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Fractional Hartley Transform on G-Boehmian Space

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ABSTRACT: Using a special type of fractional convolution, a G-Boehmian space $\mathcal{B}_\alpha$ containing integrable functions on $\mathbb{R}$ is constructed. The fractional Hartley transform (FRHT) is defined as a linear, continuous injection from $\mathcal{B}_\alpha$ into the space of all continuous functions on $\mathbb{R}$. This extension simultaneously generalizes the fractional Hartley transform on $L^1(\mathbb{R})$ as well as Hartley transform on an integrable Boehmian space.

Key Words: Fractional Hartley transform, Fractional convolution, Boehmians.

Contents

1. Introduction

Hartley introduced a Fourier-like transform in 1942, which is called Hartley transform (see [7,11]). Like the fractional Fourier transform (FRFT) [21], many integral transforms have been generalized to the corresponding fractional integral transforms. In particular, fractional Fourier cosine transform (FRFCT), fractional Fourier sine transform (FRFST) and fractional Hartley transform (FRHT) were defined and used extensively in signal processing [4,24].

In [14], Mikusiński, J. and Mikusiński, P., introduced Boehmian space, which in general, consists of convolution quotients of sequences of functions. In [15], an abstract Boehmian space $\mathcal{B}$ is constructed by using a complex topological vector space G, $S \subset G$, $\star : G \times S \to G$ and a collection Δ of sequences satisfying certain axioms. As many of these Boehmian spaces contain the respective domains of various classical integral transforms, the research on Boehmian space includes extension of integral transforms to larger domains. For example, we refer [1,2,3,6,29,5,9,10,12,22,23,25,26,27]. Meanwhile, various versions of Boehmian spaces are introduced with new assumptions or slightly weaker assumptions than that are used in the general construction of a Boehmian space given in [15], by many authors [8,13,17,18,19]. Most recently, the G-Boehmian space is introduced in [9] as a generalization of the Boehmian space and the Hartley transform is extended to a suitable G-Boehmian space. In the present article, we introduce a special type of fractional convolution to construct a G-Boehmian space $\mathcal{B}_\alpha$ containing the space of integrable functions on $\mathbb{R}$. The fractional Hartley transform (FRHT) is extended consistently as a linear, continuous injection from $\mathcal{B}_\alpha$ in to the space $C(\mathbb{R})$, of all complex-valued continuous functions on reals.

This paper is organized as follows. In Section 2, we recall fractional Hartley transform, the general construction of a G-Boehmian space and some of their properties. In Section 3, we shall prove all the preliminary results required for the construction of the G-Boehmian space $\mathcal{B}_\alpha$. In Section 4, we provide the extended FRHT on this G-Boehmian space and investigate its properties.

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2. Preliminaries

We now recall from [24], the definition of FRHT of $f \in \mathscr{L}^1(\mathbb{R})$ and some of its properties. FRHT of an arbitrary integrable function f was defined by

$$[\mathcal{H}_\alpha(f)](u) = c_\alpha \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}} f(x) e^{ia_\alpha(x^2+u^2)} \text{ Cas } (b_\alpha xu)dx, \ \forall u \in \mathbb{R}, \tag{2.1}$$

where

$$\text{Cas } (\cdot) = \cos(\cdot) + \sin(\cdot), a_\alpha = \frac{\cot\alpha}{2}, b_\alpha = \frac{1}{\sin\alpha}, \text{ and } c_\alpha = \frac{e^{i\alpha/2}}{\sqrt{i\sin\alpha}}.$$

The FRFT and FRHT are obtained from one another through the following identities:

$$\begin{aligned} \mathcal{H}_\alpha(f) &= \frac{1+i}{2}\mathcal{F}_\alpha(f) + \frac{1-i}{2}\mathcal{F}_\alpha(-f) \\ \mathcal{F}_\alpha(f) &= \frac{\mathcal{H}_\alpha(f)+\mathcal{H}_\alpha(-f)}{2} - i\frac{\mathcal{H}_\alpha(f)-\mathcal{H}_\alpha(-f)}{2}, \end{aligned}$$

where $\mathcal{F}_\alpha(f)$ is the fractional Fourier transform of f, which is defined by

$$\mathcal{F}_\alpha(f)(u) = \frac{c_\alpha}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ia_\alpha(x^2+u^2-2ux\sec\alpha)}dx, \ \forall u \in \mathbb{R}.$$

If $f \in \mathscr{L}^1(\mathbb{R})$ and $\mathcal{H}_\alpha(f) \in \mathscr{L}^1(\mathbb{R})$ then $\mathcal{H}_\alpha[\mathcal{H}_\alpha(f)] = f$.

Using these identities along with the properties of FRFT, we have $\mathcal{H}_\alpha(f) \in C_0(\mathbb{R})$ and $\|\mathcal{H}_\alpha(f)\|_\infty \leq \frac{|c_\alpha|}{\sqrt{2\pi}}\|f\|_1$. Thus the FRHT is a continuous injective mapping from $\mathscr{L}^1(\mathbb{R})$ into $C_0(\mathbb{R})$.

We shall devote the next part of this section to the general construction of a G-Boehmian space $\mathcal{B}$ given in [9]. According to [9], a G-Boehmian space is a quotient space defined as follows: Let $\mathcal{B} = \mathcal{B}(\Gamma, S, \star, \Delta)$, where Γ is a topological vector space over $\mathbb{C}$, $S \subseteq \Gamma$, $\star : \Gamma \times S \to \Gamma$ satisfies the following conditions:

A_1: $(g_1 + g_2) \star s = g_1 \star s + g_2 \star s$, $\forall g_1, g_2 \in \Gamma$ and $\forall s \in S$.

A_2: $(cg) \star s = c(g \star s)$, $\forall c \in \mathbb{C}$, $\forall g \in \Gamma$ and $\forall s \in S$.

A_3: $g \star (s \star t) = (g \star s) \star t = (g \star t) \star s$, $\forall g \in \Gamma$ and $\forall s, t \in S$.

