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Solving Fractional Differential Equations by the Ultraspherical Integration Method

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ABSTRACT: In this paper, we present a numerical method to solve a linear fractional differential equations. This new investigation is based on ultraspherical integration matrix to approximate the highest order derivatives to the lower order derivatives. By this approximation the problem is reduced to a constrained optimization problem which can be solved by using the penalty quadratic interpolation method. Numerical examples are included to confirm the efficiency and accuracy of the proposed method.

Key Words: Fractional differential equation, Ultraspherical polynomials, Ultraspherical integration matrix.

Contents

1. Introduction

In recent decades, fractional equations have gained much attention due to exact description of physical phenomena such as damping laws, electromagnetic acoustics, viscoelasticity, electro analytical chemistry neuron modeling, diffusion processing and material science(see for example [8,10]). Also many attempts have been made to find analytical and numerical solutions for the fractional problems.
These attempts include introducing finite difference [5,20], collocation-shooting [6], spline and B-spline collocation [18], Adomian decomposition [11], variational iteration [17], operational matrix [22] and many other methods.
Some authors present spectral or pseudospectral integration methods proven successful in the numerical solutions of many problems (see for example [2,3,4,12]).
In this work, we introduce a new formula of spectral integration matrix depends on using ultraspherical polynomials at some equally spaced points. The results indicate that the spectral accuracy is achieved and the effect of round off errors is limited.

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By the ultraspherical integration matrix we approximate the highest order fractional derivatives to the lower order fractional derivatives. By this work, the problem is reduced to a constrained optimization problem which can be solved by using the penalty quadratic interpolation method [13].
Consider a linear fractional differential equation in the from:

$$D^{\alpha}y(x) + f(x)y(x) = g(x),\ n-1 < \alpha < n,\ \ n \in \mathbf{N},\ 0 \le x \le 1 \tag{1.1}$$

with initial conditions:

$$y^{(k)}(0) = b_k, \qquad k = 0, 1, \ldots, n-1, \tag{1.2}$$

where $f, g : [0,1] \to R$ are given continuous function, and b_k are given constants.
The rest of the paper is organized as follows:
Basic concepts of fractional calculus problems are discussed in section 2. In section 3, we introduce ultraspherical polynomials and some of their properties. In section 4, ultraspherical spectral approximation for a given function and ultraspherical spectral integration matrix are presented. In section 5, description of the method and convergence are discussed. And finally in section 6, numerical results for confirmation of proposed method are presented.

2. Preliminaries

In this section, we recall some basic concepts and properties of the fractional calculus which are used throughout the paper.

Definition 2.1. [21] *A real function $f(x), x > 0$, is said to be in the space $C_{\mu}, \mu \in R$, if there exist a real number $p > \mu$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,1)$. If $\beta \le \mu$, then $C_{\mu} \subset C_{\beta}$.*

Definition 2.2. [21] *A function $f(x), x > 0$ is said to be in the space $C_{\mu}^{m}, m \in N \cup \{0\}$, if $f^{(m)} \in C_{\mu}$.*

Definition 2.3. [9] *The Riemann-Liouville fractional integral operator of order $\alpha \ge 0$ of a function $f \in C_{\mu}, \mu \ge -1$, is defined as:*

$$I^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt\ \alpha > 0, x > 0, \tag{2.1}$$

$$I^0 f(x) = f(x). \tag{2.2}$$

Definition 2.4. [21] *Let $f \in C_{-1}^{n}, n \in N \cup \{0\}$. Then the Caputo fractional derivative of order α is defined as:*

$$^{c}D_0^{\alpha}f(x) = I^{n-\alpha}f^{(n)}(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{f^{(n)}(t)}{(x-t)^{1-(n-\alpha)}}dt,\ n-1 < \alpha < n, n \in \boldsymbol{N}. \tag{2.3}$$

It can be shown that [6,8,9]:

1. $I_a^{\alpha}I_a^{\beta}f = I_a^{(\alpha+\beta)}f \qquad \alpha, \beta > 0, f \in C_M, M > 0.$
2. $I_a^{\alpha}x^{\beta} = \dfrac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}x^{\alpha+\beta}, \quad \alpha > 0, \quad \beta > -1, \quad x > 0$
3. $I_a^{\alpha}(^{c}D_0^{\alpha}f(x)) = f(x) - \sum_{k=0}^{n-1} f^{k}(0^{+})\dfrac{x^k}{k!}, \qquad x > 0$
4. $^{c}D_0^{\alpha}I^{\alpha}f(x) = f(x), \quad x > 0, \quad n-1 < \alpha \le n.$
5. $^{c}D_0^{\alpha}C = 0, \quad C$ *is constant.*
6. *If* $\beta < [\alpha]$, *then* $\ ^{c}D_0^{\alpha}x^{\beta} = 0, \quad x > 0.$
7. *If* $\beta > [\alpha]$, *then* $\ ^{c}D_0^{\alpha}x^{\beta} = \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, \quad x > 0.$

In this paper, we used the Caputo fractional derivative, because it allows traditional initial and boundary conditions to be included in the formulation of the problem.

