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Construction of a Normalized Basis of a Univariate Quadratic C^1 Spline Space and Application to the Quasi-interpolation *

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ABSTRACT: In this paper, we use the finite element method to construct a new normalized basis of a univariate quadratic C^1 spline space. We give a new representation of Hermite interpolant of any piecewise polynomial of class at least C^1 in terms of its polar form. We use this representation for constructing several superconvergent and super-superconvergent discrete quasi-interpolants which have an optimal approximation order. This approach is simple and provides an interesting approximation. Numerical results are given to illustrate the theoretical ones.

Key Words: Finite element, Splines, Polar form, Quasi-interpolation.

Contents

1. Introduction

In the classical Hermite interpolation, for each one of the knots $x_0 < \ldots < x_n$ we are given a set of interpolation values $f_i^{(j)}$, $j = 0, \ldots, k$ then we need to find a polynomial function $\mathcal{S}$ of degree $2k+1$ in every interval $[x_i, x_{i+1}]$, such that $\mathcal{S}(x_i) = f_i^{(j)}$, $i = 0, \ldots, n$, $j = 0, \ldots, k$. Many authors have been working in this direction like Schoenberg [31], Lee [16] and Mummy [19], who has derived an explicit formula for the control points in terms of the interpolation data. He has used the de Boor-Fix dual functionals as an effective tool for solving this problem. In [33], Seidel gave another simple and elegant proof of Mummy's result using polar forms. Another interesting paper in this area of work is Schumaker's, he gave a general treatment of the use of quadratic splines for solving a similiar Hermite interpolation problem. The author showed exactly when it was necessary to add knots to a subinterval and where they can be placed.

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Let $\tau := (a = x_0 < x_1 < \cdots < x_n = b)$ be a partition of a bounded interval $I := [a, b]$. For the sake of the paper we assume that we know the values and the derivatives of a function f. Our problem of interpolation is to find a piecewise polynomial function $\mathcal{S}$ of class C^1 and degree two such that

$$\mathcal{S}(x_i) = f(x_i),\ \mathcal{S}^{'}(x_i) = f^{'}(x_i),\ i = 0, \ldots, n.$$

To solve this problem we need to add a new knot in each subinterval $[x_i, x_{i+1}]$, $i = 0, \ldots, n-1$, and imposing the C^1 smoothness at this knot. The associated spline space is $\mathbb{P}^1_2(I, \tau_1)$, where τ_1 is a refinement of τ which is of dimension $2(n+1)$. The Hermite spline interpolants' construction by adding some additional knots to the initial partition and increasing the polynomial pieces' number is not new. This method has been recently studied in the literature (see [17,32]) also by Lamnii et al [11]. The added knots can be chosen to preserve certain geometric shape such as monotonicity and convexity.

Various methods have been developed for building a positive and stable basis of B-splines, (see [15,35], for instance). Originally Dierckx [9] presented in the bivariate case, an algorithm for calculating a suitable normalized B-spline reprentation for Powell-Sabin spline. Similar B-spline representation for bivariate Powell-Sabin splines with higher smoothness have been considered in [37] and for bivariate Clough-Tocher splines in [36]. These B-splines have been used in [18,25] for constructing some interesting quasi-interpolants with optimal approximation order. Such a representation is given in trivariate setting by Sbibih et al [24] for C^1 quadratique B-splines over a Worsey-Piper split of a tetrahedral partition. In the recent years, Speleers [38] presented a method for constructing a normalized basis for the multivariate quadratic spline space defined over a generalized Powell-Sabin refinement of a triangulation in $\mathbb{R}^s$ $(s \geq 1)$. He has indicated that the univariate version is related to the well-known classical B-splines of degree two. In addition these results, our main objectif is to construct a compact normalized basis of univariate quadratic C^1 spline space on the real interval refined by τ_1. Thanks to the new constracted normalized B-splines we were able to find the classical quadratic B-splines of class C^1 without using any recurrence formula or divided differences. The main motivation is that we establish a new B-spline representation of Hermite interpolant $\mathcal{H}f$ of any function f of class C^1 in $\mathbb{P}^1_2(I, \tau_1)$. Particularly we obtain the Marsden Identity.

A quasi-interpolant for a given function f is obtained as a linear combination of some elements of a suitable set of basis functions. In order to achieve stability and local control, these functions are required to be positive and to have small local supports. The coefficients of the linear combination are the values of linear functionals, depending on f and (or) its derivatives or integrals. Many works concerning the construction of quasi-interpolant are developed in the literature (see [1,2,3,8,4,5,14,13,21,22,23]). The main gain of these operators is that they have a direct construction without solving any system of equations and with the minimum possible computation time.

