lambda_nt^{\alpha})\right]\\
&=\left(f_0(t)\tilde{f}_0(t)\right)^{-1}[\tilde{f}_0(t)({}^cD^{\alpha}_{0+}g(t)+{}^cD^{\beta}_{0+}g(t)\\
&\quad+\sum_{n=1}^{\infty}4\lambda_n\varphi_{2n}E_{(\alpha-\beta,\alpha),1}(-t^{\alpha-\beta},-\lambda_nt^{\alpha}))\\
&\quad-f_0(t)({}^cD^{\alpha}_{0+}\tilde{g}(t)+{}^cD^{\beta}_{0+}\tilde{g}(t)+\sum_{n=1}^{\infty}4\lambda_n\tilde{\varphi}_{2n}E_{(\alpha-\beta,\alpha),1}(-t^{\alpha-\beta},-\lambda_nt^{\alpha}))]\\
&=(f_0(t))^{-1}\left({}^cD^{\alpha}_{0+}\left(g-\tilde{g}\right)(t)+{}^cD^{\beta}_{0+}\left(g-\tilde{g}\right)(t)\right)\\
&\quad+(f_0(t))^{-1}\sum_{n=1}^{\infty}4\lambda_n(\varphi_{2n}-\tilde{\varphi}_{2n})E_{(\alpha-\beta,\alpha),1}(-t^{\alpha-\beta},-\lambda_nt^{\alpha})\\
&\quad+\left(f_0(t)\tilde{f}_0(t)\right)^{-1}\left(\tilde{f}_0-f_0\right)(t)[{}^cD^{\alpha}_{0+}\tilde{g}(t)+{}^cD^{\beta}_{0+}\tilde{g}(t)\\
&\quad+\sum_{n=1}^{\infty}4\lambda_n\tilde{\varphi}_{2n}E_{(\alpha-\beta,\alpha),1}(-t^{\alpha-\beta},-\lambda_nt^{\alpha})].
\end{aligned}$$

Notice that

$$\begin{aligned}
\varphi_{2n}-\tilde{\varphi}_{2n} &= \int_0^1\left(\varphi-\tilde{\varphi}\right)(x)\sin\left(2\pi nx\right)dx\\
&= \frac{1}{16\pi^4n^4}\int_0^1\left(\varphi^{(4)}-\tilde{\varphi}^{(4)}\right)(x)\sin\left(2\pi nx\right)dx,
\end{aligned}$$

and

$$\left(\tilde{f}_0-f_0\right)(t)=\int_0^1x\left(\tilde{f}-f\right)(x,t)dx.$$

We have the estimate

$$\left\|\mathcal{L}-\tilde{\mathcal{L}}\right\|\le N_1\left\|\varphi-\tilde{\varphi}\right\|_{C^4([0,1])}+N_2\left\|f-\tilde{f}\right\|_{C^4(\Omega_T)}+N_3\left\|g-\tilde{g}\right\|_{AC([0,T])},$$

where $0<\frac{1}{M_1}\le\left|\int_0^1xf(x,t)dx\right|$, $0<\frac{1}{M_1}\le\left|\int_0^1x\tilde{f}(x,t)dx\right|$, $\sum_{n=1}^{\infty}\frac{\varrho}{2\pi^3n^3}\le M_8$, $N_1:=CM_1M_8$, $N_2:=M_1^2\left[M_7\left(\frac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}+\frac{T^{1-\beta}}{(1-\beta)\Gamma(1-\beta)}\right)+CM_5M_8\right]$, and $N_3:=M_1\left(\frac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}+\frac{T^{1-\beta}}{(1-\beta)\Gamma(1-\beta)}\right)$.

