Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 ON LINE
SPM: www.spm.uem.br/bspm

(3s.) **v. 2022 (40)** : 1–7.
ISSN-0037-8712 IN PRESS
doi:10.5269/bspm.43345

Algebraic Extension of $\mathcal{A}_n^*$ Operator

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ABSTRACT: $T \in L(H_1 \oplus H_2)$ is said to be an algebraic extension of a $\mathcal{A}_n^*$ operator if

$$T = \begin{pmatrix} T_1 & T_2 \\ O & T_3 \end{pmatrix}$$

is an operator matrix on $H_1 \oplus H_2$, where T_1 is a $\mathcal{A}_n^*$ operator and T_3 is a algebraic.

In this paper, we study basic and spectral properties of an algebraic extension of a $\mathcal{A}_n^*$ operator. We show that every algebraic extension of a $\mathcal{A}_n^*$ operator has SVEP, is polaroid and satisfies Weyl's theorem.

Key Words: Algebraic extension $\mathcal{A}_n^*$ operator, SVEP, polaroid, Weyl's theorem.

Contents

1. Introduction

Throughout this paper, let H and K be infinite dimensional complex Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$. We denote by $L(H, K)$ the set of all bounded operators from H into K. To simplify, we put $L(H) := L(H, H)$. For $T \in L(H)$, we denote by $\ker(T)$ the null space and by $T(H)$ the range of T. The null operator and the identity on H will be denoted by O and I, respectively. If T is an operator, then T^* is its adjoint, and $\|T\| = \|T^*\|$. We shall denote the set of all complex numbers by $\mathbb{C}$, the set of all non$-$negative integers by $\mathbb{N}$ and the complex conjugate of a complex number μ by $\overline{\mu}$. The closure of a set M will be denoted by $\overline{M}$ and we shall henceforth shorten $T - \mu I$ to $T - \mu$. An operator $T \in L(H)$, is a positive operator, $T \geq O$, if $\langle Tx, x \rangle \geq 0$ for all $x \in H$. We write $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ for the spectral radius. It is well known that $r(T) \leq \|T\|$, for every $T \in L(H)$. The operator T is called a normaloid operator if $r(T) = \|T\|$.

Let $Hol(\sigma(T))$ be the space of all analytic functions in an open neighborhood of $\sigma(T)$. We say that $T \in L(H)$ has the single valued extension property at $\lambda_0 \in \mathbb{C}$, (SVEP for short), if for every open neighborhood U of λ_0 the only analytic function $f : U \to \mathbb{C}$ which satisfies equation $(T - \lambda)f(\lambda) = 0$ for all $\lambda \in U$, is the constant function $f \equiv 0$. An operator $T \in L(H)$ has SVEP at every point of the resolvent $\rho(T) = \mathbb{C} \setminus \sigma(T)$, so T has SVEP if T has SVEP at every $\lambda \in \sigma(T)$. Every operator T has SVEP at an isolated point of the spectrum.

$T \in L(H)$ is said to be analytic if there exists a non-constant analytic function f on a neighborhood of $\sigma(T)$ such that $f(T) = O$. We say that $T \in L(H)$ is algebraic if there is a non-constant polynomial p such that $p(T) = O$.

An operator T is algebraic if and only if $\sigma(T)$ is a finite set consisting of the poles of the resolvent of T (*i.e.,* if and only if $\sigma(T)$ is finite set and T is polaroid), [4, Theorem 3.83]. If T is an algebraic operator, then T has SVEP.

If an operator $T \in L(H)$ is analytic, then $f(T) = O$ for some non-constant analytic function f on a neighborhood D of $\sigma(T)$. Since f cannot have infinitely many zeros in D, we write $f(z) = G(z)p(z)$,

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2010 *Mathematics Subject Classification:* 47B20, 47A10.
Submitted June 18, 2018. Published November 07, 2018

where G is an analytic function which does not vanish on D and p is a non-constant polynomial with zeros in D. By Riesz functional calculus, $G(T)$ is invertible and then $p(T) = O$, which means that T is algebraic, [5, Lemma 3.2].