In numerical analysis, the superconvergence is a phenomenon where the order of convergence of the approximant error at certain special points is higher than the order of convergence of the approximant error over the definition's domain (see [6,26,27,28,29]). Then by considering a local linear polynomial operator in the neighborhood of the support of the B-splines that reproduces the space of polynomials of degree at most $m \geq 2$, we propose a method to bluid superconvergent discrete quasi-interpolants of a function f. It satisfies an interesting property that these quasi-interpolants are globally of order 3 and of order $m+1$ at the knots of the initial partition τ. This property is not only true for approximating function values but also for approximating first derivative.

To improve the numerical results given by our operator, we introduce a new concept, called the super-superconvergence, when the local polynomial approximant is even. Thanks to this phenomenon, the quadratic spline quasi interpolant provides an improvement of the approximation order at the knots, it is of order $m+2$ instead of $m+1$.

The paper is organized as follows. In Section 2, we first define the finite element used in the construction of the normalized B-splines. In Section 3, we introduce a B-spline representation of the Hermite interpolant of any piecewise polynomial on the refinement τ_1 of class at least C^1 in terms of its polar form. In Section 4, we show how to construct superconvergent discrete quasi-interpolants. In Section 5, we give an estimate of the errors between the function f and the superconvergent discrete quasi-interpolants and between their first derivatives at the knots of τ. In Section 6, we introduce a new technique allowing us to establish the super-superconvergence phenomenon. We illustrate the theoretical results obtained in Sections 5 and 6 by some numerical tests.

2. Construction of the normalized basis

The main objective of this section is to describe a method allowing us to construct Hermite B-splines. The family of B-splines that we propose presents useful properties in the approximation. Let us denote by $\tau := (a = x_0 < x_1 < \cdots < x_n = b)$ a partition of a bounded interval $I := [a, b]$. In order to simplify our work, we choose a uniform subdivision with $x_i = a + 2ih$ where $h := \frac{b-a}{2n}$ and suppose that we know the values of a function f and its first derivatives at the knots x_i, $i = 0, \ldots, n$.

2.1. Finite element of class C^1 and degree 2

In this subsection, we define the finite element in each subinterval $[x_i, x_{i+1}]$, through the function and its first derivative values at knots x_i and x_{i+1}. So, we have four data in these knots, which make it impossible to write a quadratic spline because it requires just three data. To remedy this problem, we consider a new refinement τ_1 of τ obtained by adding an arbitrary knot $x_{i,1}$ in the interval $]x_i, x_{i+1}[$ and by imposing the C^1 smoothness at this new point, the construction of the finite element can now be completed.

Let $\phi_{i,k}$, $i = 0, \ldots, k$ be the Bernstein polynomials of degree k defined by

$$\phi_{i,k}(t) := C_k^i t^i (1-t)^{k-i}, \ t \in [0, 1]$$

For $i = 0, \ldots, n-1$, let $\tau_{i,1} := (x_i < x_{i,1} < x_{i+1})$, be a subdivision of $[x_i, x_{i+1}]$ into two subintervals $[x_i, x_{i,1}]$ and $[x_{i,1}, x_{i+1}]$, and $\mathcal{S}_i$ be a spline of degree 2 and class C^1 defined on $[x_i, x_{i+1}]$. Denote by

$$\mathcal{S}_i^g := \mathcal{S}_i|_{[x_i, x_{i,1}]} \text{ and } \mathcal{S}_i^d := \mathcal{S}_i|_{[x_{i,1}, x_{i+1}]}$$

the restrictions of the spline $\mathcal{S}_i$ in each subinterval. The polynomials $\mathcal{S}_i^g$ and $\mathcal{S}_i^d$ are written in the Bernstein basis as follows

$$\mathcal{S}_i^g(x) = \sum_{j=0}^{2} c_j \phi_{j,2}\left(\frac{x - x_i}{h}\right), \quad \mathcal{S}_i^d(x) = \sum_{j=0}^{2} d_j \phi_{j,2}\left(\frac{x - x_{i,1}}{h}\right),$$