From (3.14), we have

$$\begin{aligned} a(t)-\tilde{a}(t) =&\mathcal{L}(t)-\tilde{\mathcal{L}}(t)+(f_0(t))^{-1}\int_0^t K(t,\tau)a(\tau)d\tau \\ &-\left(\tilde{f}_0(t)\right)^{-1}\int_0^t \tilde{K}(t,\tau)\tilde{a}(\tau)d\tau \\ =&\mathcal{L}(t)-\tilde{\mathcal{L}}(t) \\ &+\left(f_0(t)\tilde{f}_0(t)\right)^{-1}\left[\tilde{f}_0(t)\int_0^t K(t,\tau)a(\tau)d\tau - f_0(t)\int_0^t \tilde{K}(t,\tau)\tilde{a}(\tau)d\tau\right] \\ =&\mathcal{L}(t)-\tilde{\mathcal{L}}(t) \\ &+\left(f_0(t)\tilde{f}_0(t)\right)^{-1}[\tilde{f}_0(t)\int_0^t \left(K(t,\tau)a(\tau)-\tilde{K}(t,\tau)\tilde{a}(\tau)\right)d\tau \\ &+\left(\int_0^t \tilde{K}(t,\tau)\tilde{a}(\tau)d\tau\right)\left(f_0(t)-\tilde{f}_0(t)\right)] \\ =&\mathcal{L}(t)-\tilde{\mathcal{L}}(t) \\ &+\left(f_0(t)\tilde{f}_0(t)\right)^{-1}[\tilde{f}_0(t)\int_0^t \left(K(t,\tau)-\tilde{K}(t,\tau)\right)a(\tau)d\tau \\ &+\tilde{f}_0(t)\int_0^t \tilde{K}(t,\tau)\left(a(\tau)-\tilde{a}(\tau)\right)d\tau \\ &+\left(\int_0^t \tilde{K}(t,\tau)\tilde{a}(\tau)d\tau\right)\left(f_0(t)-\tilde{f}_0(t)\right)]. \end{aligned}$$

We have the estimate

$$\begin{aligned} \|a-\tilde{a}\| \leq & \left\|\mathcal{L}-\tilde{\mathcal{L}}\right\| + M_1\|a\|\int_0^t \left(K(t,\tau)-\tilde{K}(t,\tau)\right)d\tau \\ & +M_1\|a-\tilde{a}\|\int_0^t \tilde{K}(t,\tau)d\tau + M_1^2\left\|f_0-\tilde{f}_0\right\|\|a\|\int_0^t \tilde{K}(t,\tau)d\tau \\ \leq & \left\|\mathcal{L}-\tilde{\mathcal{L}}\right\| + M_1\left(C\|a\|\left\|f_{2n}-\tilde{f}_{2n}\right\|\frac{T^\alpha}{\alpha}+\|a-\tilde{a}\|C_1C\frac{T^\alpha}{\alpha}\right) \\ & +M_1^2\left\|f_0-\tilde{f}_0\right\|\|\tilde{a}\|C_1C\frac{T^\alpha}{\alpha} \\ \leq & \left\|\mathcal{L}-\tilde{\mathcal{L}}\right\| + \left\|f-\tilde{f}\right\|_{C^4(\Omega_T)} M_1CN\frac{T^\alpha}{\alpha}(1+M_1C_1) \\ & +\|a-\tilde{a}\|M_1C_1C\frac{T^\alpha}{\alpha}. \end{aligned}$$

Due to the estimate of $\left\|\mathcal{L}-\tilde{\mathcal{L}}\right\|$, we have

$$\begin{aligned} (1-M_1C_1C\frac{T^\alpha}{\alpha})\|a-\tilde{a}\| \leq & \left\|f-\tilde{f}\right\|_{C^4(\Omega_T)}[M_1CN\frac{T^\alpha}{\alpha}(1+M_1C_1)+N_2] \\ & +N_1\|\varphi-\tilde{\varphi}\|_{C^4([0,1])}+N_3\|g-\tilde{g}\|_{AC([0,T])}. \end{aligned}$$

Hence

$$(1-M_1C_1C\frac{T^\alpha}{\alpha})\|a-\tilde{a}\| \leq N_5\left\|\psi-\tilde{\psi}\right\|_{\mathcal{F}},$$

where

$$N_5 := \max\left\{N_1, M_1CN\frac{T^\alpha}{\alpha}(1+M_1C_1)+N_2, N_3\right\}.$$

For $T < \left(\frac{\alpha}{M_1 C_1 C}\right)^{\frac{1}{\alpha}}$, we have

$$\|a - \tilde{a}\| \leq \frac{N_5}{1 - M_1 C_1 C \frac{T^\alpha}{\alpha}} \left\|\psi - \tilde{\psi}\right\|_{\mathcal{F}}.$$

From (3.2) a similar estimate can be obtained for $u - \tilde{u}$.
This completes the proof. □

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