An operator $T \in L(H)$ is said to be isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T, while an operator $T \in L(H)$ is said to be polaroid if every isolated point of $\sigma(T)$ is a pole of the resolvent of T. In general, if T is polaroid, then T is isoloid. However, the converse is not true.

For an operator $T \in L(H)$, as usual, $|T| = (T^*T)^{\frac{1}{2}}$.

Definition 1.1. *An operator $T \in L(H)$, is said to belong to $k-$quasi class $\mathcal{A}_n^*$ operator if*

$$T^{*k}\left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2\right)T^k \geq O,$$

for non-negative integers n and k.

If $k = 0$ then 0-quasi class $\mathcal{A}_n^*$ operators coincides with class $\mathcal{A}_n^*$ operators:

$$|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2 \geq O.$$

If $n = 1$ and $k = 1$ then $1-$quasi class $\mathcal{A}_1^*$ operators coincides with $\mathcal{Q}(\mathcal{A}^*)$ operators:

$$T^*(|T^2| - |T^*|^2)T \geq O.$$

Lemma 1.2. *[7, Hansen Inequality] If $A, B \in L(H)$, satisfying $A \geq O$ and $\|B\| \leq 1$, then*

$$(B^*AB)^\delta \geq B^*A^\delta B \text{ for all } \delta \in (0, 1].$$

2. Algebraic extension of a $\mathcal{A}_n^*$ operator

Definition 2.1. *For a positive integer n, T is f-quasi class $\mathcal{A}_n^*$ operators, if*

$$f(T)^*\left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2\right)f(T) \geq O,$$

for some non-constant analytic function f on some neighborhood of $\sigma(T)$.

If f is some non-constant polynomial p, then T is p-quasi class $\mathcal{A}_n^*$ operator. If $p(z) = z^k$, then T is $k-$quasi class $\mathcal{A}_n^*$ operator.

Definition 2.2. *$T \in L(H_1 \oplus H_2)$ is said to be an algebraic extension of a $\mathcal{A}_n^*$ operator if*

$$T = \begin{pmatrix} T_1 & T_2 \\ O & T_3 \end{pmatrix}$$

is an operator matrix on $H_1 \oplus H_2$, where T_1 is a $\mathcal{A}_n^*$ operator and T_3 is a algebraic.

Theorem 2.3. *Let $T \in L(H)$ be an f-quasi class $\mathcal{A}_n^*$ operator and M be an invariant subspace for T. Then the restriction $T|_M$ is a p-quasi class $\mathcal{A}_n^*$ operator.*

Proof. Let $T \in L(H)$ be an f-quasi class $\mathcal{A}_n^*$ operator. There exists non-constant analytic functions f on a neighborhood of $\sigma(T)$ such that

$$f(T)^*\left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2\right)f(T) \geq O.$$

Since M is a T-invariant subspace, we can write

$$T = \begin{pmatrix} T_1 & T_2 \\ O & T_3 \end{pmatrix} \text{ on } H = M \oplus M^\perp,$$

where $T_1 = T|_M$.

Since $(T^{*(n+1)}T^{n+1})^{\frac{1}{n+1}} \geq O$, from [6] we can set

$$|T^{n+1}|^{\frac{2}{n+1}} = (T^{*(n+1)}T^{n+1})^{\frac{1}{n+1}} = \begin{pmatrix} B & C \\ C^* & D \end{pmatrix},$$

where $B \geq O$, $D \geq O$ and $C = B^{\frac{1}{2}}SD^{\frac{1}{2}}$ for some contraction $S : M^{\perp} \to M$. Then

$$|T^{n+1}|^2 = \begin{pmatrix} B & C \\ C^* & D \end{pmatrix}^{n+1} = \begin{pmatrix} B^{n+1} + Z & * \\ * & * \end{pmatrix},$$

where $Z \geq O$.