where $x_{i,1} = \frac{x_i + x_{i+1}}{2} = x_i + h$ and the unknown coefficients c_j and d_j, for $j = 0, 1, 2$, are determined by the values and the first derivatives of $\mathcal{S}_i$ at the knots x_i and x_{i+1} and by the C^1 smoothness at the midpoint $x_{i,1}$. Then, we can show that

$$\begin{aligned}\mathcal{S}_i^g(x) =& f(x_i)\phi_{0,2}\left(\frac{x - x_i}{h}\right) + \big(\frac{h}{2}f'(x_i) + f(x_i)\big)\phi_{1,2}\left(\frac{x - x_i}{h}\right) \\ &+ \frac{1}{2}\big(f(x_i) + f(x_{i+1}) + \frac{h}{2}\big(f'(x_i) - f'(x_{i+1})\big)\big)\phi_{2,2}\left(\frac{x - x_i}{h}\right)\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}_i^d(x) =& \frac{1}{2}\big(f(x_i) + f(x_{i+1}) + \frac{h}{2}\big(f'(x_i) - f'(x_{i+1})\big)\big)\phi_{0,2}\left(\frac{x - x_{i,1}}{h}\right) \\ &+ \big(f(x_{i+1}) - \frac{h}{2}f'(x_{i+1})\big)\phi_{1,2}\big(\frac{x - x_{i,1}}{h}\big) + f(x_{i+1})\phi_{2,2}\left(\frac{x - x_{i,1}}{h}\right).\end{aligned}$$

The quadratic finite element $\mathcal{S}_i$ of class C^1 on $[x_i x_{i+1}]$ is defined by $\mathcal{S}_i^d$ and $\mathcal{S}_i^g$ (see Figure 1).

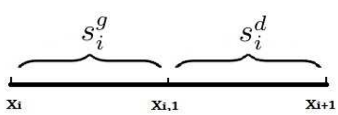

Figure 1: Finite element of class C^1 and degree 2.

Define $\tau_1 := \cup_{i=0}^{n-1}\tau_{i,1}$ as a refinement of τ. The space of C^1 quadratic splines on the interval I endowed with τ_1

$$\mathbb{P}_2^1(I,\tau_1) := \{\mathcal{S} \in C^1(I):\ \mathcal{S}_{|_{[x_i,x_{i,1}]}} \text{ and } \mathcal{S}_{|_{[x_{i,1},x_{i+1}]}} \in \mathbb{P}_2(\mathbb{R}),\ i=0,\dots,n-1\},$$

where $\mathbb{P}_2(\mathbb{R})$ is the polynomial space of degree two.

In [32], Schumaker proved that for a given data $f(x_i)$ and $f'(x_i)$, $i=0,\dots,n$, there exists a unique spline $\mathcal{S} \in \mathbb{P}_2^1(I,\tau_1)$ solution of the following Hermite interpolation problem:

$$\mathcal{S}(x_i) = f(x_i),\ \mathcal{S}^{'}(x_i) = f^{'}(x_i),\ i=0,\dots,n. \tag{2.1}$$

Therefore, the dimension of the space $\mathbb{P}_2^1(I,\tau_1)$ equals $2(n+1)$.

2.2. Hermite basis of space $\mathbb{P}_2^1(I,\tau_1)$

Let φ_i and ψ_i be the solution functions of the problem (2.1) in $\mathbb{P}_2^1(I,\tau_1)$ which satisfy the conditions

$$\varphi_i(x_j) = \delta_{ij},\ \varphi_i'(x_j) = 0,\ j=0,..,n,$$

$$\psi_i(x_j) = 0,\ \psi_i'(x_j) = \delta_{ij},\ j=0,..,n,$$

where δ_{ij} stands for the Kronecker symbol. We can easily verify that the supports of φ_i and ψ_i are $\operatorname{supp}\varphi_i = \operatorname{supp}\psi_i = [x_{i-1},x_{i+1}]$ and that the spline $\mathcal{S}$ solution of the problem (2.1) can be written as

$$\mathcal{S} = \sum_{i=0}^{n}\left(f(x_i)\varphi_i + f'(x_i)\psi_i\right).$$

Furthermore, the functions $\varphi_i,\ \psi_i,\ i=0,\dots,n$, constitute the Hermite basis of the space $\mathbb{P}_2^1(I,\tau_1)$. This basis presents a major disadvantage which is the instability caused by the non-positivity of its elements (see Figure 2). Consequently, it is in practice undesirable especially in the construction of approximants.