3. Ultraspherical polynomials and some of the properties

The ultraspherical (Gegenbauer) polynomials with the real parameter $\left(\lambda > -\dfrac{1}{2}, \lambda \neq 0\right)$, are a sequence of polynomials $C_j^{(\lambda)}(x), j = 0, 1, 2, \ldots$, in the finite domain $x \in [-1, 1]$, each respectively of degree j satisfying the orthogonality relation:

$$\int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} C_j^{(\lambda)}(x) C_k^{(\lambda)}(x) dx = \begin{cases} 0 & j \neq k \\ \psi_j^{(\lambda)} & j = k \end{cases} \tag{3.1}$$

where the normalization constant $\psi_j^{(\lambda)}$, is defined as [23]:

$$\psi_j^{(\lambda)} = 2^{1-2\lambda} \pi \frac{\Gamma(j+2\lambda)}{(j+\lambda)\big\{\Gamma(\lambda)\big\}^2 \Gamma(j+1)}, \quad \lambda \neq 0. \tag{3.2}$$

Here, the ultraspherical polynomials are standardized such that $C_j^{(\lambda)}(1) = 1$, for $j = 0, 1, \ldots$, has the desirable properties that $C_j^{(0)}(x)$ is identical with the Chebyshev polynomial of the first kind $T_j(x), C_j^{(1/2)}(x)$ is the Legender polynomial $P_j(x)$ and $C_j^{(1)}(x)$ is equal to $\left(\dfrac{1}{j+1}\right) U_j(x)$, where $U_j(x)$ is the Chebyshve polynomial of the second kind.
The polynomials $C_j^{(\lambda)}(x)$ may be generated by Rodrigue's formula given as [6]

$$C_j^{(\lambda)}(x) = \sum_{r=0}^{[\frac{1}{2}j]} (-1)^r \frac{\Gamma(j-r+\lambda)}{\Gamma(\lambda)(r!)(j-2r)!} (2x)^{j-2r}. \tag{3.3}$$

In applications, recurrence formulaes which link pairs of coefficients are often more useful than explicit formulae for the coefficients. Therefore, a general expression for an ultraspherical polynomial can be considered as:

$$C_j^{(\lambda)}(x) = \sum_{r=0}^{[\frac{1}{2}j]} G_r^{(j)}(\lambda) x^{j-2r}, \tag{3.4}$$

where

$$G_r^{(j)}(\lambda) = (-1)^r \frac{2^{j-2r}\Gamma(j-r+\lambda)}{\Gamma(\lambda)(r!)(j-2r)!}. \tag{3.5}$$

A relationship between the coefficients $G_{r+1}^{(j)}(\lambda)$ and $G_r^{(j)}(\lambda)$ is given by:

$$G_{r+1}^{(j)}(\lambda) = -\frac{(j-2r-1)(j-2r-2)}{4(r+1)(\lambda+j-r-1)} G_r^{(j)}(\lambda), \tag{3.6}$$

where

$$G_0^{(0)}(\lambda) = 1, \quad G_0^{(j)}(\lambda) = 2^j \frac{\Gamma(j+\lambda)}{\Gamma(\lambda) j!},$$

and so $C_0^{(\lambda)}(x) = 1, C_1^{(\lambda)}(x) = 2\lambda x$.

Theorem 3.1. [16] *The integral of ultraspherical polynomials is expressed in terms of ultraspherical polynomials as follows:*

$$I(x_i) = \int_{-1}^{x_i} C_j^{(\lambda)}(x) dx = \sum_{r=0}^{[\frac{1}{2}j]} \frac{1}{j-2r+1} G_r^{(j)}(\lambda) \big(x_i^{j-2r+1} - (-1)^{j-2r+1}\big). \tag{3.7}$$

Proof. see [6]. □

4. Ultrasphericlcal spectral approximations

In the remaining parts of this paper, we assume that $f(x)$ is a smooth continuous function and

$$S=\left\{x_i=\frac{i}{N},\quad i=0,\dots,N\right\}.$$

Theorem 4.1. [15] *Let $f(x)$ be approximated by ultraspherical finite expansion, namely,*

$$f(x)\simeq\sum_{j=0}^{N}a_jC_j^{(\lambda)}(x).$$

Then

$$a_j=(\psi_j^{(\lambda)})^{-1}\int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}}C_j^{(\lambda)}(x)f(x), \tag{4.1}$$

where

$$\psi_j^{(\lambda)}=2^{1-2\lambda}\pi\frac{\Gamma(j+2\lambda)}{(j+\lambda)[\Gamma(\lambda)]^2\Gamma(j+1)},\quad \lambda\neq 0.$$

Proof. See [7]. □

Theorem 4.2. *Let* $\dfrac{\phi(x)}{(t-x)^{1-\alpha}}$ $(t\neq x)$ *be approximated by ultraspherical polynomials as in* (4.1)*:*

$$a_j\simeq\sum_{k=0}^{N}\frac{2\theta_k}{N}(\psi_j^{(\lambda)})^{-1}(1-x_k^2)^{\lambda-\frac{1}{2}}C_j^{(\lambda)}(x_k)\frac{\phi(x_k)}{(t-x_k)^{1-\alpha}}, \tag{4.2}$$

where $\theta_0=\theta_N=\frac{1}{2},\quad \theta_k=1\quad for\quad k=1,2,\dots,N-1.$