Since

$$|T^{n+1}|^2 = \begin{pmatrix} T_1^{*(n+1)}T_1^{n+1} & * \\ ** & *** \end{pmatrix} \tag{2.1}$$

then $T_1^{*(n+1)}T_1^{n+1} = B^{n+1} + Z \geq B^{n+1}$. Therefore,

$$|T_1^{n+1}|^{\frac{2}{n+1}} = (T_1^{*(n+1)}T_1^{n+1})^{\frac{1}{n+1}} \geq B. \tag{2.2}$$

Also, since $|T^*|^2 = TT^* = \begin{pmatrix} T_1T_1^* + T_2T_2^* & * \\ * & * \end{pmatrix}$ we have

$$O \leq f(T)^* \left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2\right) f(T) = f(T)^* \begin{pmatrix} B - T_1T_1^* - T_2T_2^* & * \\ * & * \end{pmatrix} f(T) =$$

$$G(T)^* \begin{pmatrix} p(T_1)^*(B - T_1T_1^* - T_2T_2^*)p(T_1) & * \\ * & * \end{pmatrix} G(T),$$

by Riesz functional calculus.

Since $G(T)$ is invertible, from [6] we have

$$p(T_1)^*(B - T_1T_1^*)p(T_1) \geq O.$$

Then, from relations (2.2) we have

$$p(T_1)^*(|T_1^{n+1}|^{\frac{2}{n+1}} - T_1T_1^*)p(T_1) \geq p(T_1)^*(B - T_1T_1^*)p(T_1) \geq O.$$

So T_1 is a p-quasi class $\mathcal{A}_n^*$ operator. □

Theorem 2.4. *If T is a f-quasi class $\mathcal{A}_n^*$ operator and $f(T)$ does not have a dense range, then*

$$T = \begin{pmatrix} T_1 & T_2 \\ O & T_3 \end{pmatrix} \text{ on } H = \overline{f(T)(H)} \oplus \ker((f(T)^*),$$

$(T_1^{*(n+1)}T_1^{n+1})^{\frac{1}{n+1}} - T_1T_1^* \geq T_2T_2^*$, *$T_3$ is algebraic and $\sigma(T) = \sigma(T_1) \cup \sigma(T_3)$.*

Proof. Let Q be the orthogonal projection onto $\overline{f(T)(H)}$. Since T is f-quasi class $\mathcal{A}_n^*$ then

$$Q(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2)Q \geq O.$$

Hence, by Hansen inequality we have

$$\begin{aligned} &(T_1^{*(n+1)}T_1^{n+1})^{\frac{1}{n+1}} \\ &\quad = (QT^{*(n+1)}T^{n+1}Q)^{\frac{1}{n+1}} \geq Q(T^{*(n+1)}T^{n+1})^{\frac{1}{n+1}}Q \\ &\quad \geq Q|T^*|^2Q = T_1T_1^* + T_2T_2^*. \end{aligned}$$

On the other hand, for any $x = (x_1, x_2) \in H$ we have

$$\langle f(T_3)x_2, x_2\rangle = \langle f(T)(I - Q)x, (I - Q)x\rangle = \langle (I - Q)x, f(T)^*x\rangle = 0,$$

therefore, T_3 is an algebraic operator.