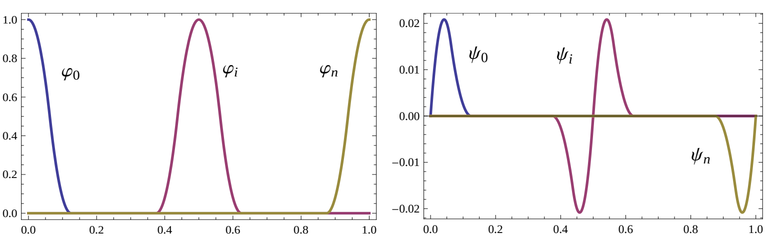

Figure 2: Hermite basis of $\mathbb{P}_2^1([0,1],\tau_1)$, $\tau_1 = \{0, \frac{1}{16}, \frac{2}{16}, \frac{3}{16}, \dots, 1\}$.

To remedy this problem we are going to build a normalized basis of the spline space $\mathbb{P}_2^1(I,\tau_1)$.

2.3. Normalized basis of $\mathbb{P}_2^1(I,\tau_1)$

In this subsection, we present the construction of new normalized B-splines. For $i=0,\dots,n$, let $[a_i,b_i]$ be an interval such that $x_i \in\ ]a_i,b_i[,\ i=1,\dots,n-1$, $a_0 = x_0$ and $b_n = x_n$. Let $\phi_{j,1},\ j=0,1$, be the Bernstein polynomials of degree 1 associated with the interval $[a_i,b_i]$, where they have been defined in subsection 2.1. Then, we construct new B-splines $H_{i,j}$ as follows

$$H_{i,0}(x) := \alpha_i\varphi_i(x) + \beta_i\psi_i(x),\ H_{i,1}(x) := (1-\alpha_i)\varphi_i(x) - \beta_i\psi_i(x), \tag{2.2}$$

where $\alpha_i = \phi_{0,1}(\frac{x_i-a_i}{b_i-a_i})$ and $\beta_i = \dfrac{d}{dx}\left(\phi_{0,1}(\frac{x-a_i}{b_i-a_i})\right)(x_i)$. It is easy to see that $H_{i,s}$, $i=0,\dots,n$, $s=0,1$, are linearly independents and therefore form a basis of the space $\mathbb{P}_2^1(I,\tau_1)$. We will prove that this basis can be constructed in such a way that its elements have a local support $[x_{i-1},x_{i+1}]$ with modifications when $i=0,n,$, are nonnegative and form a partition of unity, i.e., for each $x \in I$

$$H_{i,s}(x) \geq 0,\ \sum_{i=0}^{n}\sum_{s=0}^{1} H_{i,s}(x) = 1.$$

Remark 2.1. *The basis elements $H_{i,s}$, $s = 0, 1$ and the Bernstein basis elements of degree 1, $\phi^i_{s,1}$, $s = 0, 1$, defined in the interval $[a_i, b_i]$ take same values and first derivatives at the knot x_i, i.e.*

$$H_{i,s}(x_i) = \phi_{s,1}(\frac{x_i - a_i}{b_i - a_i}), \quad H'_{i,s}(x_i) = \frac{d}{dx}\left(\phi_{s,1}(\frac{x - a_i}{b_i - a_i})\right)(x_i), \ \forall i = 0, \ldots, n$$

The following theorem gives the non-negativity conditions.

Theorem 2.2. *The B-splines $H_{i,s}$, $s = 0, 1$, are non-negative if and only if*

$$a_i \leq x_i - \frac{h}{2} \text{ and } b_i \geq x_i + \frac{h}{2} \text{ for } i = 0, \ldots, n.$$

Proof. The B-coefficients of the basis elements $H_{i,s}$ are easily computed on the subintervals of the support. They appear in Table 1.

	$[x_{i-1}, x_i]$	$[x_i, x_{i+1}]$
φ_i	$[0\ 0\ \frac{1}{2}; \frac{1}{2}\ 1\ 1]$	$[1\ 1\ \frac{1}{2}; \frac{1}{2}\ 0\ 0]$
ψ_i	$[0\ 0\ \frac{-h}{4}; \frac{-h}{4}\ \frac{-h}{2}\ 0]$	$[0\ \frac{h}{2}\ \frac{h}{4}; \frac{h}{4}\ 0\ 0]$

Table 1: B-coefficients of the functions φ_i and ψ_i $i = 1, \ldots, n-1$ on the subintervals of $[x_{i-1}, x_{i+1}]$.

