



## One Solution For Nonlocal Fourth Order Equations

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ABSTRACT: A critical point result for differentiable functionals is exploited in order to prove that a suitable class of fourth-order boundary value problem of Kirchhoff-type possesses at least one weak solution under an asymptotical behaviour of the nonlinear datum at zero. Some examples to illustrate the results are given.

Key Words: Weak solution, Fourth-order problem, Kirchhoff-type elliptic problem, Variational methods.

### Contents

<b>1 Introduction</b>	<b>1</b>
<b>2 Preliminaries</b>	<b>3</b>
<b>3 Main results</b>	<b>4</b>

### 1. Introduction

In this article, we are concerned with the fourth-order Kirchhoff-type elliptic problem

$$\begin{cases} \Delta \left( |\Delta u|^{p-2} \Delta u \right) - [M \left( \int_{\Omega} |\nabla u|^p dx \right)]^{p-1} \Delta_p u + \rho |u|^{p-2} u = f(x, u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega \end{cases} \quad (P^f)$$

where  $p > \max \{1, \frac{N}{2}\}$ ,  $\Omega \subset \mathbb{R}^N$  ( $N \geq 1$ ) is a bounded smooth domain,  $\rho > 0$ ,  $M : [0, +\infty[ \rightarrow \mathbb{R}$  is a continuous function and  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is an  $L^1$ -Carathéodory function.

Partial differential equations (PDEs) have been used in many fields such as physics, biology, engineering, economics, and finance to model and analyse dynamic systems. Recently, partial differential equations (PDEs) have become important in socioeconomics, as descriptive tools in the qualitative and quantitative sense. Very often PDE modeling of socioeconomic processes is based on principles carried over from the physical and chemical sciences.

The problem  $(P^f)$  is related to the stationary analogue of the Kirchhoff equation,

$$\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0 \quad (1.1)$$

where  $\rho$  is the mass density,  $\rho_0$  is the initial tension,  $h$  is the area of the crosssection,  $E$  is the Young modulus of the material and  $L$  is the length of the string, proposed by Kirchhoff [27] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the length changes of the string produced by transverse vibrations. Nonlinear Kirchhoff model can also be used for describing the dynamics of an axially moving string. In recent years, axially moving string-like continua such as wires, belts, chains, band-saws have been subjects of the study of researchers (see [41]).

Latter (1.1) was developed to form

$$u_{tt} - M \left( \int_{\Omega} |\nabla u|^2 dx \right) \Delta u = f(x, u) \quad (1.2)$$

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where  $M(s) = as + b$ ,  $a, b > 0$ . After that, many authors studied the nonlocal elliptic boundary value problem

$$-M\left(\int_{\Omega} |\nabla u|^2 dx\right) \Delta u = f(x, u). \quad (1.3)$$

Problems like (1.3) can be used for modeling several physical and biological systems where  $u$  describes a process which depends on the average of itself, such as the population density, see [2]. There are a number of papers concerned with Kirchhoff-type boundary value problem, for instance see [10,13,19,21,32,33,37,40,45,46]. For example, Ricceri in an interesting paper [40] established the existence of at least three weak solutions to a class of Kirchhoff-type doubly eigenvalue boundary value problem. In [19] employing a three critical point theorem due to Ricceri, the existence of at least three weak solutions for Kirchhoff-type problems involving two parameters was discussed. The existence and multiplicity of stationary problems of Kirchhoff type were also studied in some recent papers, via variational methods like the symmetric mountain pass theorem in [11] and via a three critical point theorem in [5]. Moreover, in [3,4] some evolutionary higher order Kirchhoff problems, mainly focusing on the qualitative properties of the solutions were treated.

Fourth-order boundary value problems which describe the deformations of an elastic beam in an equilibrium state whose both ends are simply supported have been extensively studied in the literature. For classical results obtained on elastic beam equations we refer to [6,12,14,15,38,44], in particular [44] is one of the pioneering works on extensible beams), while [12] settles the existence and multiplicity question for  $(P^f)$  in the physical situation  $p = 2, \rho = 0$  and  $M$  of the form  $M(s) = as + b$ . Recently, the existence of solutions to fourth-order boundary value problems have been studied in many papers and we refer the reader to the papers [8,9,20,22,26,28,29,30,31,36] and the references therein. For example, Candito and Livrea in [9] by using critical point theory, established the existence of infinitely many weak solutions for a class of elliptic Navier boundary value problems depending on two parameters and involving the  $p$ -biharmonic operator. Liu et al. in [30] employing variational methods, studied the existence and multiplicity of nontrivial solutions for fourth-order elliptic equations. In [20,26] based on variational methods and critical point theory, the existence of multiple solutions for  $(p_1, \dots, p_n)$ -biharmonic systems was discussed. Molica Bisci and Repovš in [36] exploiting variational methods, investigated the existence of multiple weak solutions for a class of elliptic Navier boundary problems involving the  $p$ -biharmonic operator, and presented a concrete example of an application.

The problem  $(P^f)$  models the bending equilibrium of simply supported extensible beams on non-linear foundations. The function  $f$  represents the force that the foundation exerts on the beam and  $M\left(\int_{\Omega} |\nabla u|^p dx\right)$  models the effects of the small changes in the length of the beam. Recently, many researchers have paid their attention to fourth-order Kirchhoff-type problems, we refer the reader to [16,34,42,43] and the references therein. In [42], using the mountain pass theorem, Wang and An established the existence and multiplicity of solutions for the following fourth-order nonlocal elliptic problem

$$\begin{cases} \Delta^2 u - M\left(\int_{\Omega} |\nabla u|^2 dx\right) \Delta u = \lambda f(x, u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega. \end{cases}$$

In particular, in [16] using variational methods and critical point theory, multiplicity results of nontrivial and nonnegative solutions for the parametric version of the problem  $(P^f)$  were established.