Proof. By using the Trapezoidal rule

$$\int_a^b f(x)dx=h\sum_{j=0}^{N}{}''f(x_j)-\frac{(b-a)h^2}{12}f''(\xi),\ h=\frac{b-a}{N},$$

we obtain from (5.1):

$$a_j=\frac{2}{N}(\psi_j^{(\lambda)})^{-1}\sum_{k=0}^{N}{}''(1-x_k^2)^{\lambda-\frac{1}{2}}C_j^{(\lambda)}(x_k)\frac{\phi(x_k)}{(t-x_k)^{1-\alpha}}\frac{-2}{3N^2}(\psi_j^{(\lambda)})^{-1}H_j^{(2)}(\xi)$$

with

$$H_j(x)=(1-x^2)^{\lambda-\frac{1}{2}}C_j^{(\lambda)}(x)\frac{\phi(x)}{(t-x)^{1-\alpha}}.$$

Hence

$$a_j\simeq\frac{2}{N}(\psi_j^{(\lambda)})^{-1}\sum_{k=0}^{N}{}''(1-x_k^2)^{\lambda-\frac{1}{2}}C_j^{\lambda}(x_k)\frac{\phi(x_k)}{(t-x_k)^{1-\alpha}}.$$

With $\theta_0=\theta_N=\frac{1}{2},\quad \theta_k=1\quad for\quad k=1,2,\dots,N-1,$
and

$$\frac{\phi(x)}{(t-x)^{1-\alpha}}\simeq\sum_{j=0}^{N}\sum_{k=0}^{N}\frac{2\theta_k(\psi_j^{(\lambda)})^{-1}}{N}(1-x_k^2)^{\lambda-\frac{1}{2}}C_j^{(\lambda)}(x_k)\frac{\phi(x_k)}{(t-x_k)^{1-\alpha}}C_j^{\lambda}(x),\quad t,x_k\in S.$$

□

4.1. Ultraspherical integration matrix

In this section, we approximate integral of a function $f \in C[-1,1]$ by interpolating the function with ultraspherical polynomial at the points given in S.

Theorem 4.3. *Let* $\dfrac{\phi(x)}{(t-x)^{1-\alpha}}$ *($t \in S, t \neq x$) be approximated by ultraspherical polynomial. Then there exists a matrix $Q = [q_{ij}]$, $i, j = 0 \dots N$, satisfying*

$$\int_{-1}^{x_i} \frac{\phi(x)}{(x_i - x)^{1-\alpha}} dx \simeq \sum_{\substack{k=0 \\ i \neq k}}^{N} q_{ik}(\lambda) \frac{\phi(x_k)}{(x_i - x_k)^{1-\alpha}}, \tag{4.3}$$

where

$$q_{ik}(\lambda) = \sum_{j=0}^{N} \sum_{r=0}^{[\frac{1}{2}j]} \frac{2\theta_k G_r^j(\lambda)(\psi_j^{(\lambda)})^{-1}}{N(j-2r+1)} (1-x_k^2)^{\lambda-\frac{1}{2}} C_j^{(\lambda)}(x_k)\big(x_i^{j-2r+1} - (-1)^{j-2r+1}\big), \tag{4.4}$$

for $x_i, x_k \in S$, with $\theta_0 = \theta_N = \frac{1}{2}$, $\theta_k = 1$ for $k = 1, 2, \dots, N-1$.

Proof. From theorem 3, we have

$$\frac{\phi(x)}{(x_i - x)^{1-\alpha}} \simeq \sum_{j=0}^{N} \sum_{\substack{k=0 \\ i \neq k}}^{N} \frac{2\theta_k \big(\psi_j(\lambda)\big)^{-1}}{N} (1-x_k^2)^{\lambda-\frac{1}{2}} C_j^{(\lambda)}(x_k) \frac{\phi(x_k)}{(x_i - x_k)^{1-\alpha}} C_j^{(\lambda)}(x), \tag{4.5}$$

therefor

$$\int_{-1}^{x_i} \frac{\phi(x)}{(x_i - x)^{1-\alpha}} dx =$$

$$\sum_{j=0}^{N} \sum_{\substack{k=0 \\ k \neq i}}^{N} \frac{2\theta_k \big(\psi_j(\lambda)\big)^{-1}}{N} (1-x_k^2)^{\lambda-\frac{1}{2}} C_j^{(\lambda)}(x_k) \left[\frac{\phi(x_k)}{(x_i - x_k)^{1-\alpha}} \int_{-1}^{x_i} C_j^{(\lambda)}(x) dx \right]$$

$$= \sum_{\substack{k=0 \\ k \neq i}}^{N} \left[\sum_{j=0}^{N} \frac{2\theta_k (\psi_j^{(\lambda)})^{-1}}{N} (1-x_k^2)^{\lambda-\frac{1}{2}} C_j^{(\lambda)}(x_k) \int_{-1}^{x_i} C_j^{(\lambda)}(x) dx \right] \frac{\phi(x_k)}{(x_i - x_k)^{1-\alpha}},$$

substituting from (3.7) yields

$$\int_{-1}^{x_i} \frac{\phi(x)}{(x_i - x_k)^{1-\alpha}} = \sum_{\substack{k=0 \\ i \neq k}}^{N} q_{ik}(\lambda) \frac{\phi(x_k)}{(x_i - x_k)^{1-\alpha}},$$

with

$$q_{ik}(\lambda) = \sum_{j=0}^{N} \sum_{r=0}^{[\frac{1}{2}j]} \frac{2\theta_k G_r^j(\lambda)(\psi_j^{(\lambda)})^{-1}}{N(j-2r+1)} (1-x_k^2)^{\lambda-\frac{1}{2}} C_j^{(\lambda)}(x_k)\big(x_i^{j-2r+1} - (-1)^{j-2r+1}\big),$$

for $x_i, x_k \in S$, with $\theta_0 = \theta_N = \frac{1}{2}$, $\theta_k = 1$ *for* $k = 1, 2, \dots, N-1$, which completes the proof. □