A_4: If $g_n \to g$ as $n \to \infty$ in Γ and $s \in S$, then $g_n \star s \to g \star s$ as $n \to \infty$ in Γ.

and let Δ be a collection of sequences from S such that

(Δ_1) If $(s_n), (t_n) \in \Delta$, then $(s_n \star t_n) \in \Delta$.

(Δ_2) If $g \in \Gamma$ and $(s_n) \in \Delta$, then $g \star s_n \to g$ as $n \to \infty$ in Γ.

Let $\mathcal{A} = \{((g_n), (s_n))/g_n \in \Gamma, (s_n) \in \Delta \text{ and } g_n \star s_m = g_m \star s_n, \ \forall m, n \in \mathbb{N}\}$. An equivalence relation $\sim$ on $\mathcal{A}$ is defined as follows:

$$((g_n), (s_n)) \sim ((h_n), (t_n)) \text{ if } g_n \star t_m = h_m \star s_n, \ \forall m, n \in \mathbb{N}$$

Denoting the equivalence class $\left[\frac{g_n}{s_n}\right]$ containing $((g_n), (s_n))$, we define the G-Boehmian space $\mathcal{B}$ as the set of all equivalence classes $\left[\frac{g_n}{s_n}\right]$ induced by the equivalence relation $\sim$ on $\mathcal{A}$. It is clear that $\mathcal{B}$ is a vector space with respect to the addition and scalar multiplication defined as follows.

$$\left[\frac{g_n}{s_n}\right] + \left[\frac{h_n}{t_n}\right] = \left[\frac{g_n \star t_n + h_n \star s_n}{s_n \star t_n}\right], \ c\left[\frac{g_n}{s_n}\right] = \left[\frac{cg_n}{s_n}\right].$$

Every member $g \in \Gamma$ can be uniquely identified as a member of $\mathcal{B}$ by $\left[\frac{g \star s_n}{s_n}\right]$, where $(s_n) \in \Delta$ is arbitrary and the operation $\star$ is also extended to $\mathcal{B} \times S$ by $\left[\frac{g_n}{\phi_n}\right] \star t = \left[\frac{g_n \star t}{\phi_n}\right]$. There are two notions of convergence on $\mathcal{B}$ namely δ-convergence and Δ-convergence which are defined as follows.

Definition 2.1. *[9, δ-convergence] We say that $X_m \xrightarrow{\delta} X$ as $m \to \infty$ in $\mathcal{B}$, if there exist $g_{m,n}, g_n \in \Gamma$, $m, n \in \mathbb{N}$ and $(s_n) \in \Delta$ such that $X_m = \left[\frac{g_{m,n}}{s_n}\right]$, $X = \left[\frac{g_n}{s_n}\right]$ and for each $n \in \mathbb{N}$, $g_{m,n} \to g_n$ as $m \to \infty$ in Γ.*

Definition 2.2. *[9, Δ-convergence] We say that $X_m \xrightarrow{\Delta} X$ as $m \to \infty$ in $\mathcal{B}$, if there exist $g_n \in \Gamma$ and $(s_n) \in \Delta$ such that $X_m - X = \left[\frac{g_m \star s_n}{s_n}\right]$ and $g_m \to 0$ as $m \to \infty$ in Γ.*

The major difference between a Boehmian space and a G-Boehmian space is that $\star$ should be commutative on S for a Boehmian space, which is not required for a G-Boehmian space.

3. Fractional Convolutions and Fractional Hartley Transform

In this section, we introduce a new type of fractional convolution $\#_\alpha$, using which, we establish all the results required for constructing the fractional Hartley transformable G-Boehmian space $\mathcal{B}_\alpha$.

Definition 3.1. *For $f, g \in \mathscr{L}^1(\mathbb{R})$, we define the convolution $\#_\alpha$ as follows:*

$$(f\#_\alpha g)(x) = \frac{c_\alpha}{\sqrt{2\pi}} \int_{\mathbb{R}} g(y) e^{2ia_\alpha y^2}[f(x+y)e^{2ia_\alpha xy} + f(x-y)e^{-2ia_\alpha xy}]dy, \ \forall x \in \mathbb{R}.$$

We first point out that $\#_\alpha$ is not commutative on $\mathscr{L}^1(\mathbb{R})$. Indeed, we can give a pair of functions similar to that given in [10, Example 3.7].

Lemma 3.2. *If $f, g \in \mathscr{L}^1(\mathbb{R})$, then $\|f\#_\alpha g\|_1 \le |c_\alpha|\sqrt{\frac{2}{\pi}}\|f\|_1\|g\|_1$ and hence $f\#_\alpha g \in \mathscr{L}^1(\mathbb{R})$.*

Proof. By using Fubini's theorem, we obtain

$$\begin{aligned}
\|f\#_\alpha g\|_1 &= \frac{|c_\alpha|}{\sqrt{2\pi}} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} g(y)e^{2ia_\alpha y^2}[f(x+y)e^{2ia_\alpha xy} + f(x-y)e^{-2ia_\alpha xy}]dy \right| dx \\
&\le \frac{|c_\alpha|}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} |g(y)e^{2ia_\alpha y^2}[f(x+y)e^{2ia_\alpha xy} + f(x-y)e^{-2ia_\alpha xy}]| dy\, dx \\
&\le \frac{|c_\alpha|}{\sqrt{2\pi}} \int_{\mathbb{R}} |g(y)| \int_{\mathbb{R}} |f(x+y)e^{2ia_\alpha xy} + f(x-y)e^{-2ia_\alpha xy}| dx\, dy \\
&\le \frac{|c_\alpha|}{\sqrt{2\pi}} \int_{\mathbb{R}} |g(y)| \left[\int_{\mathbb{R}} |f(x+y)|dx + \int_{\mathbb{R}} |f(x-y)|dx \right] dy \\
&\le \frac{|c_\alpha|}{\sqrt{2\pi}} \int_{\mathbb{R}} |g(y)| \left[\int_{\mathbb{R}} |f(z)|dz + \int_{\mathbb{R}} |f(z)|dz \right] dy \\
&\le |c_\alpha|\sqrt{\frac{2}{\pi}}\|f\|_1\|g\|_1
\end{aligned}$$

and hence $f\#_\alpha g \in \mathscr{L}^1(\mathbb{R})$. □

Lemma 3.3. *If $f, g, h \in \mathscr{L}^1(\mathbb{R})$, then $(f\#_\alpha g)\#_\alpha h = f\#_\alpha(g\#_\alpha h)$.*