Consequently, with the boundary conditions $x_0 = a_0$ and $x_n = b_n$, the normalized B-splines coefficients in the Bernstein basis on their support are given in Table 2.

	$[x_{i-1}, x_i]$	$[x_i, x_{i+1}]$
$H_{i,0}$	$[0\ 0\ \frac{2b_i-2x_i+h}{4(b_i-a_i)}; \frac{2b_i-2x_i+h}{4(b_i-a_i)}\ \frac{2b_i-2x_i+h}{2(b_i-a_i)}\ \frac{b_i-x_i}{b_i-a_i}]$	$[\frac{b_i-x_i}{b_i-a_i}\ \frac{2b_i-2x_i-h}{2(b_i-a_i)}\ \frac{2b_i-2x_i-h}{4(b_i-a_i)}; \frac{2b_i-2x_i-h}{4(b_i-a_i)}\ 0\ 0]$
$H_{i,1}$	$[0\ 0\ \frac{2x_i-2a_i-h}{4(b_i-a_i)}; \frac{2x_i-2a_i-h}{4(b_i-a_i)}\ \frac{2x_i-2a_i-h}{2(b_i-a_i)}\ \frac{x_i-a_i}{b_i-a_i}]$	$[\frac{x_i-a_i}{b_i-a_i}\ \frac{2x_i-2a_i+h}{2(b_i-a_i)}\ \frac{2x_i-2a_i+h}{4(b_i-a_i)}; \frac{2x_i-2a_i+h}{4(b_i-a_i)}\ 0\ 0]$

Table 2: B-coefficients of $H_{i,s}$, $i = 1, \ldots, n-1$ $s = 0, 1$ on the subintervals of $[x_{i-1}, x_{i+1}]$.

For $s = 0, 1$, the B-splines $H_{i,s}$ are nonnegatives since all their B-cofficients are no-negatives, if

$$a_i \leq x_i - \frac{h}{2} \text{ and } b_i \geq x_i + \frac{h}{2} \text{ for } i = 0, \ldots, n.$$

From Table 2, we see that these conditions are also necessary. □

In Figure 3, we show four cases illustrating the form of the normalized B-splines on $[x_{i-1}, x_{i+1}]$ according to given values of a_i and b_i. Then, we can see that where the interval $[a_i, b_i]$ contains the points $x_i - \frac{h}{2}$ and $x_i + \frac{h}{2}$, we ensure the positivity of the B-splines. Otherwise, at least one of the B-splines is non-positive.

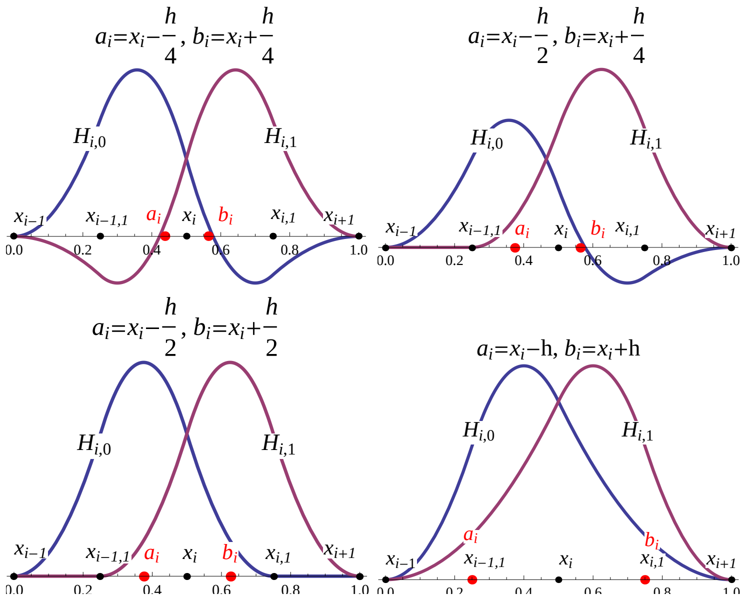

Figure 3: Normalized B-splines according to values of a_i and b_i of the space $\mathbb{P}_2^1([0,1],\tau_1)$ with $\tau_1 = \{0,\frac{1}{4},\frac{1}{2},\frac{3}{4},1\}$.

The form of the boundary B-splines change according to the boudary conditions $x_0 = a_0$ and $x_n = b_n$(see Figure 4).