5. Shifted ultraspherical polynomials

The shifted ultraspherical polynomials are defined on $[0,1]$ by

$$\tilde{C}_n^{(\lambda)}(x) = C_n^{(\lambda)}(2x-1).$$

All results of ultraspherical polynomials can be easily obtained for their shifted. The orthogonality relation for $\tilde{C}_n^{(\lambda)}(x)$ with respect to the weight function $(x-x^2)^{\lambda-1/2}$ is given by

$$\int_0^1 (x-x^2)^{\lambda-\frac{1}{2}} \tilde{C}_j^{(\lambda)}(x)\tilde{C}_k^{(\lambda)}(x)dx = \begin{cases} 0 & j \neq k \\ 4^{-\lambda}\psi_j^{(\lambda)} & j = k, \end{cases}$$

where $\psi_j^{(\lambda)}$ is given in (3.2)(see [4])
. As a direct consequence of Theorem 4, we have the following corollary.

1. Let $f(x) \in L^2[0,1]$, then

$$f(x) \simeq \sum_{j=0}^{N} a_j \tilde{C}_j^{(\lambda)}(x), \tag{5.1}$$

   where
   $a_j = 4^{\lambda}(\psi_j^{(\lambda)})^{-1} \int_0^1 (x-x^2)^{\lambda-\frac{1}{2}} \tilde{C}_j^{(\lambda)}(x)f(x).$

2. Let $f(x)$ be approximated by shifted ultraspherical polynomials, then there exists a matrix $\tilde{Q} = [\tilde{q}_{ij}], \quad i,j = 0\dots N$, satisfying

$$\int_0^{x_i} f(x)dx \simeq \sum_{\substack{k=0\\ t\neq k}}^{N} \tilde{q}_{ik}(\lambda)f(x_k), \tag{5.2}$$

   where

$$\tilde{q}_{ik}(\lambda) = \sum_{j=0}^{N}\sum_{r=0}^{[\frac{1}{2}j]} \frac{4^{\lambda}2\theta_k \tilde{G}_r^j(\lambda)(\psi_j^{(\lambda)})^{-1}}{N(j-2r+1)}(x_k - x_k^2)^{\lambda-\frac{1}{2}}\tilde{C}_j^{(\lambda)}(x_k)\big((2x_i-1)^{j-2r+1} - (-1)^{j-2r+1}\big), \tag{5.3}$$

   for $x_i, x_k \in S$, with $\theta_0 = \theta_N = \frac{1}{2}$, $\theta_k = 1 \quad for \quad k = 1,2,\dots,N-1.$

6. Description of the method

In this section, we shall present the ultraspherical spectral integration method for solving the problems $(1.1)-(1.2)$. For this, purpose we give ultraspherical integration matrix for the highest order fractional derivative in the problem (1.1), i.e

$$\varphi(x) = {}^{c}D_0^{\alpha}y(x), \quad n-1 < \alpha \leq n. \tag{6.1}$$

An application of the integral operator I^{α} to both sides of (5.1) and using the initial conditions (1.2) and part3 of (2.4), yield (for $a=0$)

$$y(x) = \sum_{k=0}^{n-1} b_k \frac{x^k}{k!} + \frac{1}{\Gamma(\alpha)}\int_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}dt.$$

Hence, for $x \in S$ and use the theorem 4, we get

$$y(x_i) \simeq \sum_{k=0}^{n-1} b_k \frac{x_i^k}{k!} + \frac{1}{\Gamma(\alpha)}\sum_{\substack{k=0\\ i\neq k}}^{N} \frac{\varphi(x_k)q_{ik}}{(x_i-x_k)^{1-\alpha}}. \tag{6.2}$$

Substituting (6.1) and (6.2) into (1.1), it can be written as:

$$\varphi(x_i) \simeq F(x_i)\left(\sum_{k=0}^{n-1} b_k \frac{x_i^k}{k!} + \frac{1}{\Gamma(\alpha)}\sum_{\substack{k=0\\ i\neq k}}^{N} \frac{\varphi(x_k)q_{ik}(\lambda)}{(x_i-x_k)^{1-\alpha}}\right), \quad 0 \le i \le N,$$

or

$$\varphi_i - F(x_i)\left(\sum_{k=0}^{n-1} b_k \frac{x_i^k}{k!} + \frac{1}{\Gamma(\alpha)}\sum_{\substack{k=0\\ i\neq k}}^{N} \frac{\varphi(x_k)q_{ik}(\lambda)}{(x_i-x_k)^{1-\alpha}}\right) \simeq 0.$$

This can be written as NLP problem:

$$F_k = (\varphi_0, \varphi_1, ..., \varphi_n, \lambda) = 0, \quad k = 0, ...N.$$

By using partial quadratic interpolation method [13], we can solve the above unconstrained optimization problem.