Proof. For a fixed $\alpha \in \mathbb{R}$, let $k_\alpha = \frac{c_\alpha^2}{2\pi}$. For $x \in \mathbb{R}$, we obtain that

$$\begin{aligned}
[f\#_\alpha(g\#_\alpha h)](x) &= \frac{c_\alpha}{\sqrt{2\pi}} \int_{\mathbb{R}} (g\#_\alpha h)(y) e^{2ia_\alpha y^2}[f(x+y)e^{2ia_\alpha xy} + f(x-y)e^{-2ia_\alpha xy}]dy \\
&= k_\alpha \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} h(z)e^{2ia_\alpha z^2}[g(y+z)e^{2ia_\alpha yz} + g(y-z)e^{-2ia_\alpha yz}]dz \right\} e^{2ia_\alpha y^2} \\
&\quad [f(x+y)e^{2ia_\alpha xy} + f(x-y)e^{-2ia_\alpha xy}]dy \\
&= k_\alpha \int_{\mathbb{R}} h(z)e^{2ia_\alpha z^2} \left\{ \int_{\mathbb{R}} f(x+y)g(y+z)e^{2ia_\alpha(y^2+xy+yz)}dy \right. \\
&\quad + \int_{\mathbb{R}} f(x+y)g(y-z)e^{2ia_\alpha(y^2+xy-yz)}dy \\
&\quad + \int_{\mathbb{R}} f(x-y)g(y+z)e^{2ia_\alpha(y^2-xy+yz)}dy \\
&\quad \left. \int_{\mathbb{R}} + f(x-y)g(y-z)e^{2ia_\alpha(y^2-xy-yz)}dy \right\} dz \text{ (by Fubini's theorem)}
\end{aligned}$$

$$
\begin{aligned}
&= k_\alpha \int_{\mathbb{R}} h(z)e^{2ia_\alpha z^2}\Big\{\int_{\mathbb{R}}[f(x+u-z)g(u)e^{2ia_\alpha[(u-z)^2+x(u-z)+(u-z)z]}du\\
&+\int_{\mathbb{R}} f(x+u+z)g(u)e^{2ia_\alpha[(u+z)^2+x(u+z)-(u+z)z]}du\\
&+\int_{\mathbb{R}} f(x+z-u)g(u)e^{2ia_\alpha[(u-z)^2-x(u-z)+(u-z)z]}du\\
&+\int_{\mathbb{R}} f(x-z-u)g(u)e^{2ia_\alpha[(z+u)^2-x(z+u)-(u+z)z]}du\}dz\\
&= k_\alpha \int_{\mathbb{R}} h(z)e^{2ia_\alpha z^2}\Big\{\int_{\mathbb{R}}[f(x+u-z)g(u)e^{2ia_\alpha[u^2-uz+xu-xz]}du\\
&+\int_{\mathbb{R}} f(x+z+u)g(u)e^{2ia_\alpha[u^2+uz+xu+xz]}du\\
&+\int_{\mathbb{R}} f(x+z-u)g(u)e^{2ia_\alpha[u^2-uz-xu+xz]}du\\
&+\int_{\mathbb{R}} f(x-z-u)g(u)e^{2ia_\alpha[u^2+uz-xu-xz]}du\}dz\\
&= \frac{c_\alpha}{\sqrt{2\pi}}\int_{\mathbb{R}} h(z)e^{2ia_\alpha z^2}\Bigg\{\Bigg[\frac{c_\alpha}{\sqrt{2\pi}}\int_{\mathbb{R}} g(u)e^{2ia_\alpha u^2}f(x+z+u)e^{2ia_\alpha(x+z)u}du\\
&+\frac{c_\alpha}{\sqrt{2\pi}}\int_{\mathbb{R}} g(u)e^{2ia_\alpha u^2}f(x+z-u)e^{-2ia_\alpha(x+z)u}du\Bigg]e^{2ia_\alpha xz}\\
&+\Bigg[\frac{c_\alpha}{\sqrt{2\pi}}\int_{\mathbb{R}} g(u)e^{2ia_\alpha u^2}f(x-z+u)e^{2ia_\alpha(x-z)u}du\\
&+\frac{c_\alpha}{\sqrt{2\pi}}\int_{\mathbb{R}} g(u)e^{2ia_\alpha u^2}f(x-z-u)e^{-2ia_\alpha(x-z)u}du\Bigg]e^{-2ia_\alpha xz}\Bigg\}dz\\
&= \frac{c_\alpha}{\sqrt{2\pi}}\int_{\mathbb{R}} h(z)e^{2ia_\alpha z^2}\left[(f\#_\alpha g)(x+z)e^{2ia_\alpha xz}-(f\#_\alpha g)(x-z)e^{-2ia_\alpha xz}\right]dz\\
&= [(f\#_\alpha g)\#_\alpha h](x).
\end{aligned}
$$

Since $x \in \mathbb{R}$ is arbitrary, the proof follows. □

Lemma 3.4. *If $f, g, h \in \mathscr{L}^1(\mathbb{R})$, then $(f\#_\alpha g)\#_\alpha h = (f\#_\alpha h)\#_\alpha g$.*