Since $\sigma(T_3)$ is a finite set, $\sigma(T_1) \cap \sigma(T_3)$ is also finite, which implies $\sigma(T_1) \cap \sigma(T_3)$ has no interior points. By [10, Corollary 8], $\sigma(T) = \sigma(T_1) \cup \sigma(T_3)$. □

Proposition 2.5. *Let T be f-quasi class $\mathcal{A}_n^*$ operator. If $T|_{\overline{f(T)(H)}}$ is a normal operator, then $\overline{f(T)(H)}$ reduces T.*

Proof. We may assume T is a f-quasi class $\mathcal{A}_n^*$ operator and $\overline{f(T)(H)}$ is not dense. Then

$$T = \begin{pmatrix} T_1 & T_2 \\ O & T_3 \end{pmatrix} \text{ on } H = \overline{f(T)(H)} \oplus \ker((f(T)^*),$$

where $T_1 = T|_{\overline{f(T)(H)}}$ is a normal operator. Let Q be the orthogonal projection onto $\overline{f(T)(H)}$. Since T is f-quasi class $\mathcal{A}_n^*$ then

$$Q\left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2\right)Q \geq O.$$

By Hansen inequality, relations (2.1) and the normality of T_1 we have

$$\begin{aligned} \begin{pmatrix} T_1T_1^* & T_2T_2^* \\ O & O \end{pmatrix} & \\ &= Q|T^*|^2Q \leq Q|T^{n+1}|^{\frac{2}{n+1}}Q \leq \left(Q|T^{n+1}|^2Q\right)^{\frac{1}{n+1}} \\ &= \begin{pmatrix} |T_1^{n+1}|^2 & O \\ O & O \end{pmatrix}^{\frac{2}{n+1}} = \begin{pmatrix} T_1T_1^* & O \\ O & O \end{pmatrix} \end{aligned}$$

So $T_2 = O$ and $\overline{f(T)(H)}$ is reduced of T.

□

Theorem 2.6. *If $T \in L(H,K)$ is an algebraic extension class $\mathcal{A}_n^*$ operator, then T has SVEP.*

Proof. Suppose that $T \in L(H,K)$ is algebraic extension class $\mathcal{A}_n^*$. Then $T = \begin{pmatrix} T_1 & T_2 \\ O & T_3 \end{pmatrix}$ where T_1 is class $\mathcal{A}_n^*$ operator and T_3 is algebraic.

Assume $(T-z)f(z) = 0$ and put $f(z) = f_1(z) \oplus f_2(z)$ on $H \oplus K$. Then

$$\begin{pmatrix} T_1 - z & T_2 \\ O & T_3 - z \end{pmatrix}\begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} = \begin{pmatrix} (T_1 - z)f_1(z) + T_2f_2(z) \\ (T_3 - z)f_2(z) \end{pmatrix} = 0.$$

Since T_3 is algebraic then T_3 has SVEP, so $f_2(z) = 0$. We have $(T_1 - z)f_1(z) = 0$ and since T_1 is class $\mathcal{A}_n^*$ then T_1 have SVEP, [9, Corollary 3.9]. Therefore $f_1(z) = 0$ and $f(z) = 0$. Consequently, T has SVEP. □

For $T \in L(H)$, the smallest nonnegative integer p such that $\ker(T^p) = \ker(T^{p+1})$ is called the ascent of T and is denoted by $p(T)$. If no such integer exists, we set $p(T) = \infty$. We say that $T \in L(H)$ is of finite ascent (finitely ascensive) if $p(T) < \infty$. For $T \in L(H)$, the smallest nonnegative integer q, such that $T^q(H) = T^{q+1}(H)$, is called the descent of T and is denoted by $q(T)$. If no such integer exists, we set $q(T) = \infty$. We say that $T \in L(H)$ is of finite descent if $q(T - \lambda) < \infty$, for all $\lambda \in \mathbb{C}$. Moreover, $0 < p(T - \lambda) = q(T - \lambda) < \infty$ precisely when λ is a pole of the resolvent of T; see [8, Proposition 50.2].

Lemma 2.7. *If $T \in L(H,K)$ is an algebraic extension class $\mathcal{A}_n^*$ operator, then T is polaroid.*

Proof. Let be $\lambda \in \text{iso}\sigma(T)$. Then $\lambda \in \sigma(T_1) \cup \sigma(T_3)$, so $\lambda \in \sigma(T_1)$ or $\lambda \in \sigma(T_3)$.