7. Partial Quadratic Interpolation Method(PQI)

The essential steps to apply the partial quadratic interpolation method [13] can be summarized as follows:
1)Choose some starting point $x^0 \in \mathbf{R}_m$and $\nu = 1$.
2)Approximate the function $F(x)$about in the quadratic form

$$F(X) = a + [B_m(x^r)]^{Tr}[x - x^r] + \frac{1}{2}[x - x^r]^{Tr}[A_m(x^r)][x - x^r]$$

where $A_m$and B_m represent the gradient vector and the Hessian matrix of the function $F(x)$ respectively. To compute particular values for a, A_m, B_m we choose a set of interpolation points as follows:
i) m points $[x_{i+}^r], i = 1, ..., m$

$$[x_{i+}^r] = (x_1^r, x_2^r, ..., x_{i-1}^r, x_i^r + l_i, x_{i+1}^r, ..., x_m^r)$$

ii)m points $[x_{i-}^r], i = 1, ..., m$

$$[x_{i-}^r] = (x_1^r, x_2^r, ..., x_{i-1}^r, x_i^r - l_i, x_{i+1}^r, ..., x_m^r)$$

iii)$\frac{m(m-1)}{2}$ point $[x_{ij}^r], i = 1, ..., m-1, j - i + 1, ..., m$ where

$$[x_{ij}^r] = (x_1^r, x_2^r, ..., x_{i-1}^r, x_i^r + l_i, x_{i+1}^r, ..., x_j^r + l_j, ..., x_m^r)$$

Using these interpolation points it can be shown that $a = F(x^r)$and the elements b_i, a_j,of A_m, B_m respectively are given by

$$b_i = \frac{F(l_{i+}) - F(x_{i-1})}{2l_i}, a_{ii} = \frac{F(x_{ij}^r) - 2F(x^r) + F(x_{i-}^r)}{l_i^2}, a_{ij} = \frac{F(x_{ij}^r) - F(x_{i+}^r) - F(x_{j+}^r) + F(x^r)}{l_i l_j}$$

The l_i are a set of constants which determine the accuracy of the inerpolation.
3)Extract the symmetric positive definite matrix $[A_q]$ from the symmetric matrix $[A_m]$ using Choliski method, $q \leqslant m$ by cancelling certain rows and columns. Essentially we write $A_m = [S_m][S_m]^{Tr}$. From this we have

$$s_{11}^2 = a_{11}$$

If $a_{11} \leqslant 0$ then we eliminate the first row and column in each of $[A_m], [S_m] and [S_m]^{Tr}$ and perform the calculation on the $[A_{m-1}], [S_{m-1}] and [S_{m-1}]^{Tr}$. If $a_{11} \geqslant 0$ then we have

$$s_{11}^2 = \sqrt{a_{11}}, s_{1j} = \frac{a_{1j}}{s_{11}}, j = 2, ..., m.$$

Let us now suppose that we have operated on the first $j-1$ columns of $[S]$, i.e. We have either calculated the elementes or eliminated them. The operation on the j^{th} column gives

$$s_{jj}^2 = a_{jj} - \sum_{i\in k1, i\leqslant j} s_{ij}^2.$$

where $k1$ is the set of indices of rows and columns not eliminated. If $s_{jj} \leqslant 0$ we eliminate the j^{th} columns and rows $[A_q]$ and $[S_q]$ where $[A_q]$ and $[S_q]$ are the current reduced matrices derived to date from $[A_q]$ and $[S_q]$, $q \leqslant m$. Otherwise we take

$$s_{jj} = \sqrt{a_{jj} - \sum_{i\in k1, i\leqslant j} s_{ij}^2}, s_{ij} = \frac{[a_{ij} - \sum_{i\in k1, i\leqslant j} s_{ij}^2]}{s_{jj}}$$

This process is repeated for each column until we finally obtain the reduced matrix $[A_q]$ given by

$$[A_q] = [S_q][S_q]^{Tr}$$

4)Solve the system of the linear equations

$$[A_q][\Delta x_i] = [B_q]$$

where B_q is the reduced form of gradiant vector corresponding to A_q.
5)Compute a new point $x^{\nu+1}$ from

$$x_i^{\nu+1} = \begin{cases} x_i^{\nu+1} + \beta \triangle x_i, & for x_i \in \mathbf{R}_q \\ x_i^{\nu}, & for x_i \in \mathbf{R}_q \end{cases}$$

where β is a parameter which takes values $1, \frac{1}{2}, \frac{1}{4}, \dots$ and we use the first value of β which satisfies $F(x^{\nu+1}) < F(x^{\nu})$. If β becomes too small without satisfying this condition, the calculation can be restarted with a finer approximation of the matrices $[A_m]$ and $[B_m]$, i.e. smaler values l_i.

8. Modified PQI Method

In PQI method we set $t = 1, \frac{1}{2}, \frac{1}{4}, \dots$ and we take the first value of t which satisfies the condition

$$f(x^{r+1}) < f(x^r), x^r \in \mathbf{R}_n$$

However, the value of t taken by this way may not be the optimal value of t. Since there is a great possibility that the optimal value t^* lies between these values, i.e. bwtween$t = 1, \frac{1}{2}, \frac{1}{4}, \dots$ to get the optimal step size t^* we suggest the following modification:
Let us approximate $f(t) = f(x^r + t\delta x^r)$ by a polynomial of second degree $p_2(t)$ over the interval $[0, 1]$ as following:

$$f(t) = p_2(t) = \begin{bmatrix} 1 & \frac{t}{h} & (\frac{t}{h})^2 \end{bmatrix} [L_2] \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix}$$