The objective of the present paper is to establish the existence of at least one weak solution for the problem  $(P^f)$ . Precisely, in Theorem 3.1 using a smooth version of Theorem 2.1 of [7] which is a more precise version of Ricceri's Variational Principle [39] which we recall in the section we establish the existence of at least one weak solution for the problem  $(P^f)$  requiring an algebraic condition on the nonlinear term  $f$ . We present Example 3.2 in which the hypotheses of Theorem 3.1 are fulfilled. Also in Theorem 3.3 a parametric version of the result of Theorem 3.1 is successively discussed in which, for small values of the parameter and requiring an additional asymptotical behaviour of the potential at zero if  $f(x, 0) = 0$  for every  $x \in \Omega$ , the existence of one nontrivial weak solution is established; see Proposition 3.10. As a consequence of Theorem 3.3, we obtain Corollary 3.4 for the autonomous case. We also present Example 3.5 in which the hypotheses of Corollary 3.4 are fulfilled. Moreover, we deduce the existence of solutions for small positive values of the parameter  $\lambda$  such that the corresponding solutions have smaller

and smaller energies as the parameter goes to zero; see Remark 3.11. We present the concrete Example 3.12 as an application of Theorem 3.3, Proposition 3.10 and Remark 3.11.

## 2. Preliminaries

We shall prove our main results applying the following smooth version of Theorem 2.1 of [7] which is a more precise version of Ricceri's Variational Principle [39, Theorem 2.1].

**Theorem 2.1.** *Let  $X$  be a reflexive real Banach space, let  $\Phi, \Psi : X \rightarrow \mathbb{R}$  be two Gâteaux differentiable functionals such that  $\Phi$  is sequentially weakly lower semicontinuous, strongly continuous and coercive in  $X$  and  $\Psi$  is sequentially weakly upper semicontinuous in  $X$ . Let  $I_\lambda$  be the functional defined as  $I_\lambda := \Phi - \lambda\Psi$ ,  $\lambda \in \mathbb{R}$ , and for every  $r > \inf_X \Phi$ , let  $\varphi$  be the function defined as*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{\sup_{v \in \Phi^{-1}(-\infty, r)} \Psi(v) - \Psi(u)}{r - \Phi(u)}.$$

*Then, for every  $r > \inf_X \Phi$  and every  $\lambda \in (0, \frac{1}{\varphi(r)})$ , the restriction of the functional  $I_\lambda$  to  $\Phi^{-1}(-\infty, r)$  admits a global minimum, which is a critical point (precisely a local minimum) of  $I_\lambda$  in  $X$ .*

We refer the interested reader to the papers [1,17,18,23,24,25,35] in which Theorem 2.1 has been successfully employed to the existence of at least one non-trivial solution for boundary value problems.

Here and in the sequel,  $X$  will denote the space  $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$  endowed with the norm

$$\|u\| := \left( \int_{\Omega} (|\Delta u(x)|^p + |\nabla u(x)|^p + |u(x)|^p) dx \right)^{\frac{1}{p}}.$$

Put

$$k = \sup_{u \in X \setminus \{0\}} \frac{\max_{x \in \overline{\Omega}} |u(x)|}{\|u\|}. \quad (2.1)$$

For  $p > \max\{1, \frac{N}{2}\}$ , since the embedding  $X \hookrightarrow C^0(\overline{\Omega})$  is compact, one has  $k < +\infty$ .

**Remark 2.2.** *We recall that a function  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is said to be  $L^1$ -Carathéodory if*

- (a)  $x \mapsto f(x, t)$  is measurable for every  $t \in \mathbb{R}$ ,
- (b)  $t \mapsto f(x, t)$  is continuous for a.e.  $x \in \Omega$ ,
- (c) for every  $s > 0$  there exists a function  $l_s \in L^1(\Omega)$  such that

$$\sup_{|t| \leq s} |f(x, t)| \leq l_s(x)$$

for almost every  $x \in \Omega$ .

Let  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be an  $L^1$ -Carathéodory function and  $M : [0, +\infty[ \rightarrow \mathbb{R}$  be a continuous function such that there are two positive constants  $m_0$  and  $m_1$  with  $m_0 \leq M(t) \leq m_1$  for all  $t \geq 0$ .

Set

$$F(x, t) := \int_0^t f(x, \xi) d\xi, \quad \text{for all } (x, t) \in \Omega \times \mathbb{R},$$

$$\widetilde{M}(t) := \int_0^t [M(s)]^{p-1} ds \quad \text{for all } t \geq 0,$$

$$M^- := \min \{1, m_0^{p-1}, \rho\}$$

and

$$M^+ := \max \{1, m_1^{p-1}, \rho\}.$$

We say that a function  $u \in X$  is a (weak) solution of the problem  $(P^f)$  if

$$\int_{\Omega} |\Delta u(x)|^{p-2} \Delta u(x) \Delta v(x) dx$$

$$\begin{aligned}
& + \left[ M \left( \int_{\Omega} |\nabla u(x)|^p dx \right) \right]^{p-1} \int_{\Omega} |\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x) dx \\
& + \rho \int_{\Omega} |u(x)|^{p-2} u(x) v(x) dx - \int_{\Omega} f(x, u(x)) v(x) dx = 0
\end{aligned}$$

for every  $v \in X$ .

### 3. Main results

In the sequel  $\text{meas}(\Omega)$  denotes the Lebesgue measure of the set  $\Omega$ .

We formulate our main result as follows.

**Theorem 3.1.** *Assume that*

$$\sup_{\gamma > 0} \frac{\gamma^p}{\int_{\Omega} \sup_{|t| \leq \gamma} F(x, t) dx} > \frac{pk^p}{M^-}. \quad (D_F)$$

Then, the problem  $(P^f)$  possesses at least one weak solution in  $X$ .