Proof. From the proof of Lemma 3.3, we have

$$
\begin{aligned}
[(f\#_\alpha g)\#_\alpha h](x) &= \frac{c_\alpha}{\sqrt{2\pi}}\int_{\mathbb{R}} h(z)e^{2ia_\alpha z^2}\Bigg\{\Bigg[\frac{c_\alpha}{\sqrt{2\pi}}\int_{\mathbb{R}} g(u)e^{2ia_\alpha u^2}f(x+z+u)e^{2ia_\alpha(x+z)u}du\\
&+\frac{c_\alpha}{\sqrt{2\pi}}\int_{\mathbb{R}} g(u)e^{2ia_\alpha u^2}f(x+z-u)e^{-2ia_\alpha(x+z)u}du\Bigg]e^{2ia_\alpha xz}\\
&+\Bigg[\frac{c_\alpha}{\sqrt{2\pi}}\int_{\mathbb{R}} g(u)e^{2ia_\alpha u^2}f(x-z+u)e^{2ia_\alpha(x-z)u}du\\
&+\frac{c_\alpha}{\sqrt{2\pi}}\int_{\mathbb{R}} g(u)e^{2ia_\alpha u^2}f(x-z-u)e^{-2ia_\alpha(x-z)u}du\Bigg]e^{-2ia_\alpha xz}\Bigg\}dz\\
&= \frac{c_\alpha}{\sqrt{2\pi}}\int_{\mathbb{R}} g(u)e^{2ia_\alpha u^2}\Bigg\{\Bigg[\frac{c_\alpha}{\sqrt{2\pi}}\int_{\mathbb{R}} h(z)e^{2ia_\alpha z^2}f(x+u+z)e^{2ia_\alpha(x+u)z}dz\\
&+\frac{c_\alpha}{\sqrt{2\pi}}\int_{\mathbb{R}} h(z)e^{2ia_\alpha z^2}f(x+u-z)e^{-2ia_\alpha(x+u)z}dz\Bigg]e^{2ia_\alpha xz}\\
&+\Bigg[\frac{c_\alpha}{\sqrt{2\pi}}\int_{\mathbb{R}} h(z)e^{2ia_\alpha z^2}f(x-u+z)e^{2ia_\alpha(x-u)z}dz\\
&+\frac{c_\alpha}{\sqrt{2\pi}}\int_{\mathbb{R}} h(z)e^{2ia_\alpha z^2}f(x-u-z)e^{-2ia_\alpha(x-u)z}dz\Bigg]e^{-2ia_\alpha xz}\Bigg\}du\\
&\quad \text{(applying Fubini's theorem)}\\
&= \frac{c_\alpha}{\sqrt{2\pi}}\int_{\mathbb{R}} g(u)e^{2ia_\alpha u^2}[(f\#_\alpha h)(x+u)e^{2ia_\alpha xu}+(f\#_\alpha h)(x-u)e^{-2ia_\alpha xu}]du\\
&= [(f\#_\alpha h)\#_\alpha g](x).
\end{aligned}
$$

Since $x \in \mathbb{R}$ is arbitrary, the proof follows. □

Lemma 3.5. *If $f_n \to f$ as $n \to \infty$ in $\mathscr{L}^1(\mathbb{R})$ and if $g \in \mathscr{L}^1(\mathbb{R})$, then $f_n \#_\alpha g \to f \#_\alpha g$ as $n \to \infty$ in $\mathscr{L}^1(\mathbb{R})$.*

Proof. From the proof of Lemma 3.2, we have the estimate

$$\|(f_n - f)\#_\alpha g\|_1 \leq |c_\alpha| \sqrt{\frac{2}{\pi}}\, \|f_n - f\|_1 \|g\|_1. \tag{3.1}$$

Since $f_n \to f$ as $n \to \infty$ in $\mathscr{L}^1(\mathbb{R})$, the right hand side of (3.1) tends to zero as $n \to \infty$. Hence the lemma follows. □

Lemma 3.6. *If $f \in \mathscr{L}^1(\mathbb{R})$ and if $\xi(y) = \int\limits_{\mathbb{R}} |f(x \pm y)e^{ia_\alpha(y^2 \pm 2xy)} - f(x)|dx$, $\forall y \in \mathbb{R}$, then $\lim\limits_{y \to 0} \xi(y) = 0$.*

Proof. Since $C_c(\mathbb{R})$ is dense in $\mathscr{L}^1(\mathbb{R})$, we find $g \in C_c(\mathbb{R})$ such that $\|f - g\|_1 < \epsilon$. Using the continuity of the mapping $y \mapsto g_y$ from $\mathbb{R}$ in to $\mathscr{L}^1(\mathbb{R})$, (see [28, Theorem 9.5]), we choose $0 < \delta < \epsilon$ such that

$$\|g_s - g\|_1 < \epsilon \text{ whenever } |s| < \delta, \tag{3.2}$$

where $g_y(x) = g(x - y)$, $\forall x \in \mathbb{R}$. Let K be the compact support of g and $C = \sup\limits_{x \in K} |x|$. Applying mean-value theorem, for the function $y \mapsto e^{ia_\alpha(y^2 \pm 2xy)}$ on $|y| < \delta$, for each fixed $x \in K$, we have

$$|e^{ia_\alpha(y^2 \pm 2xy)} - 1| \leq |y||2y \pm 2x| \leq 2\,(\delta + C)\,|y|.$$

For $|y| < \delta$, we get that

$$\begin{aligned}
\xi(y) &= \int\limits_{\mathbb{R}} |f(x \pm y)e^{ia_\alpha(y^2 \pm 2xy)} - f(x)|dx \\
&= \int\limits_{\mathbb{R}} |f(x \pm y)e^{ia_\alpha(y^2 \pm 2xy)} - g(x \pm y)e^{ia_\alpha(y^2 \pm 2xy)} + g(x \pm y)e^{ia_\alpha(y^2 \pm 2xy)} \\
&\quad -g(x \pm y) + g(x \pm y) - g(x) + g(x) - f(x)|dx \\
&\leq \int\limits_{\mathbb{R}} |f(x \pm y) - g(x \pm y)|dx + \int\limits_{\mathbb{R}} |g(x \pm y)|\,|e^{ia_\alpha(y^2 \pm 2xy)} - 1|dx \\
&\quad + \int\limits_{\mathbb{R}} |g(x \pm y) - g(x)|dx + \int\limits_{\mathbb{R}} |g(x) - f(x)|dx \\
&= 2\|f - g\|_1 + \|g_{\mp y} - g\|_1 + \int\limits_{K} |g(x \pm y)|\,|e^{ia_\alpha(y^2 \pm 2xy)} - 1|dx \\
&< 3\epsilon + \|g\|_\infty \int\limits_{K} |y| 2\,(\delta + C)\,dx \\
&< 3\epsilon + \|g\|_\infty 2\,(\delta + C)\,m(K)\epsilon.
\end{aligned}$$

where $m(K)$ is the Lebesgue measure of the compact set K. This completes the proof of the lemma. □