If $\lambda \in \sigma(T_3)$, then $\lambda \in \text{iso}\sigma(T_3)$. Since T_3 is polaroid, then λ is pole of resolvente T_3, consequently

$$0 < p(T_3 - \lambda) = q(T_3 - \lambda) < \infty. \tag{2.3}$$

If $\lambda \in \sigma(T_1)$ and $\lambda \notin \sigma(T_3)$, then $\lambda \in \text{iso}\sigma(T_1)$. From [9, Lemma 2.8], T is polaroid and

$$0 < p(T_1 - \lambda) = q(T_1 - \lambda) < \infty. \tag{2.4}$$

From inequalities $p(T - \lambda) \leq p(T_1 - \lambda) + p(T_3 - \lambda)$, $q(T - \lambda) \leq q(T_1 - \lambda) + q(T_3 - \lambda)$ and relations (2.3) and (2.4) we have $0 < p(T - \lambda) = q(T - \lambda) < \infty$. Consequently λ is a pole of the resolvente of T.

□

Corollary 2.8. *If $T \in L(H,K)$ is an algebraic extension class $\mathcal{A}_n^*$ operator, then T is isoloid.*

Proof. From above Lemma. □

3. Weyl's theorem

We write $\alpha(T) = \dim\ker(T)$, $\beta(T) = \dim(H/T(H))$. An operator $T \in L(H)$ is called an upper semi-Fredholm, if it has a closed range and $\alpha(T) < \infty$, while T is called a lower semi-Fredholm if $\beta(T) < \infty$. However, T is called a semi-Fredholm operator if T is either an upper or a lower semi−Fredholm, and T is said to be a Fredholm operator if it is both an upper and a lower semi-Fredholm. If $T \in L(H)$ is semi-Fredholm, then the index is defined by

$$\text{ind}(T) = \alpha(T) - \beta(T).$$

An operator $T \in L(H)$ is said to be upper semi-Weyl operator if it is upper semi−Fredholm and $\text{ind}(T) \leq 0$, while $T \in L(H)$ is said to be lower semi-Weyl operator if it is lower semi-Fredholm and $\text{ind}(T) \geq 0$. An operator is said to be Weyl operator if it is Fredholm of index zero. The Weyl spectrum and the essential approximate spectrum are defined by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$$

and

$$\sigma_{uw}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not upper semi-Weyl}\}.$$

An operator $T \in L(H)$ is said to be upper semi-Browder operator, if it is upper semi−Fredholm and $p(T) < \infty$. An operator $T \in L(H)$ is said to be lower semi-Browder operator, if it is lower semi-Fredholm and $q(T) < \infty$. An operator $T \in L(H)$ is said to be Browder operator, if it is Fredholm of finite ascent and descent. The Browder spectrum and the upper semi-Browder spectrum are defined by

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\}$$

and

$$\sigma_{ub}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not upper semi-Browder}\}.$$

For $T \in L(H)$ we will denote $p_{00}(T)$ the set of all poles of finite rank of T. We have $\sigma(T) \setminus \sigma_b(T) = p_{00}(T)$ and we say that T satisfies Browder's theorem if

$$\sigma_w(T) = \sigma_b(T) \text{ or } \sigma(T) \setminus \sigma_w(T) = p_{00}(T).$$

For $T \in L(H)$ we write $\pi_{00}(T) = \{\lambda \in iso\sigma(T) : 0 < \alpha(T - \lambda) < \infty\}$ for the isolated eigenvalues of finite multiplicity. We say that T satisfies Weyl's theorem, if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T).$$