*Proof.* In order to apply Theorem 2.1 to our problem, we introduce the functionals  $\Phi, \Psi : X \rightarrow \mathbb{R}$  for each  $u \in X$ , as follows

$$\Phi(u) = \frac{1}{p} \int_{\Omega} |\Delta u(x)|^p dx + \frac{1}{p} \widetilde{M} \left[ \int_{\Omega} |\nabla u(x)|^p dx \right] + \frac{\rho}{p} \int_{\Omega} |u(x)|^p dx \quad (3.1)$$

and

$$\Psi(u) = \int_{\Omega} F(x, u(x)) dx, \quad (3.2)$$

and we put  $I(u) = \Phi(u) - \Psi(u)$  for every  $u \in X$ . Let us prove that the functionals  $\Phi$  and  $\Psi$  satisfy the required conditions. It is well known that  $\Psi$  is a differentiable functional whose differential at the point  $u \in X$  is

$$\Psi'(u)(v) = \int_{\Omega} f(x, u(x)) v(x) dx$$

for every  $v \in X$  as well as is sequentially weakly upper semicontinuous. Moreover,  $\Phi$  is continuously differentiable whose differential at the point  $u \in X$  is

$$\begin{aligned}
\Phi'(u)(v) &= \int_{\Omega} |\Delta u(x)|^{p-2} \Delta u(x) \Delta v(x) dx \\
&+ \left[ M \left( \int_{\Omega} |\nabla u(x)|^p dx \right) \right]^{p-1} \int_{\Omega} |\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x) dx \\
&+ \rho \int_{\Omega} |u(x)|^{p-2} u(x) v(x) dx
\end{aligned}$$

for every  $v \in X$ . Moreover, since  $m_0 \leq K(s) \leq m_1$  for all  $s \in [0, +\infty[$ , from (3.1), we have

$$\frac{M^-}{p} \|u\|^p \leq \Phi(u) \leq \frac{M^+}{p} \|u\|^p \quad (3.3)$$

for all  $u \in X$ , it follows  $\lim_{\|u\| \rightarrow +\infty} \Phi(u) = +\infty$ , namely  $\Phi$  is coercive. Furthermore,  $\Phi$  is sequentially weakly lower semicontinuous. Therefore, we see that the regularity assumptions on  $\Phi$  and  $\Psi$ , as requested in Theorem 2.1 are verified. The condition  $(D_F)$  ensures that there exists  $\bar{\gamma} > 0$  such that

$$\frac{\bar{\gamma}^p}{\int_{\Omega} \sup_{|t| \leq \bar{\gamma}} F(x, t) dx} > \frac{pk^p}{M^-}. \quad (3.4)$$

Choosing  $r = \frac{M^-}{p}(\frac{\bar{\gamma}}{k})^p$ , in view of (3.3) and bearing (2.1) in mind, we see that

$$\Phi^{-1}(-\infty, r) \subseteq \left\{ u \in X; \frac{M^-}{p} |u|^p \leq r \right\} \subseteq \{u \in X; |u(x)| \leq \bar{\gamma} \text{ for each } x \in \Omega\},$$

and it follows that

$$\sup_{u \in \Phi^{-1}(-\infty, r)} \Psi(u) = \sup_{u \in \Phi^{-1}(-\infty, r)} \int_{\Omega} F(x, u(x)) dx \leq \int_{\Omega} \sup_{|t| \leq \bar{\gamma}} F(x, t) dx.$$

By simple calculations and from the definition of  $\varphi(r)$ , since  $0 \in \Phi^{-1}(-\infty, r)$  and  $\Phi(0) = \Psi(0) = 0$ , one has

$$\begin{aligned} \varphi(r) &= \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{(\sup_{v \in \Phi^{-1}(-\infty, r)} \Psi(v)) - \Psi(u)}{r - \Phi(u)} \leq \frac{\sup_{v \in \Phi^{-1}(-\infty, r)} \Psi(v)}{r} \\ &\leq \frac{pk^p}{M^-} \frac{\int_{\Omega} \sup_{|t| \leq \bar{\gamma}} F(x, t) dx}{\bar{\gamma}^p}. \end{aligned}$$

At this point, observe that

$$\varphi(r) \leq \frac{pk^p}{M^-} \frac{\int_{\Omega} \sup_{|t| \leq \bar{\gamma}} F(x, t) dx}{\bar{\gamma}^p}. \quad (3.5)$$

Consequently, from (3.4) and (3.5) one has  $\varphi(r) < 1$ . Hence, since  $1 \in (0, \frac{1}{\varphi(r)})$ , applying Theorem 2.1 the functional  $I$  possesses at least one critical point (local minima)  $\tilde{u} \in \Phi^{-1}(-\infty, r)$ , and since any weak solution of the problem ( $P^f$ ) is exactly a critical point of the functional  $I$ , we have the conclusion.  $\square$

Here we present an example in which the hypotheses of Theorem 3.1 are satisfied.

**Example 3.2.** Let  $p = 3$  and  $\rho = 1$ . Consider the problem

$$\begin{cases} \Delta(|\Delta u| \Delta u) - \left[ M \left( \int_{\Omega} |\nabla u|^3 dx \right) \right]^2 \Delta_3 u + |u|u = f(u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega \end{cases} \quad (3.6)$$

where  $\Omega \subset \mathbb{R}^2$  with  $\text{meas}(\Omega) = 1$ ,  $M(t) = 2 + \cos(t)$  for all  $t \in [0, +\infty)$  and

$$f(t) = \frac{1}{10^3 k^3} (3t^2 + e^t)$$

for all  $t \in \mathbb{R}$ . By the expression of  $f$  we have

$$F(t) = \frac{1}{10^3 k^3} (t^3 + e^t - 1)$$

for every  $t \in \mathbb{R}$ . Taking into account that  $m_0 = 1$  and

$$\sup_{\gamma > 0} \frac{\gamma^3}{\sup_{|t| \leq \gamma} F(t)} > 3k^3 = \frac{pk^p}{M^-},$$

we observe that all assumptions of Theorem 3.1 are fulfilled. Hence, Theorem 3.1 implies that the problem (3.6) possesses at least one nontrivial weak solution in  $W^{2,3}(\Omega) \cap W_0^{1,3}(\Omega)$ .

We note that Theorem 3.1 can be exploited showing the existence of at least one solution for the parametric version of the problem ( $P^f$ ),

$$\begin{cases} \Delta \left( |\Delta u|^{p-2} \Delta u \right) - \left[ M \left( \int_{\Omega} |\nabla u|^p dx \right) \right]^{p-1} \Delta_p u + \rho |u|^{p-2} u = \lambda f(x, u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega \end{cases} \quad (3.7)$$

where  $\lambda$  is a positive parameter. Precisely, we have the following existence result.