Definition 3.7. *The collection of all sequences (δ_n) from $\mathscr{L}^1(\mathbb{R})$, satisfying the conditions*

Δ_1 : $c_\alpha \sqrt{\frac{2}{\pi}} \int\limits_{\mathbb{R}} e^{ia_\alpha t^2}\, \delta_n(t)\, dt = 1 \ \forall n \in \mathbb{N}$;

Δ_2 : $\int\limits_{\mathbb{R}} |\delta_n(t)|\, dt \leq M$, $\forall n \in \mathbb{N}$, *for some* $M > 0$;

Δ_3 : *supp* $\delta_n \to 0$ *as* $n \to \infty$, *where supp* δ_n *is the support of* δ_n;

is denoted by Δ^α.

Lemma 3.8. *If $(\delta_n), (\psi_n) \in \Delta^\alpha$ then $(\delta_n \#_\alpha \psi_n) \in \Delta^\alpha$.*

Proof. Let $(\delta_n), (\psi_n) \in \Delta^\alpha$. By using Fubini's theorem, we get that

$$\begin{aligned}
\int_{\mathbb{R}} e^{ia_\alpha y^2}[\delta_n \#_\alpha \psi_n](y)\, dy &= \tfrac{c_\alpha}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ia_\alpha y^2} \int_{\mathbb{R}} \psi_n(u) e^{2ia_\alpha u^2}[\delta_n(y+u)e^{2ia_\alpha yu} + \delta_n(y-u)e^{-2ia_\alpha yu}]dudy \\
&= \tfrac{c_\alpha}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi_n(u) e^{2ia_\alpha u^2} \int_{\mathbb{R}} e^{ia_\alpha y^2}[\delta_n(y+u)e^{2ia_\alpha yu} + \delta_n(y-u)e^{-2ia_\alpha yu}]dydu \\
&= \tfrac{c_\alpha}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi_n(u) e^{ia_\alpha u^2} \int_{\mathbb{R}} e^{ia_\alpha(y^2+u^2+2yu)}\delta_n(y+u)dy \\
&\quad + \int_{\mathbb{R}} e^{ia_\alpha(y^2+u^2-2yu)}\delta_n(y-u)dydu \\
&= \tfrac{c_\alpha}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi_n(u) e^{ia_\alpha u^2} \int_{\mathbb{R}} e^{ia_\alpha(y+u)^2}\delta_n(y+u)dy \\
&\quad + \int_{\mathbb{R}} e^{ia_\alpha(y-u)^2}\delta_n(y-u)dydu \\
&= \tfrac{2c_\alpha}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi_n(u) e^{ia_\alpha u^2} \int_{\mathbb{R}} e^{ia_\alpha s^2}\delta_n(s)ds\, du \\
&= \tfrac{2c_\alpha}{\sqrt{2\pi}} \left(c_\alpha\sqrt{\tfrac{2}{\pi}}\right)^{-1} \int_{\mathbb{R}} \psi_n(u) e^{ia_\alpha u^2}\, du, \text{ (by condition } \Delta_2 \text{ of } \Delta^\alpha) \\
&= \tfrac{2c_\alpha}{\sqrt{2\pi}} \left(c_\alpha\sqrt{\tfrac{2}{\pi}}\right)^{-1} \left(c_\alpha\sqrt{\tfrac{2}{\pi}}\right)^{-1} = \left(c_\alpha\sqrt{\tfrac{2}{\pi}}\right)^{-1}.
\end{aligned}$$

Next, by Lemma 3.2 and property (Δ_2) of Δ^α, we get that

$$\|\delta_n \#_\alpha \psi_n\|_1 \leq |c_\alpha|\sqrt{\frac{2}{\pi}}\|\delta_n\|_1\|\psi_n\|_1 < |c_\alpha|\sqrt{\frac{2}{\pi}}\, P_1\, P_2,\ \forall n \in \mathbb{N},$$

where $P_1 > 0$ and $P_2 > 0$ are such that

$$\int_{\mathbb{R}} |\delta_n(t)|(t)\, dt \leq P_1 \text{ and } \int_{\mathbb{R}} |\psi_n(t)|(t)\, dt \leq P_2,\ \forall n \in \mathbb{N}.$$

For a given $\epsilon > 0$, we choose $N \in \mathbb{N}$ such that supp δ_n, supp $\psi_n \subset (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$ for all $n \geq N$. Using the fact that

$$\text{supp } (\delta_n \#_\alpha \psi_n) \subset [\text{supp } \delta_n + \text{supp } \psi_n] \cup [\text{ supp } \delta_n - \text{ supp } \psi_n],$$

we get that supp $(\delta_n \#_\alpha \psi_n) \subset (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) + (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) = (-\epsilon, \epsilon)$, for all $n \geq N$. Hence it follows that $(\delta_n \#_\alpha \psi_n) \in \Delta^\alpha$. □

Theorem 3.9. *Let $f \in \mathscr{L}^1(\mathbb{R})$ and let $(\varphi_n) \in \Delta^\alpha$, then $f \#_\alpha \varphi_n \to f$ as $n \to \infty$ in $\mathscr{L}^1(\mathbb{R})$.*