Let $\pi_{00}^a(T) = \{\lambda \in iso\sigma_a(T) : 0 < \alpha(T-\lambda) < \infty\}$ be the set of all eigenvalues of T of finite multiplicity, which are isolated in the approximate point spectrum. We say that T satisfies a-Weyl's theorem, if

$$\sigma_a(T) \setminus \sigma_{uw}(T) = \pi_{00}^a(T).$$

We will denote $p_{00}^a(T)$ the set of all left poles of finite rank of T. We have

$$\sigma_a(T) \setminus \sigma_{ub}(T) = p_{00}^a(T)$$

and we say that T satisfies a-Browder's theorem, if

$$\sigma_{uw}(T) = \sigma_{ub}(T) \text{ or } \sigma_a(T) \setminus \sigma_{uw}(T) = p_{00}^a(T).$$

Lemma 3.1. *If T is algebraic extension k-quasi class $\mathcal{A}_n^*$, then $\sigma_w(f(T)) = f(\sigma_w(T))$ for all $f \in Hol(\sigma(T))$.*

Proof. The inclusion $f(\sigma_w(T)) \subseteq \sigma_w(f(T))$ holds for any operator. Since T is an algebraic extension k-quasi class $\mathcal{A}_n^*$, operator, T has SVEP, then from [1, Theorem 4.19] holds $\sigma_w(f(T)) \subseteq f(\sigma_w(T))$. □

Theorem 3.2. *If T is an algebraic extension k-quasi class $\mathcal{A}_n^*$, then Weyl's theorem holds for $f(T)$ for every $f \in Hol(\sigma(T))$.*

Proof. Suppose T is an algebraic extension k-quasi class $\mathcal{A}_n^*$. From Lemma 2.7 we have T is polaroid. Since T is an algebraic extension k-quasi class $\mathcal{A}_n^*$, therefore T has SVEP by Theorem 2.6. Then, from [2, Theorem 3.3], T satisfies Weyl's theorem.

Since T is isoloid from [11] we have

$$f(\sigma(T) \setminus \pi_{00}(T)) = \sigma(f(T)) \setminus \pi_{00}(f(T)).$$

Then, by Lemma 3.1 we have

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T) \setminus \pi_{00}(T)) = f(\sigma_w(T)) = \sigma_w(f(T)),$$

which implies that Weyl's theorem holds for $f(T)$. □

4. Property (ω)

In this section we will show under which conditions, that algebraic extension k-quasi class $\mathcal{A}_n^*$ operator T, satisfies property (ω).

Definition 4.1. *[3] A bounded linear operator $T \in L(H)$ is said to satisfy property (ω), if*

$$\sigma_a(T) \setminus \sigma_{uw}(T) = \pi_{00}(T).$$

Examples of operators satisfying Weyl's theorem but not property (ω) may be found in [3]. Property (ω) is independent from a-Weyl's theorem: in [3] there are examples of operators $T \in L(H)$ satisfying property (ω) but not a-Weyl's theorem and vice versa.

Example 5.2 given in [12], shows that a-Weyl's theorem and Weyl's theorem does not imply property (ω).

Lemma 4.2. *[3] Let us suppose that $T \in L(H)$, then*

1. *If T^* has the SVEP, then $\sigma_{uw}(T) = \sigma_b(T)$.*
2. *If T has SVEP, then $\sigma_{uw}(T^*) = \sigma_b(T)$.*

Techniques of the proof of the following Theorem are similar to that of Theorem 5.4, given in [12].

Theorem 4.3. *Let $T \in L(H)$.*

1. *If T^* is an algebraic extension k-quasi class $\mathcal{A}_n^*$, then property (ω) holds for T.*
2. *If T is an algebraic extension k-quasi class $\mathcal{A}_n^*$, then property (ω) holds for T^*.*

Proof. (i) Since T^* is an algebraic extension k-quasi class $\mathcal{A}_n^*$ operator, then T^* has the SVEP and T is a polaroid operator by Theorem 2.7 because T is polaroid if and only if T^* is polaroid. Consequently $\sigma(T) = \sigma_a(T)$.