**Theorem 3.3.** *For every  $\lambda$  small enough, i.e.*

$$\lambda \in \left(0, \frac{M^-}{pk^p} \sup_{\gamma>0} \frac{\gamma^p}{\int_{\Omega} \sup_{|t|\leq\gamma} F(x,t) dx}\right),$$

the problem (3.7) possesses at least one weak solution  $u_{\lambda} \in X$ .

*Proof.* Fix  $\lambda$  as in the conclusion. We define  $\Phi$  and  $\Psi$  be as given in (3.1) and (3.2), respectively, and we put  $I_{\lambda}(u) = \Phi(u) - \lambda\Psi(u)$  for every  $u \in X$ . Let us pick

$$0 < \lambda < \frac{M^-}{pk^p} \sup_{\gamma>0} \frac{\gamma^p}{\int_{\Omega} \sup_{|t|\leq\gamma} F(x,t) dx}.$$

So, there exists  $\bar{\gamma} > 0$  such that

$$\lambda \frac{pk^p}{M^-} < \frac{\bar{\gamma}^p}{\int_{\Omega} \sup_{|t|\leq\bar{\gamma}} F(x,t) dx}.$$

Choosing  $r = \frac{M^-}{p} \left(\frac{\bar{\gamma}}{k}\right)^p$ , by the same notations as in the proof of Theorem 3.1, one has

$$\varphi(r) \leq \frac{\sup_{v \in \Phi^{-1}(-\infty, r)} \Psi(v)}{r} \leq \frac{pk^p}{M^-} \frac{\int_{\Omega} \sup_{|t|\leq\bar{\gamma}} F(x,t) dx}{\bar{\gamma}^p} < \frac{1}{\lambda}.$$

Hence, since  $\lambda \in (0, \frac{1}{\varphi(r)})$ , Theorem 2.1 ensures that the functional  $I_{\lambda}$  admits at least one critical point (local minima)  $u_{\lambda} \in \Phi^{-1}(-\infty, r)$  and since the critical points of the functional  $I_{\lambda}$  are the solutions of the problem (3.7) we have the conclusion.  $\square$

Now we present the following consequence of Theorem 3.3.

**Corollary 3.4.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a nonnegative continuous function and denote*

$$F(t) = \int_0^t f(\xi) d\xi \text{ for all } t \in \mathbb{R}.$$

Assume that

$$\lim_{\xi \rightarrow 0^+} \frac{F(\xi)}{|\xi|^p} = +\infty.$$

Then, for each

$$\lambda \in \Lambda = \left(0, \frac{M^-}{\text{meas}(\Omega)pk^p} \sup_{\gamma>0} \frac{\gamma^p}{F(\gamma)}\right),$$

the problem

$$\begin{cases} \Delta \left( |\Delta u|^{p-2} \Delta u \right) - [M \left( \int_{\Omega} |\nabla u|^p dx \right)]^{p-1} \Delta_p u + \rho |u|^{p-2} u = \lambda f(u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega \end{cases} \quad (3.8)$$

possesses at least one nontrivial weak solution  $u_{\lambda} \in X$  such that

$$\lim_{\lambda \rightarrow 0^+} \|u_{\lambda}\| = 0$$

and the real function

$$\begin{aligned} \lambda \rightarrow & \frac{1}{p} \int_{\Omega} |\Delta u_{\lambda}(x)|^p dx + \frac{1}{p} \widetilde{M} \left[ \int_{\Omega} |\nabla u_{\lambda}(x)|^p dx \right] + \frac{\rho}{p} \int_{\Omega} |u_{\lambda}(x)|^p dx \\ & - \lambda \int_{\Omega} F(u_{\lambda}(x)) dx \end{aligned}$$

is negative and strictly decreasing in  $\Lambda$ .

We exhibit an example in which the hypotheses of Corollary 3.4 are satisfied.

**Example 3.5.** *We consider the problem*

$$\begin{cases} \Delta \left( |\Delta u|^3 \Delta u \right) - \left[ M \left( \int_{\Omega} |\nabla u|^5 dx \right) \right]^4 \Delta_5 u + |u|^3 u = \lambda f(u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega \end{cases} \quad (3.9)$$

where  $\Omega \subset \mathbb{R}^3$  with  $\text{meas}(\Omega) = 1$ ,  $M(t) = 2 + \sin(t)$  for all  $t \in [0, +\infty)$  and

$$f(t) = \frac{1}{k^5} (5t^4 + 4e^t)$$

for all  $t \in \mathbb{R}$ . A direct calculation yields

$$F(t) = \frac{1}{k^5} (t^5 + 4e^t - 4)$$

for every  $t \in \mathbb{R}$ . Taking into account that  $m_0 = 1$ , all the assumptions of Corollary 3.4 are satisfied, and it implies that the problem (3.9) for each  $\lambda \in (0, \frac{1}{5})$ , possesses at least one nontrivial weak solution  $W^{2,5}(\Omega) \cap W_0^{1,5}(\Omega)$ , such that

$$\lim_{\lambda \rightarrow 0^+} \|u_\lambda\| = 0$$

and the real function

$$\begin{aligned} \lambda \rightarrow & \frac{1}{5} \int_{\Omega} |\Delta u_\lambda(x)|^5 dx + \frac{1}{5} \widetilde{M} \left[ \int_{\Omega} |\nabla u_\lambda(x)|^5 dx \right] + \frac{1}{5} \int_{\Omega} |u_\lambda(x)|^5 dx \\ & - \frac{\lambda}{k^5} \int_{\Omega} (u_\lambda^5(x) + 4e^{u_\lambda(x)} - 4) dx \end{aligned}$$

is negative and strictly decreasing in  $(0, \frac{1}{5})$ .

Now we want to give some remarks on our results.