where h is the interval of the interpolation, $f_{2i} = f(u^r + i\delta u^r), i = 0, \frac{1}{2}, 1$ and L_2 is a Lagrange matrix where

$$L_2 = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ -3 & 4 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

from this and $p_2(t)$ we have

$$p_2(t) = \begin{bmatrix} 1 & \frac{t}{h} & (\frac{t}{h})^2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ -3 & 4 & -1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix}$$

then,

$$p_2'(t) = \frac{1}{h}(-\frac{2}{3}f_0 + 2f_1 - \frac{1}{2}f_2) + \frac{2t^*}{h^2}(\frac{1}{2}f_0 - 4f_1 + f_2) = 0$$

and taking $h = \frac{1}{2}$ we obtain

$$t^* = \frac{3f_0 - 4f_1 + f_2}{4(f_0 - 2f_1 + f_2)}$$

9. Error estimation and convergence

Theorem 9.1. [15] *Let* $f(x) \in C^\infty[-1, 1]$ *be approximated by*

$$f(x) \cong \sum_{k=0}^{N} a_k C_k^{(\lambda)}(x), \tag{9.1}$$

then for each x *in* $[-1, 1]$, *there exists a number* $\xi(x) \in [-1, 1]$, *such that*

$$R_N^{[\lambda]}(x, \xi) = f(x) - P_N(x) = \frac{f^{N+1}(\xi)}{(N+1)!K_{N+1}^{(\lambda)}} C_{N+1}^{\lambda}(x) \tag{9.2}$$

and

$$\|R_N^{(\lambda)}(x, \xi)\| \leq \max_{1\leq\delta\leq 1} \frac{1}{(N+1)!K_{N+1}^{(\lambda)}} \|f^{N+1}(\xi)\|,$$

where

$$K_N^{\lambda} = 2^N \frac{\Gamma(N+\lambda)\Gamma(2\lambda+1)}{\Gamma(N+2\lambda)\Gamma(\lambda)}.$$

Proof. See [7]. □

Theorem 9.2. *Let* $\frac{\varphi(x)}{(t-x)^{1-\alpha}} (t \in S, t \neq x)$ *be approximated by* (9.1), *then there exists a number* ξ *in* $[-1, 1]$ *such that*

$$\int_{-1}^{x_i} \frac{\varphi(x)}{(x_i - x)^{1-\alpha}} dx = \sum_{\substack{k=0 \\ i\neq k}}^{N} \frac{\varphi(x_k) q_{ik}(\lambda)}{(x_i - x_k)^{1-\alpha}} + E_N^{(\lambda)}(x_i, \xi), \tag{9.3}$$

where $x_i, x_k \in S,\ 0 \leq i \leq N$,

$$E_N^{(\lambda)}(x_i, \xi) = \frac{f^{(N+1)}(\xi)}{(N+1)!K_{N+1}^{(\lambda)}} \int_{-1}^{x_i} C_{N+1}^{[\lambda]}(x) dx \tag{9.4}$$

$$-\frac{2}{3N^2} \sum_{j=0}^{N} [\psi_j^{(\lambda)}]^{-1} H_j^{(2)}(\xi) \int_{-1}^{x_i} C_j^{(\lambda)}(x) dx. \tag{9.5}$$

By $f(x) = \frac{\varphi(x)}{(x_i - x)^{1-\alpha}}$, *the* (5.6) *is convergence.*

Proof. Let $\frac{\varphi(x)}{(t-x)^{1-\alpha}}$ be approximated on S by ultraspherical polynomials, then by (5.1), (5.1), (5.3) and the trapezoidal rule, we have

$$\frac{\varphi(x)}{(x_i - x)^{1-\alpha}} \simeq \sum_{j=0}^{N} \frac{2}{N} \left[\psi_j^{(\lambda)}\right]^{-1} \sum_{k=0}^{N}{}'' (1 - x_k^2)^{\lambda - \frac{1}{2}} C_j^{(\lambda)}(x_k) \frac{\varphi(x_k)}{(x_i - x_k)^{1-\alpha}} C_j^{(\lambda)}(x)$$
$$-\frac{2}{3N^2} (\psi_j(\lambda))^{-1} H_j^{(2)}(\xi))^r C_j^{(\lambda)}(x) + \frac{f^{(N+1)}(\xi)}{(N+1)!K_{N+1}^{(\lambda)}} C_{N+1}^{(\lambda)}(x).$$

then integration of both sides gives

$$\int_{-1}^{x_i} \frac{\varphi(x)}{(x_i-x)^{1-\alpha}}dx = \sum_{j=0}^{N} \frac{2}{N}\left[\psi_j^{(\lambda)}\right]^{-1} \sum_{k=0}^{N}{}'' (1-x_k^2)^{\lambda-\frac{1}{2}} C_j^{(\lambda)}(x_k) \frac{\varphi(x_k)}{(x_i-x_k)^{1-\alpha}}$$

$$\int_{-1}^{x_i} C_j^{(\lambda)}(x)dx - \frac{2}{3N^2}\sum_{j=0}^{N}\left(\psi_j^{(\lambda)}\right)^{-1} H_j^{(2)}(\xi)\int_{-1}^{x_i} C_j^{(\lambda)}(x)dx$$

$$+\frac{f^{(N+1)}(\xi)}{(N+1)!K_{N+1}^{[\lambda]}}\int_{-1}^{x_i} C_{N+1}^{(\lambda)}(x)dx$$