Consider two cases:

Case I: If iso$\sigma(T) = \emptyset$, then $\pi_{00}(T) = \emptyset$. We show that $\sigma_a(T) \setminus \sigma_{uw}(T)$ -is empty. By Lemma 4.2 we have $\sigma_a(T) \setminus \sigma_{uw}(T) = \sigma(T) \setminus \sigma_b(T)$ and the last set is empty, since $\sigma(T)$ has no isolated points. Therefore, T satisfies property (ω).

Case II: If iso$\sigma(T) \neq \emptyset$. Suppose that $\lambda \in \pi_{00}(T)$. Then λ is isolated in $\sigma(T)$ and hence, by the polaroid condition, λ is a pole of the resolvent of T, i.e. $p(T-\lambda) = q(T-\lambda) < \infty$. By assumption

$\alpha(T-\lambda) < \infty$, so by [4, Theorem 3.4] $\beta(T-\lambda) < \infty$, and hence $T-\lambda$ is a Browder operator. Therefore, by Lemma 4.2,

$$\lambda \in \sigma(T) \setminus \sigma_b(T) = \sigma_a(T) \setminus \sigma_{uw}(T).$$

Conversely, if $\lambda \in \sigma_a(T) \setminus \sigma_{uw}(T) = \sigma(T) \setminus \sigma_b(T)$ then λ is an isolated point of $\sigma(T)$. Clearly, $0 < \alpha(T-\lambda) < \infty$, so $\lambda \in \pi_{00}(T)$ and hence T satisfies property (ω).

(ii) First note that since T has SVEP then

$$\sigma_a(T^*) = \{\lambda \in \mathbb{C} : T-\lambda \text{ is not onto } \} = \sigma(T) = \sigma(T^*).$$

Suppose first that $\text{iso}\sigma(T) = \text{iso}\sigma(T^*) = \emptyset$. Then $\pi_{00}(T^*) = \emptyset$. By Lemma 4.2 we have $\sigma_a(T^*) \setminus \sigma_{uw}(T) = \sigma(T) \setminus \sigma_b(T) = \emptyset$, so T^* satisfies property (ω).

Suppose that $\text{iso}\sigma(T) \neq \emptyset$ and let $\lambda \in \pi_{00}(T^*)$. Then λ is isolated in $\sigma(T) = \sigma(T^*)$, hence a pole of the resolvent of T^*, since T^* is a polaroid operator by Theorem 2.7. By assumption $\alpha(T^*-\lambda) < \infty$ and since the ascent and the descent of $T^*-\overline{\lambda}$ are both finite it then follows by [4, Theorem 3.4] that $\alpha(T^*-\overline{\lambda}) = \beta(T^*-\overline{\lambda}) < \infty$, so $T^*-\overline{\lambda}$ is a Browder operator and hence also $T-\lambda$ is a Browder operator. Therefore, $\lambda \in \sigma(T) \setminus \sigma_b(T)$ and by Lemma 4.2 it then follows that $\lambda \in \sigma_a(T^*) \setminus \sigma_{uw}(T^*)$. Conversely, if $\lambda \in \sigma_a(T^*) \setminus \sigma_{uw}(T^*) = \sigma(T) \setminus \sigma_b(T)$, then λ is an isolated point of the spectrum of $\sigma(T) = \sigma(T^*)$. Hence $T-\lambda$ is a Browder operator, or equivalently $T^*-\overline{\lambda}$ is a Browder operator. Since $\alpha(T^*-\overline{\lambda}) = \beta(T^*-\overline{\lambda})$ we then have $\alpha(T^*-\overline{\lambda}) > 0$ (otherwise $\lambda \notin \sigma(T^*)$). Clearly, $\alpha(T^*-\overline{\lambda}) < \infty$, since by assumption $T^*-\overline{\lambda}$ is a semi-upper Weyl operator, so that $\lambda \in \pi_{00}(T^*)$. Thus T^* satisfies property (ω). □

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