**Remark 3.6.** *In Theorem 3.3 we looked for the existence of at least one critical point of the functional  $I_\lambda$  naturally associated with the problem (3.7). We note that, in general,  $I_\lambda$  can be unbounded below in  $X$ . Indeed, for example, in the case when  $f(\xi) = 1 + |\xi|^{\gamma-2}\xi$  for every  $\xi \in \mathbb{R}$  with  $\gamma > p$ , for any fixed  $u \in X \setminus \{0\}$  and  $\iota \in \mathbb{R}$ , we obtain*

$$\begin{aligned} I_\lambda(\iota u) &= \Phi(\iota u) - \lambda \int_{\Omega} F(\iota u(x)) dx \\ &\leq \iota^p \frac{M^+}{p} \|u\|^p - \lambda \iota \|u\|_{L^1(\Omega)} - \lambda \frac{\iota^\gamma}{\gamma} \|u\|_{L^\gamma(\Omega)}^\gamma \rightarrow -\infty \end{aligned}$$

as  $\iota \rightarrow +\infty$ . Hence, we can not use direct minimization to find critical points of the functional  $I_\lambda$ .

**Remark 3.7.** *We want to point out that the energy functional  $I_\lambda$  associated with the problem (3.7) is not coercive. Indeed, when  $f(t) = |t|^{s-1}$  with  $s \in (p, +\infty)$  for every  $t \in \mathbb{R}$ , for any fixed  $u \in X \setminus \{0\}$  and  $\iota \in \mathbb{R}$ , we have*

$$I_\lambda(\iota u) = \Phi(\iota u) - \lambda \int_{\Omega} F(\iota u(x)) dx \leq \iota^p \frac{M^+}{p} \|u\|^p - \lambda \frac{\iota^s}{s} \|u\|_{L^s(\Omega)}^s \rightarrow -\infty$$

as  $\iota \rightarrow +\infty$ .

**Remark 3.8.** *For fixed  $\bar{\gamma} > 0$  let*

$$\frac{\bar{\gamma}^p}{\int_{\Omega} \sup_{|t| \leq \bar{\gamma}} F(x, t) dx} > \frac{pk^p}{M^-}.$$

Then the result of Theorem 3.3 holds with  $\|u_\lambda\|_\infty \leq \bar{\gamma}$  where  $u_\lambda$  is the ensured weak solution in  $X$ .

**Remark 3.9.** If in Theorem 3.1 the function  $f(x, \xi) \geq 0$  for every  $(x, \xi) \in \Omega \times \mathbb{R}$ , then the condition  $(D_F)$  assumes the more simple form

$$\sup_{\gamma > 0} \frac{\gamma^p}{\int_{\Omega} F(x, \gamma) dx} > \frac{pk^p}{M^-}. \quad (D'_F)$$

Moreover, if the assumption

$$\limsup_{\gamma \rightarrow +\infty} \frac{\gamma^p}{\int_{\Omega} F(x, \gamma) dx} > \frac{pk^p}{M^-}$$

is verified, then the condition  $(D'_F)$  automatically holds.

**Proposition 3.10.** If in Theorem 3.3,  $f(x, 0) \neq 0$  for all  $x \in \Omega$ , then the ensured weak solution is obviously non-trivial. On the other hand, the nontriviality of the weak solution can be achieved also in the case  $f(x, 0) = 0$  for a.e.  $x \in \Omega$  requiring the extra condition at zero, that is there are a nonempty open set  $D \subseteq \Omega$  and a set  $B \subset D$  of positive Lebesgue measure such that

$$\limsup_{\xi \rightarrow 0^+} \frac{\text{ess inf}_{x \in B} F(x, \xi)}{|\xi|^p} = +\infty \quad (3.10)$$

and

$$\liminf_{\xi \rightarrow 0^+} \frac{\text{ess inf}_{x \in D} F(x, \xi)}{|\xi|^p} > -\infty. \quad (3.11)$$

*Proof.* Let  $0 < \bar{\lambda} < \lambda^*$  where

$$\lambda^* = \frac{M^-}{pk^p} \sup_{\gamma > 0} \frac{\gamma^p}{\int_{\Omega} \sup_{|t| \leq \gamma} F(x, t) dx}.$$

Then, there exists  $\bar{\gamma} > 0$  such that

$$\bar{\lambda} \frac{pk^p}{M^-} < \frac{\bar{\gamma}^p}{\int_{\Omega} \sup_{|t| \leq \bar{\gamma}} F(x, t) dx}.$$

Take  $\Phi$  and  $\Psi$  as given in the proof of Theorem 3.1. Due to Theorem 2.1, for every  $\lambda \in (0, \bar{\lambda})$  there exists a critical point of  $I_{\lambda} = \Phi - \lambda\Psi$  such that  $u_{\lambda} \in \Phi^{-1}(-\infty, r_{\lambda})$  where  $r_{\lambda} = \frac{M^-}{p} (\frac{\bar{\gamma}}{k})^p$ . In particular,  $u_{\lambda}$  is a global minimum of the restriction of  $I_{\lambda}$  to  $\Phi^{-1}(-\infty, r_{\lambda})$ . We will prove that the function  $u_{\lambda}$  cannot be trivial. Let us show that

$$\limsup_{\|u\| \rightarrow 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty. \quad (3.12)$$

Owing to the assumptions (3.10) and (3.11), we can consider a sequence  $\{\xi_n\} \subset \mathbb{R}^+$  converging to zero and two constants  $\sigma, \kappa$  (with  $\sigma > 0$ ) such that

$$\lim_{n \rightarrow +\infty} \frac{\text{ess inf}_{x \in B} F(x, \xi_n)}{|\xi_n|^p} = +\infty$$

and

$$\text{ess inf}_{x \in D} F(x, \xi) \geq \kappa |\xi|^p$$

for every  $\xi \in [0, \sigma]$ . We consider a set  $\mathcal{G} \subset B$  of positive measure and a function  $v \in X$  such that

- (k<sub>1</sub>)  $v(x) \in [0, 1]$  for every  $x \in \Omega$ ,
- (k<sub>2</sub>)  $v(x) = 1$  for every  $x \in \mathcal{G}$ ,
- (k<sub>3</sub>)  $v(x) = 0$  for every  $x \in \Omega \setminus D$ .