Proof. Let $\epsilon > 0$ be given. By the property (Δ_2) of (φ_n), there exists $M > 0$ with $\int_{\mathbb{R}} |\varphi_n(t)|dt \leq M$, $\forall\, n \in \mathbb{N}$. Using Lemma 3.6, choose $\delta > 0$ such that

$$\int_{\mathbb{R}} |f(x \pm y)e^{ia_\alpha(y^2 \pm 2xy)} - f(x)|dx < \frac{\epsilon}{2} \tag{3.3}$$

whenever $|y| < \delta$. By the property (Δ_3) of (φ_n), there exists $N \in \mathbb{N}$ with supp $\varphi_n \subset [-\delta, \delta]$, $\forall n \geq N$. For $x \in \mathbb{R}$, we have

$$
\begin{aligned}
(f\#_\alpha\varphi_n)(x)-f(x) &= \tfrac{c_\alpha}{\sqrt{2\pi}}\int_{\mathbb{R}}\varphi_n(y)e^{2ia_\alpha y^2}[f(x+y)e^{2ia_\alpha xy}+f(x-y)e^{-2ia_\alpha xy}]dy-f(x)\\
&= \tfrac{c_\alpha}{\sqrt{2\pi}}\int_{\mathbb{R}}\varphi_n(y)e^{2ia_\alpha y^2}[f(x+y)e^{2ia_\alpha xy}+f(x-y)e^{-2ia_\alpha xy}]dy\\
&\quad -c_\alpha\sqrt{\tfrac{2}{\pi}}\int_{\mathbb{R}}f(x)e^{ia_\alpha y^2}\varphi_n(y)dy\\
&= \tfrac{c_\alpha}{\sqrt{2\pi}}\int_{\mathbb{R}}\varphi_n(y)e^{ia_\alpha y^2}[f(x+y)e^{ia_\alpha(y^2+2xy)}+f(x-y)e^{ia_\alpha(y^2-2xy)}]dy\\
&\quad -2\int_{\mathbb{R}}f(x)e^{ia_\alpha y^2}\varphi_n(y)dy\\
&= \tfrac{c_\alpha}{\sqrt{2\pi}}\int_{-\delta}^{\delta}\varphi_n(y)e^{ia_\alpha y^2}[f(x+y)e^{ia_\alpha(y^2+2xy)}-f(x)\\
&\quad +f(x-y)e^{ia_\alpha(y^2-2xy)}-f(x)]dy.
\end{aligned}
$$

This implies that for each $n\geq N$,

$$
\begin{aligned}
\|f\#_\alpha\varphi_n-f\|_1 &= \int_{\mathbb{R}}|(f\#_\alpha\varphi_n)(x)-f(x)|dx\\
&\leq \tfrac{|c_\alpha|}{\sqrt{2\pi}}\int_{-\delta}^{\delta}|\varphi_n(y)|\int_{\mathbb{R}}|f(x+y)e^{ia_\alpha(y^2+2xy)}-f(x)|dx\\
&\quad +\int_{\mathbb{R}}|f(x-y)e^{ia_\alpha(y^2-2xy)}-f(x)|dxdy\\
&\leq \tfrac{|c_\alpha|}{\sqrt{2\pi}}\int_{-\delta}^{\delta}|\varphi_n(y)|\{\tfrac{\epsilon}{2}+\tfrac{\epsilon}{2}\}dy,\ \text{by using equation (3.3)}\\
&\leq \tfrac{M\epsilon|c_\alpha|}{\sqrt{2\pi}},\ \text{by using the property } (\Delta_3) \text{ of } (\varphi_n).
\end{aligned}
$$

Since $\epsilon>0$ is arbitrary, it now follows that $f\#_\alpha\varphi_n\to f$ as $n\to\infty$ in $\mathscr{L}^1(\mathbb{R})$. □

Lemma 3.10. *If $f_n\to f$ as $n\to\infty$ in $\mathscr{L}^1(\mathbb{R})$ and $(\varphi_n)\in\Delta^\alpha$, then $f_n\#_\alpha\varphi_n\to f$ as $n\to\infty$ in $\mathscr{L}^1(\mathbb{R})$.*

Proof. For any $n\in\mathbb{N}$, using Lemma 3.2, we have

$$
\begin{aligned}
\|f_n\#_\alpha\varphi_n-f\|_1 &= \|f_n\#_\alpha\varphi_n-f\#_\alpha\varphi_n+f\#_\alpha\varphi_n-f\|_1\\
&\leq \|(f_n-f)\#_\alpha\varphi_n\|_1+\|f\#_\alpha\varphi_n-f\|_1\\
&\leq |c_\alpha|\sqrt{\frac{2}{\pi}}\ \|f_n-f\|_1\ \|\varphi_n\|_1+\|f\#_\alpha\varphi_n-f\|_1,\\
&\leq M|c_\alpha|\sqrt{\frac{2}{\pi}}\|f_n-f\|_1+\|f\#_\alpha\varphi_n-f\|_1.
\end{aligned}
$$

Since $f_n\to f$ as $n\to\infty$ in $\mathscr{L}^1(\mathbb{R})$ and by Theorem 3.9, the right hand side of the last inequality tends to zero as $n\to\infty$. Hence the lemma follows. □

Thus the G-Boehmian space $\mathcal{B}_\alpha=\mathcal{B}(\mathscr{L}^1(\mathbb{R}),\mathscr{L}^1(\mathbb{R}),\#_\alpha,\Delta^\alpha)$ has been constructed.

4. Fractional Hartley Transform on a G-Boehmian space

In order to extend the FRHT to the Boehmian space $\mathcal{B}_\alpha$, we have to first obtain a suitable convolution theorem for fractional Hartley transform. For this purpose, we introduce the function $\mathcal{C}_\alpha$ on $\mathscr{L}^1(\mathbb{R})$, defined as

$$[\mathcal{C}_\alpha(f)](t)=c_\alpha\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}f(x)e^{ia_\alpha x^2}\cos(b_\alpha xt)\,dx,\ \forall t\in\mathbb{R}. \tag{4.1}$$

As the limits of the integration varies over the entire real line, this function $\mathcal{C}_\alpha$ differs from the usual fractional Fourier cosine transform.