$$= \sum_{\substack{k=0 \\ i\neq k}}^{N} q_{ik}(\lambda)\frac{\varphi(x_k)}{(x_i-x_k)^{1-\alpha}} + E_N^{(\lambda)}(x_i,\xi),$$

where $E_N^{(\lambda)}(x_i,\xi)$ is defined by (5.7) and

$$f^{(N+1)}(x) = \left(\frac{\varphi(x)}{(t-x)^{1-\alpha}}\right)^{(N)} = \varphi^N(t-x)^{\alpha-1} + N\varphi^{(N-1)}(-1)(\alpha-1)(t-x)^{\alpha-2} + \dots$$

$$+\varphi(-1)^N\alpha(\alpha-1)\dots(\alpha-N)(t-x)^{\alpha-(N+1)}$$

$$= \sum_{k=0}^{N} \varphi^k(-1)^k\alpha(\alpha-1)\dots(\alpha-k)\cdot(t-x)^{\alpha-(k+1)}$$

Since, the first term in (5.7) is bounded, it is enough for showing the convergence of (5.6), to show that the second term is bounded. For this purpose, we first show that $\| C_j^{(\lambda)} \| = \psi_j^{(\lambda)} < 1$. From

$$\psi_j^{(\lambda)} = 2^{1-2\lambda}\pi\frac{\Gamma(j+2\lambda)}{(j+\lambda)[\Gamma(\lambda)]^2\Gamma(j+1)}, \quad \lambda \neq 0.$$

and

$$2\int_0^{\pi/2} \cos^{(2x-1)}(t)\sin^{(2y-1)}(t)dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \Gamma(j+1) = j\Gamma(j),$$

we have:

$$\psi_j^{(\lambda)} \leq \frac{\pi 2^{1-2\lambda}}{\pi^2 j(j+\lambda)} < 1. \tag{9.6}$$

Since $x \in [-1,1]$

$$H_j(x) = (1-x^2)^{\lambda-\frac{1}{2}} C_j^{(\lambda)}(x)\frac{\phi(x)}{(t-x)^{1-\alpha}}$$

is a twice continuously differentiable function, then there exists a real number M such that $\| H_j^{(2)}(x) \| \leq M$. Finally,

$$\| \frac{2}{3N^2}\left(\psi_j^{(\lambda)}\right)^{-1} H_j(x)^{(2)} \int_{-1}^{x_i} C_j^{\lambda}(x)dx \| \leq \frac{2}{3N^2}\left(\psi_j^{(\lambda)}\right)^{-1} \| H_j(x)^{(2)} \| 2\psi_j^{(\lambda)} \leq \frac{4M}{3N^2},$$

and so $E_N^{(\lambda)}(x_i,\xi)$ is bounded. □

10. Numerical examples

In this section, we give computational results of some examples, to support our theoretical results. let us consider an initial value problem for one of the simplest fractional order differential equations appearing in applied problems:

$$ {}^{c}D_{x}^{\alpha}y(x)+Ay(x)=f(x),\quad 0<x \tag{10.1}$$

with initial conditions:

$$ y^{(k)}(0)=0,\qquad k=0,1,\ldots,n-1, \tag{10.2}$$

where A is a positive constant and $n-1<\alpha<n$. For $0<\alpha\leq 2$ this equation is called the relaxation-oscillation equation.

Example 10.1. *As the first example, we consider the following initial value problem:*

$$ {}^{c}D_{0}^{(\alpha)}y(x)+y(x)=x^{4}+\frac{24x^{4-\alpha}}{\Gamma(5-\alpha)},\quad 0\leq x\leq 1 \tag{10.3}$$

with the initial condition $y(0)=0$ *and the exact solution* $y(x)=x^{4}$. *Let* $\alpha=\frac{1}{8}$ *and* $\varphi(x)={}^{c}D_{0}^{\frac{1}{8}}y(x)$. *By using the shifted ultraspherical polynomials and system* (4.7), (4.8) *and corollary* 1, 2 *we have the form,*

$$\varphi_{i}+\frac{1}{\Gamma\left(\frac{1}{8}\right)}\sum_{\substack{k=0\\ i\neq k}}^{N}\frac{\varphi_{k}\tilde{q}_{ik}(\lambda)}{(x_{i}-x_{k})^{\frac{7}{8}}}-x_{i}^{4}-\frac{24x_{i}^{\frac{3}{8}}}{\Gamma\left(\frac{39}{8}\right)}=0,$$

where $\tilde{q}_{ik}(\lambda)$ *is defined by* (4.9), *for* $x_{i}\in S, i=0,1,...,N$.

This can be written as NLP problem:

$$F_{k}=(\varphi_{0},\varphi_{1},...,\varphi_{n},\lambda)=0,\ \ k=0,....N.$$

By using quadratic interpolation method, we can solve the above unconstrained optimization problem. The results are computed at different numbers of α on the $[0,1]$. They show the efficiency and spectral accuracy of ultraspherical integration method. In table [1], we compare the approximate solutions of Eq (6.1) obtained by the ultraspherical integration method with the exact solution for the $N=5$, $\alpha=\frac{1}{16},\alpha=\frac{1}{8},\alpha=\frac{1}{2}$and $\lambda=0.55$.