Hence, fix  $M > 0$  and consider a real positive number  $\eta$  with

$$M < \frac{\eta \text{ meas}(\mathcal{G}) + \kappa \int_{D \setminus \mathcal{G}} |v(x)|^p dx}{\frac{M^+}{p} \|v\|^p}.$$



Then, there is  $n_0 \in \mathbb{N}$  such that  $\xi_n < \sigma$  and

$$\operatorname{ess\,inf}_{x \in B} F(x, \xi_n) \geq \eta |\xi_n|^p$$

for every  $n > n_0$ . Now, for every  $n > n_0$ , by considering the properties of the function  $v$  (that is  $0 \leq \xi_n v(x) < \sigma$  for  $n$  large enough), by (3.3), one has

$$\begin{aligned} \frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} &= \frac{\int_{\mathcal{G}} F(x, \xi_n) dx + \int_{D \setminus \mathcal{G}} F(x, \xi_n v(x)) dx}{\Phi(\xi_n v)} \\ &> \frac{\eta \operatorname{meas}(\mathcal{G}) + \kappa \int_{D \setminus \mathcal{G}} |v(x)|^p dx}{\frac{M^+}{p} \|v\|^p} > M. \end{aligned}$$

Since  $M$  could be arbitrarily large, we get

$$\lim_{n \rightarrow \infty} \frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} = +\infty,$$

from which (3.12) clearly follows. So, there exists a sequence  $\{w_n\} \subset X$  strongly converging to zero such that, for  $n$  large enough,  $w_n \in \Phi^{-1}(-\infty, r)$  and

$$I_\lambda(w_n) = \Phi(w_n) - \lambda \Psi(w_n) < 0.$$

Since  $u_\lambda$  is a global minimum of the restriction of  $I_\lambda$  to  $\Phi^{-1}(-\infty, r)$ , we obtain

$$I_\lambda(u_\lambda) < 0, \tag{3.13}$$

so that  $u_\lambda$  is not trivial.  $\square$

**Remark 3.11.** By using (3.13), we without difficulty observe that the map

$$(0, \lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda) \tag{3.14}$$

is negative. Also, one has

$$\lim_{\lambda \rightarrow 0^+} \|u_\lambda\| = 0.$$

Indeed, taking into account the fact that  $\Phi$  is coercive and for every  $\lambda \in (0, \lambda^*)$  the solution  $u_\lambda \in \Phi^{-1}(-\infty, r)$ , one has that there exists a positive constant  $L$  such that  $\|u_\lambda\| \leq L$  for every  $\lambda \in (0, \lambda^*)$ . After that, it is easy to see that there exists a positive constant  $N$  such that

$$\left| \int_{\Omega} f(x, u_\lambda(x)) u_\lambda(x) dx \right| \leq N \|u_\lambda\| \leq NL \tag{3.15}$$

for every  $\lambda \in (0, \lambda^*)$ . Since  $u_\lambda$  is a critical point of  $I_\lambda$ , we have  $I'_\lambda(u_\lambda)(v) = 0$  for every  $v \in X$  and every  $\lambda \in (0, \lambda^*)$ . In particular  $I'_\lambda(u_\lambda)(u_\lambda) = 0$ , that is,

$$\Phi'(u_\lambda)(u_\lambda) = \lambda \int_{\Omega} f(x, u_\lambda(x)) u_\lambda(x) dx \tag{3.16}$$

for every  $\lambda \in (0, \lambda^*)$ . Then, since

$$0 \leq M^- \|u_\lambda\|^p \leq \Phi'(u_\lambda)(u_\lambda),$$

by considering (3.16), it follows that

$$0 \leq M^- \|u_\lambda\|^p \leq \lambda \int_{\Omega} f(x, u_\lambda(x)) u_\lambda(x) dx \tag{3.17}$$

for any  $\lambda \in (0, \lambda^*)$ . Letting  $\lambda \rightarrow 0^+$ , by (3.17) together with (3.15) we get

$$\lim_{\lambda \rightarrow 0^+} \|u_\lambda\| = 0.$$

Then, we have obviously the desired conclusion. At last, we have to show that the map

$$\lambda \mapsto I_\lambda(u_\lambda)$$

is strictly decreasing in  $(0, \lambda^*)$ . For our goal we see that for any  $u \in X$ , one has

$$I_\lambda(u) = \lambda \left( \frac{\Phi(u)}{\lambda} - \Psi(u) \right). \quad (3.18)$$

Now, let us fix  $0 < \lambda_1 < \lambda_2 < \lambda^*$  and let  $u_{\lambda_i}$  be the global minimum of the functional  $I_{\lambda_i}$  restricted to  $\Phi(-\infty, r)$  for  $i = 1, 2$ . Also, set

$$m_{\lambda_i} = \left( \frac{\Phi(u_{\lambda_i})}{\lambda_i} - \Psi(u_{\lambda_i}) \right) = \inf_{v \in \Phi^{-1}(-\infty, r)} \left( \frac{\Phi(v)}{\lambda_i} - \Psi(v) \right)$$

for every  $i = 1, 2$ . Clearly, (3.14) together with (3.18) and the positivity of  $\lambda$  implies that

$$m_{\lambda_i} < 0 \text{ for } i = 1, 2. \quad (3.19)$$

Moreover,

$$m_{\lambda_2} \leq m_{\lambda_1}, \quad (3.20)$$

due to the fact that  $0 < \lambda_1 < \lambda_2$ . Then, by (3.18)-(3.20) and again by the fact that  $0 < \lambda_1 < \lambda_2$ , we get that

$$I_{\lambda_2}(u_{\lambda_2}) = \lambda_2 m_{\lambda_2} \leq \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = I_{\lambda_1}(u_{\lambda_1}),$$

so that the map  $\lambda \mapsto I_\lambda(u_\lambda)$  is strictly decreasing in  $\lambda \in (0, \lambda^*)$ . The arbitrariness of  $\lambda < \lambda^*$  shows that  $\lambda \mapsto I_\lambda(u_\lambda)$  is strictly decreasing in  $(0, \lambda^*)$ .