Theorem 4.1 (Convolution theorem)**.** *If $f, g \in \mathscr{L}^1(\mathbb{R})$, then $\mathcal{H}_\alpha(f\#_\alpha g) = \mathcal{H}_\alpha(f) \cdot \mathcal{C}_\alpha(g)$.*

Proof. Let $t \in \mathbb{R}$ be arbitrary and $k_\alpha = \frac{c_\alpha^2}{2\pi}$ be define as in the proof of Lemma 3.3. By using Fubini's theorem, we obtain that

$$\begin{aligned}
[\mathcal{H}_\alpha(f\#_\alpha g)](u) &= c_\alpha\sqrt{\tfrac{2}{\pi}}\int_{\mathbb{R}}(f\#_\alpha g)(x)e^{ia_\alpha(x^2+u^2)} \text{ Cas } (b_\alpha xu)dx \\
&= 2k_\alpha\int_{\mathbb{R}} e^{ia_\alpha(x^2+u^2)} \text{ Cas } (b_\alpha xu)\int_{\mathbb{R}} g(y)e^{2ia_\alpha y^2}[f(x+y)e^{2ia_\alpha xy} \\
&\quad +f(x-y)e^{-2ia_\alpha xy}]dydx \\
&= 2k_\alpha\int_{\mathbb{R}} g(y)e^{ia_\alpha(y^2+u^2)}\int_{\mathbb{R}} \text{ Cas } (b_\alpha xu)[f(x+y)e^{ia_\alpha(x+y)^2} \\
&\quad +f(x-y)e^{ia_\alpha(x-y)^2}]dxdy \\
&= 2k_\alpha\int_{\mathbb{R}} g(y)e^{ia_\alpha(y^2+u^2)}\int_{\mathbb{R}} \text{ Cas } (b_\alpha(z-y)u)f(z)e^{ia_\alpha z^2}dz \\
&\quad +\int_{\mathbb{R}} \text{ Cas } (b_\alpha(z+y)u)f(z)e^{ia_\alpha z^2}dzdy \\
&= 2k_\alpha\int_{\mathbb{R}} g(y)e^{ia_\alpha(y^2+u^2)}\int_{\mathbb{R}}[\text{ Cas } (b_\alpha(z-y)u) \\
&\quad + \text{ Cas } (b_\alpha(z+y)u)]f(z)e^{ia_\alpha z^2}dzdy \\
&= 4k_\alpha\int_{\mathbb{R}} g(y)e^{ia_\alpha(y^2+u^2)}\int_{\mathbb{R}}[\cos(b_\alpha zu)\cos(b_\alpha yu) \\
&\quad +\sin(b_\alpha zu)\cos(b_\alpha yu)]f(z)e^{ia_\alpha z^2}dzdy \\
&= 4k_\alpha\int_{\mathbb{R}} g(y)e^{ia_\alpha(y^2+u^2)}\int_{\mathbb{R}} \text{ Cas } (b_\alpha zu)\cos(b_\alpha yu)f(z)e^{ia_\alpha z^2}dzdy \\
&= 4k_\alpha\int_{\mathbb{R}} g(y)e^{ia_\alpha y^2}\cos(b_\alpha yu)\int_{\mathbb{R}} \text{ Cas } (b_\alpha zu)f(z)e^{ia_\alpha(z^2+u^2)}dzdy \\
&= c_\alpha\sqrt{\tfrac{2}{\pi}}[\mathcal{H}_\alpha(f)](u)\int_{\mathbb{R}} g(y)e^{ia_\alpha y^2}\cos(b_\alpha yu)dy \\
&= [\mathcal{H}_\alpha(f)](u)\cdot[\mathcal{C}_\alpha(g)](u).
\end{aligned}$$

Thus we have $\mathcal{H}_\alpha(f\#_\alpha g) = \mathcal{H}_\alpha(f) \cdot \mathcal{C}_\alpha(g)$. □

Theorem 4.2. *If $f, g \in \mathscr{L}^1(\mathbb{R})$, then $\mathcal{C}_\alpha(f\#_\alpha g) = \mathcal{C}_\alpha(f) \cdot \mathcal{C}_\alpha(g)$.*

Proof. Proof of this theorem is much similar to that of the previous theorem and hence we prefer to omit the details. □

Lemma 4.3. *If $(\varphi_n) \in \Delta^\alpha$ then $\mathcal{C}_\alpha(\varphi_n) \to 1$ as $n \to \infty$ uniformly on each compact subset of $\mathbb{R}$.*

Proof. Let $\epsilon > 0$ and a compact subset K of $\mathbb{R}$ be given. Choose $M_1 > 0$, $M_2 > 0$ and $N \in \mathbb{N}$ such that $\int_{-\infty}^{\infty}|\varphi_n(t)|\,dt \le M_1$, $\forall n \in \mathbb{N}$, $K \subset [-M_2, M_2]$ and supp $\varphi_n \subset [-\epsilon, \epsilon]$ for all $n \ge N$. For $u \in K$ and $n \ge N$, we have

$$\begin{aligned}
|[\mathcal{C}_\alpha(\varphi_n)](u) - 1| \le& |c_\alpha|\sqrt{\frac{2}{\pi}}\int_{\mathbb{R}}|\varphi_n(s)|\,|\cos(b_\alpha us) - 1|\,ds \\
=& |c_\alpha|\sqrt{\frac{2}{\pi}}\int_{-\epsilon}^{\epsilon}|\varphi_n(s)|\,|\cos(b_\alpha us) - 1|\,ds, \ \forall\, n \ge N
\end{aligned}$$

$$\leq \int_{-\epsilon}^{\epsilon} |\varphi_n(s)|\,|b_\alpha|\,|us||sinz|\,ds,$$

(where $0 < z < |b_\alpha us|$ exists, by mean-value theorem)

$$\leq |b_\alpha| M_2 \epsilon \int_{-\epsilon}^{\epsilon} |\varphi_n(s)|\,ds$$

$$\leq |b_\alpha| M_2 M_1 \epsilon.$$

This completes the proof. □

Definition 4.4. *For $\beta = \left[\frac{f_n}{\varphi_n}\right] \in \mathcal{B}_\alpha$, we define the extended fractional Hartley transform of β by*

$$[\mathscr{H}(\beta)](t) = \lim_{n\to\infty} [\mathcal{H}_\alpha(f_n)](t), \quad (t \in \mathbb{R}).$$