Table 1: Absolute error of example 10.1

x_i	$\alpha=\frac{1}{16}$	$\alpha=\frac{1}{8}$	$\alpha=\frac{1}{2}$
x_0	0.0000	0.0000	0.0000
x_1	0.0001	0.0002	0.0007
x_2	0.00019	0.00020	0.00074
x_3	0.00023	0.00033	0.000305
x_4	0.0029	0.0034	0.00835
x_5	0.00717	0.0084	0.087

Example 10.2. *Our second example covers the linear equations:*

$$ {}^{c}D_{0}^{(\alpha)}y(x)+y(x)=x^{8}+\frac{\Gamma(9)x^{8-\alpha}}{\Gamma(9-\alpha)},\quad 0\leq x\leq 1,\quad 0<x\leq 1,$$

with the initial conditions $y(0) = 0$ *and the exact solution* $y(x) = x^8$.
For $\alpha = 0.8$, *Let* ${}^cD_0^{(0.8)} y = \varphi(x)$ *by fractional integration we have,*

$$y(x) = \frac{1}{\Gamma(0.8)} \int_0^x \frac{\varphi(t)}{(x-t)^{0.2}} dt \tag{10.4}$$

$$y(x_i) \simeq \frac{1}{\Gamma(2/3)} \sum_{k=0}^{N} \frac{\varphi(x_k)\tilde{q}_{ik}}{(x_i - x_k)^{1/2}}. \tag{10.5}$$

By using the shifted ultraspherical polynomials and the relations (4.7), (4.8) *and corollary* 1, 2 *we have the system :*

$$\varphi_i + \frac{1}{\Gamma\left(\frac{8}{10}\right)} \sum_{\substack{k=0 \\ i \neq k}}^{N} \frac{\varphi_k \tilde{q}_{ik}(\lambda)}{(x_i - x_k)^{1-\frac{2}{10}}} - x_i^8 - \frac{\Gamma(9) x_i^{7.2}}{\Gamma(8.2)} = 0$$

where $\tilde{q}_{ik}(\lambda)$ *is defined by* (4.9).

This can be written as NLP problem:

$$F_k = (\varphi_0, \varphi_1, ..., \varphi_n, \lambda) = 0, \quad k = 0, ...N.$$

By using quadratic interpolation method, we can solve the above unconstrained optimization problem. The results are computed at different numbers of α on the $[0, 1]$. They show the efficiency and spectral accuracy of ultraspherical integration method. In table [2], we compare the approximate solutions of Eq (6.2) obtained by the ultraspherical integration method with the exact solution for the $\alpha = 0.8, \alpha = 0.5, \alpha = 0.1$ and $\lambda = 0.75, N = 5$.

Table 2: Absolute error of example 10.2

x_i	$\alpha = 0.8$	$\alpha = 0.5$	$\alpha = 0.1$
x_0	0.0000	0.0000	0.0000
x_1	0.0000	0.0000	0.0000
x_2	0.0000	0.0000	0.0001
x_3	0.0009	0.0003	0.0009
x_4	0.0073	0.0030	0.0052
x_5	0.036	0.015	0.021

Example 10.3. *Consider the following linear initial value problem*

$${}^cD_0^{(\alpha)} y(x) + y(x) = 1, \tag{10.6}$$

with initial condition:

$$y(0) = 0. \tag{10.7}$$

The exact solution is $E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+\alpha k)}$, *where* $E_\alpha(x)$ *is a one-parameter function of the Mittag-Leffler type.*

Let

$$\varphi(x) = {}^cD_0^\alpha y(x), \tag{10.8}$$

by the fractional integration we have $y(x) = \dfrac{1}{\Gamma(\alpha)} \int_0^x \dfrac{\varphi(t)}{(x-t)^{1-\alpha}} dt$, for the $x_i \in S$;

$$y(x_i) = \frac{1}{\Gamma(\alpha)} \int_0^{x_i} \frac{\varphi(t)}{(x_i - t)^{1-\alpha}} dt, \tag{10.9}$$

by substituting and shifted ultraspherical polynomials and relations of (4.6), (4.7) and using corollary 1, 2 we have

$$\varphi_i + \frac{1}{\Gamma(\alpha)} \sum_{\substack{k=0 \\ i \neq k}}^{N} \frac{\varphi_k \tilde{q}_{ik}(\lambda)}{(x_i - x_k)^{1-\alpha}} - 1 = 0,$$

where $\tilde{q}_{ik}(\lambda)$ is defined by (4.9). This can be written as NLP problem:

$$F_k = (\varphi_0, \varphi_1, ..., \varphi_n, \lambda) = 0 \ \ k = 0, ...N.$$

By using quadratic interpolation method, we can solve the above unconstrained optimization problem. The results are computed at different numbers of α on the $[0, 1]$. They show the efficiency and spectral accuracy of ultraspherical integration method. Table [3] display absolute error function with various values of α and $N = 5, N = 10, N = 15$, and $\lambda = 0.55$.

Table 3: Absolute error of example 10.3

α	$N = 5$	$N = 10$	$N = 15$
$\alpha = 0.25$	0.00041	0.00011	0.000029
$\alpha = 0.5$	0.00153	0.00012	0.0000105
$\alpha = 0.75$	0.0073	0.000724	0.000124

11. Conclusion

In this paper we presented a numerical approach for solving the fractional differential equations. The ultraspherical functions were employed. The obtained results showed that this approach can solve the problem effectively.

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