In the following, we give a direct application of Theorem 3.3, Proposition 3.10 and Remark 3.11.

**Example 3.12.** Consider the problem

$$\begin{cases} \Delta \left( |\Delta u|^{p-2} \Delta u \right) - [M \left( \int_\Omega |\nabla u|^p dx \right)]^{p-1} \Delta_p u + \rho |u|^{p-2} u \\ = \lambda (\alpha(x) |u|^{r-2} u + \sigma(x) |u|^{s-2} u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega \end{cases} \quad (3.21)$$

where  $r \in (1, p)$ ,  $s \in (p, +\infty)$  and  $\alpha, \sigma : \Omega \rightarrow \mathbb{R}$  are two continuous positive functions. Thanks to Theorem 3.3 taking Proposition 3.10 and Remark 3.11 into account, the problem (3.21) possesses at least one nontrivial weak solution  $u_\lambda \in X$  such that

$$\lim_{\lambda \rightarrow 0^+} \|u_\lambda\| = 0$$

and the real function

$$\begin{aligned} \lambda \rightarrow & \frac{1}{p} \int_\Omega |\Delta u_\lambda(x)|^p dx + \frac{1}{p} \widetilde{M} \left[ \int_\Omega |\nabla u_\lambda(x)|^p dx \right] + \frac{\rho}{p} \int_\Omega |u_\lambda(x)|^p dx \\ & - \lambda \left( \frac{\|\alpha\|_{L^1(\Omega)} \|u_\lambda\|_{L^r(\Omega)}^r}{r} + \frac{\|\sigma\|_{L^1(\Omega)} \|u_\lambda\|_{L^s(\Omega)}^s}{s} \right) \end{aligned}$$

is negative and strictly decreasing in  $\left( 0, \frac{M^-}{pk^p} \sup_{\gamma > 0} \frac{\gamma^p}{\int_\Omega \sup_{|t| \leq \gamma} F(x, t) dx} \right)$ .

**Remark 3.13.** We observe that Theorem 3.3 is a bifurcation result in the sense that the pair  $(0, 0)$  belongs to the closure of the set

$$\{(u_\lambda, \lambda) \in X \times (0, +\infty) : u_\lambda \text{ is a nontrivial weak solution of (3.7)}\}$$

in  $X \times \mathbb{R}$ . Indeed, by Theorem 3.3 we have that

$$\|u_\lambda\| \rightarrow 0 \quad \text{as } \lambda \rightarrow 0.$$

Hence, there exist two sequences  $\{u_j\}$  in  $X$  and  $\{\lambda_j\}$  in  $\mathbb{R}^+$  (here  $u_j = u_{\lambda_j}$ ) such that

$$\lambda_j \rightarrow 0^+ \quad \text{and} \quad \|u_j\| \rightarrow 0,$$

as  $j \rightarrow +\infty$ . Moreover, we emphasize that due to the fact that the map

$$(0, \lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda)$$

is strictly decreasing, for every  $\lambda_1, \lambda_2 \in (0, \lambda^*)$ , with  $\lambda_1 \neq \lambda_2$ , the solutions  $u_{\lambda_1}$  and  $u_{\lambda_2}$  ensured by Theorem 3.3 are different.

## References

1. G.A. Afrouzi, A. Hadjian, G. Molica Bisci, *Some remarks for one-dimensional mean curvature problems through a local minimization principle*, Adv. Nonlinear Anal. **2**, 427-441, (2013).
2. C.O. Alves, F.S.J.A. Corrêa, T.F. Ma, *Positive solutions for a quasilinear elliptic equations of Kirchhoff type*, Comput. Math. Appl. **49**, 85-93, (2005).
3. G. Autuori, F. Colasuonno, P. Pucci, *Blow up at infinity of solutions of polyharmonic Kirchhoff systems*, Complex Var. Elliptic Eqs. **57**, 379-395, (2012).
4. G. Autuori, F. Colasuonno, P. Pucci, *Lifespan estimates for solutions of polyharmonic Kirchhoff systems*, Math. Mod. Meth. Appl. Sci. **22**, 1150009 [36 pages], (2012).
5. G. Autuori, F. Colasuonno, P. Pucci, *On the existence of stationary solutions for higher-order  $p$ -Kirchhoff problems*, Commun. Contemp. Math. **16**, 1450002 [43 pages], (2014).
6. J. M. Ball, *Stability theory for an extensible beam*, J. Differ. Equ. **14**, 399-418, (1973).
7. G. Bonanno, G. Molica Bisci, *Infinitely many solutions for a boundary value problem with discontinuous nonlinearities*, Bound. Value Probl. **2009**, 1-20, (2009).
8. A. Cabada, J.A. Cid, L. Sanchez, *Positivity and lower and upper solutions for fourth-order boundary value problems*, Nonlinear Anal. TMA **67**, 1599-1612, (2007).
9. P. Candito, R. Livrea, *Infinitely many solutions for a nonlinear Navier boundary value problem involving the  $p$ -biharmonic*, Studia Univ. "Babeş-Bolyai", Mathematica, Volume LV, Number 4, December (2010).
10. M. Chipot, B. Lovat, *Some remarks on non local elliptic and parabolic problems*, Nonlinear Anal. TMA **30**, 4619-4627, (1997).
11. F. Colasuonno, P. Pucci, *Multiplicity of solutions for  $p(x)$ -polyharmonic elliptic Kirchhoff equations*, Nonlinear Anal. TMA **74**, 5962-5974, (2011).
12. M. Coti Zelati, C. Giorgi, V. Pata, *Steady states of the hinged extensible beam with external load*, Math. Models Methods Appl. Sci. **20**, 43-58, (2010).
13. L. Duan, L. Huang, *Existence of nontrivial solutions for Kirchhoff-type variational inclusion system in  $R^N$* , Appl. Math. Comput. **235**, 174-186, (2014).
14. R.W. Dickey, *Dynamic stability of equilibrium states of the extensible beam*, Proc. Amer. Math. Soc. **41**, 94-102, (1973).
15. R.W. Dickey, *Free vibrations and dynamic buckling of the extensible beam*, J. Math. Anal. Appl. **29**, 443-454, (1970).
16. M. Ferrara, S. Khademloo, S. Heidarkhani, *Multiplicity results for perturbed fourth-order Kirchhoff type elliptic problems*, Appl. Math. Comput. **234**, 316-325, (2014).
17. M. Ferrara, G. Molica Bisci, *Existence results for elliptic problems with Hardy potential*, Bull. Sci. Math. **138**, 846-859, (2014).
18. M. Galewski, G. Molica Bisci, *Existence results for one-dimensional fractional equations*, Math. Meth. Appl. Sci. **39**, 1480-1492, (2016).