The above limit exists and is independent of the representative $((f_n), (\varphi_n))$ of β. Indeed, for $t \in \mathbb{R}$, choose k such that $[\mathcal{C}_\alpha(\varphi_k)](t) \neq 0$. Then, applying Theorem 4.1, we obtain that

$$[\mathcal{H}_\alpha(f_n)](t) = \frac{[\mathcal{H}_\alpha(f_n \#_\alpha \varphi_k)](t)}{[\mathcal{C}_\alpha(\varphi_k)](t)} = \frac{[\mathcal{H}_\alpha(f_k \#_\alpha \varphi_n)](t)}{[\mathcal{C}_\alpha(\varphi_k)](t)} = \frac{[\mathcal{H}_\alpha(f_k)](t)}{[\mathcal{C}_\alpha(\varphi_k)](t)} [\mathcal{C}_\alpha(\varphi_n)](t).$$

Therefore, using Lemma 4.3, we get $[\mathcal{H}_\alpha(f_n)](t) \to \frac{[\mathcal{H}_\alpha(f_k)](t)}{[\mathcal{C}_\alpha(\varphi_k)](t)}$, as $n \to \infty$ uniformly on each compact subset of $\mathbb{R}$. If $((f_n), (\varphi_n)) \sim ((g_n), (\psi_n))$, then $f_n \#_\alpha \psi_m = g_m \#_\alpha \varphi_n$ for all $m, n \in \mathbb{N}$. Again using Theorem 4.1, we get

$$\lim_{n\to\infty} [\mathcal{H}_\alpha(f_n)](t) = \frac{[\mathcal{H}_\alpha(f_k)](t)}{[\mathcal{C}_\alpha(\varphi_k)](t)} = \frac{[\mathcal{H}_\alpha(g_k)](t)}{[\mathcal{C}_\alpha(\psi_k)](t)} = \lim_{n\to\infty} [\mathcal{H}_\alpha(g_n)](t).$$

If $f \in \mathscr{L}^1(\mathbb{R})$ and $\beta = \left[\frac{f \#_\alpha \varphi_n}{\varphi_n}\right]$, then

$$[\mathscr{H}(\beta)](t) = \lim_{n\to\infty} [\mathcal{H}_\alpha(f \#_\alpha \varphi_n)](t) = [\mathcal{H}_\alpha(f)](t) \lim_{n\to\infty} [\mathcal{C}_\alpha(\varphi_n)](t) = [\mathcal{H}_\alpha(f)](t),$$

as $[\mathcal{C}_\alpha(\varphi_n)](t) \to 1$ as $n \to \infty$ uniformly on each compact subset of $\mathbb{R}$. This shows that the extended fractional Hartley transform is consistent with the FRHT on $\mathscr{L}^1(\mathbb{R})$.

Theorem 4.5. *If $\beta \in \mathcal{B}_\alpha$, then the extended fractional Hartley transform $\mathscr{H}(\beta) \in C(\mathbb{R})$.*

Proof. As $\mathscr{H}(\beta)$ is the uniform limit of $\{\mathcal{H}_\alpha(f_n)\}$ on each compact subset of $\mathbb{R}$ and each $\mathcal{H}_\alpha(f_n)$ is a continuous function on $\mathbb{R}$, $\mathscr{H}(\beta)$ is a continuous function on $\mathbb{R}$. □

Theorem 4.6. *The extended fractional Hartley transform $\mathscr{H} : \mathcal{B}_\alpha \to C(\mathbb{R})$ is linear.*

Proof. Proof is straightforward from the convolution theorem and the linearity of $\mathcal{H}_\alpha$ on $\mathscr{L}^1(\mathbb{R})$. □

Theorem 4.7. *The extended fractional Hartley transform $\mathscr{H} : \mathcal{B}_\alpha \to C(\mathbb{R})$ is one-to-one.*

Proof. Let $\beta = \left[\frac{f_n}{\varphi_n}\right] \in \mathcal{B}_\alpha$ be such that $[\mathscr{H}(\beta)](s) = 0, \ \forall\, s \in \mathbb{R}$. For $t \in \mathbb{R}$, choose $k \in \mathbb{N}$ with $[\mathcal{C}_\alpha(\varphi_k)](t) \neq 0$. Then

$$0 = [\mathcal{C}_\alpha(\varphi_k)](t) \cdot [\mathscr{H}(\beta)](t) = [\mathcal{C}_\alpha(\varphi_k)](t) \cdot \lim_{n\to\infty} [\mathcal{H}_\alpha(f_n)](t) = [\mathcal{H}_\alpha(f_k)](t).$$

Using $[\mathcal{H}_\alpha(f_m)](t)[\mathcal{C}_\alpha(\varphi_k)](t) = [\mathcal{H}_\alpha(f_k)](t)[\mathcal{C}_\alpha(\varphi_m)](t)$, it follows that $\mathcal{H}_\alpha(f_m) = 0, \ \forall\, m \in \mathbb{N}$. Since FRHT is injective, we get $f_m = 0$ in $\mathscr{L}^1(\mathbb{R})$, for every $m \in \mathbb{N}$. Thus $\beta = 0$, proving the result. □

Theorem 4.8. *The extended fractional Hartley transform $\mathscr{H} : \mathcal{B}_\alpha \to C(\mathbb{R})$ is continuous with respect to δ-convergence and Δ-convergence.*

Proof. As the proof of this result is routine, we prefer not to give the details. □

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