19. J.R. Graef, S. Heidarkhani, L. Kong, *A variational approach to a Kirchhoff-type problem involving two parameters*, Results. Math. **63**, 877-889, (2013).
20. J.R. Graef, S. Heidarkhani, L. Kong, *Multiple solutions for a class of  $(p_1, \dots, p_n)$ -biharmonic systems*, Commun. Pure Appl. Anal. (CPAA) **12**, 1393-1406, (2013).
21. X. He, W. Zou, *Infinitely many positive solutions for Kirchhoff-type problems*, Nonlinear Anal. TMA **70**, 1407-1414, (2009).
22. S. Heidarkhani, *Existence of non-trivial solutions for systems of  $n$  fourth order partial differential equations*, Math. Slovaca **64**, 1249-1266, (2014).
23. S. Heidarkhani, G.A. Afrouzi, M. Ferrara, G. Caristi, S. Moradi, *Existence results for impulsive damped vibration systems*, Bull. Malays. Math. Sci. Soc. **41**, 1409-1428, (2018).
24. S. Heidarkhani, G.A. Afrouzi, S. Moradi, G. Caristi, B. Ge, *Existence of one weak solution for  $p(x)$ -biharmonic equations with Navier boundary conditions*, Zeitschrift fuer Angewandte Mathematik und Physik **67**, 1-13, (2016).
25. S. Heidarkhani, M. Ferrara, G.A. Afrouzi, G. Caristi, S. Moradi, *Existence of solutions for Dirichlet quasilinear systems including a nonlinear function of the derivative*, Electronic J. Diff. Equ., Vol. **2016**, No. 56, pp. 1-12, (2016).
26. S. Heidarkhani, Y. Tian, C.-L. Tang, *Existence of three solutions for a class of  $(p_1, \dots, p_n)$ -biharmonic systems with Navier boundary conditions*, Ann. Polon. Math. **104**, 261-277, (2012).
27. G. Kirchhoff, *Vorlesungen uber mathematische Physik: Mechanik*, Teubner, Leipzig (1883).
28. L. Li, C.-L. Tang, *Existence of three solutions for  $(p, q)$ -biharmonic systems*, Nonlinear Anal. TMA **73**, 796-805, (2010).
29. C. Li, C.-L. Tang, *Three solutions for a Navier boundary value problem involving the  $p$ -biharmonic*, Nonlinear Anal. TMA **72**, 1339-1347, (2010).
30. J. Liu, S.X. Chen, X. Wu, *Existence and multiplicity of solutions for a class of fourth-order elliptic equations in  $\mathbb{R}^N$* , J. Math. Anal. Appl. **395**, 608-615, (2012).
31. H. Liu, N. Su, *Existence of three solutions for a  $p$ -biharmonic problem*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **15**, 445-452, (2008).
32. T.F. Ma, *Remarks on an elliptic equation of Kirchhoff type*, Nonlinear Anal. TMA **63**, e1967-e1977, (2005).
33. A. Mao, Z. Zhang, *Sign-changing and multiple solutions of Kirchhoff type problems without the P.S. condition*, Nonlinear Anal. TMA **70**, 1275-1287, (2009).
34. M. Massar, E.M. Hssini, N. Tsouli, M. Talbi, *Infinitely many solutions for a fourth-order Kirchhoff type elliptic problem*, J. Math. Comput. Sci. **8**, 33-51, (2014).
35. G. Molica Bisci, V. Rădulescu, *Bifurcation analysis of a singular elliptic problem modelling the equilibrium of anisotropic continuous media*, Topol. Methods Nonlinear Anal. **45**, 493-508, (2015).
36. G. Molica Bisci, D. Repovš, *Multiple solutions of  $p$ -biharmonic equations with Navier boundary conditions*, Complex Var. Elliptic Equ. **59**, 271-284, (2014).
37. K. Perera, Z.T. Zhang, *Nontrivial solutions of Kirchhoff-type problems via the Yang index*, J. Differ. Equ. **221**, 246-255, (2006).
38. E.L. Reiss, B.J. Matkowsky, *Nonlinear dynamic buckling of a compressed elastic column*, Quart. Appl. Math. **29**, 245-260, (1971).
39. B. Ricceri, *A general variational principle and some of its applications*, J. Comput. Appl. Math. **113**, 401-410, (2000).
40. B. Ricceri, *On an elliptic Kirchhoff-type problem depending on two parameters*, J. Global Optim. **46**, 543-549, (2010).
41. S.M. Shahruz, S.A. Parasurama, *Suppression of vibration in the axially moving Kirchhoff string by boundary control*, J. Sound and Vibration **214**, 567-575, (1998).
42. F. Wang, Y. An, *Existence and multiplicity of solutions for a fourth-order elliptic equation*, Bound. Value Probl. **2012**, 1-9, (2012).
43. F. Wang, M. Avci, Y. An, *Existence of solutions for fourth order elliptic equations of Kirchhoff-type*, J. Math. Anal. Appl. **409**, 140-146, (2014).
44. S. Woinowsky-Krieger, *The effect of an axial force on the vibration of hinged bars*, J. Appl. Mech. **17**, 35-36, (1950).
45. J. Zhang, *The critical Neumann problem of Kirchhoff type*, Appl. Math. Comput. **274**, 519-530, (2016).
46. J. Zhang, X. Tang, W. Zhang, *Existence of multiple solutions of Kirchhoff type equation with sign-changing potential*, Appl. Math. Comput. **242**, 491-499, (2014).

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