Remark 2.3. *The segment* $[a_k, b_k] = [x_k - \dfrac{h}{2}, x_k + \dfrac{h}{2}]$ *represents the minimal segment ensuring the positivity of the corresponding B-splines.*

2.4. Basis of the classical B-splines

In this subsection, We prove that for a particular choice of the normalized B-splines, we can construct the classical quadratic B-splines of class C^1.
If we cancel the first non zero B-coefficient of $H_{i,1}$ and the last non zero B-coefficient of $H_{i,0}$, we obtain $a_i = x_i - \frac{h}{2}$ and $b_i = x_i + \frac{h}{2}$. Then for $i = 0,\dots,n$ we have

$$H_{i,0}(x) = \frac{1}{2}\varphi_i(x) - \frac{1}{h}\psi_i(x),\;\; H_{i,1}(x) = \frac{1}{2}\varphi_i(x) + \frac{1}{h}\psi_i(x).$$

Hence, defining the partition with knots X_i given by $X_{2i} := x_i$ and $X_{2i+1} := x_{i,1}$, the normalized B-splines $H_{i,0}$ and $H_{i,1}$ provide the classical C^1 quadratic B-splines $B_{j,2}$, $j = -2,\dots,2n-1$ associated with the subdivision X_j, $j = 0,\dots,2n$, with $X_{-2} = X_{-1} = X_0$ and $X_{2n} = X_{2n+1} = X_{2n+2}$. The support of each B-spline is $\mathrm{supp}(B_{j,2}) = [X_j, X_{j+3}]$.

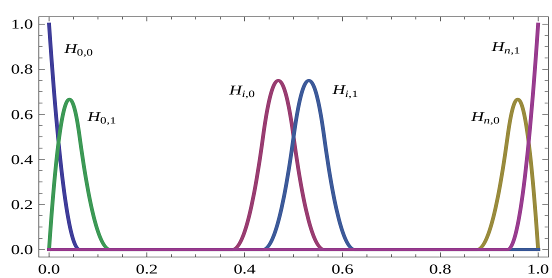

Figure 4: Classical B-spline basis of degree 2 on the interval $[0,1]$.

Remark 2.4. *The normalized B-splines are then a generalization of the classical B-splines without using the recurrence relation or divided differences.*

3. Representation of Hermite interpolant of polynomials or splines in the normalized basis

3.1. The polar form

An interesting and powerful tool in the approximation which is based on new polar approach has been emerged from the work of de Casteljau, Ramshaw and others (see [7,20,34]). The polar form of a polynomial is a transformation that reduces its complexity by adding new variables while having a certain symmetry property. In this subsection, we review some basic properties of the blossoming principle.

Definition 3.1. *Let $m \in \mathbb{N}$ and $u_1, u_2, \ldots, u_m \in \mathbb{R}$. For each $p \in \mathbb{P}_m(\mathbb{R})$, the polar form $\hat{p}$ of p (or the blossom $\mathcal{B}[p]$ of p) is a function of m variables satisfying the following properties:*

- *Multi-affine: for any index i and any real number λ, it holds*

$$\begin{aligned}\hat{p}(u_1, \ldots, u_{i-1}, \lambda u + \overline{\lambda} v, u_{i+1}, \ldots, u_m) &= \lambda \hat{p}(u_1, \ldots, u_{i-1}, u, u_{i+1}, \ldots, u_m)\\ &+ \overline{\lambda} \hat{p}(u_1, \ldots, u_{i-1}, v, u_{i+1}, \ldots, u_m),\end{aligned}$$

  where $\overline{\lambda} := 1 - \lambda$.

- *Symmetry: for any permutation σ of the set $\{1, 2, \ldots, m\}$ it holds*

$$\hat{p}(u_1, \ldots, u_m) = \hat{p}(u_{\sigma(1)}, \ldots, u_{\sigma(m)}).$$

- *Diagonal: $\hat{p}$ reduces to p when evaluated on its diagonal for each real number u, i.e.,*

$$\hat{p}(u, \ldots, u) = p(u).$$

In order to express the polar form of a product of polynomials of the first degree, we have the following result.

Proposition 3.2. *Let $l_1, l_2, \ldots, l_m$ be m polynomials in $\mathbb{P}_1$, and let $\mathcal{P}_m$ denote the symmetric group of all permutations of the set $\{1, 2, \ldots, m\}$. If*

$$p(x) = \prod_{i=1}^{m} l_i(x),$$

then we have

$$\hat{p}(u_1, u_2, \ldots, u_m) = \frac{1}{m!} \sum_{\sigma \in \mathcal{P}_m} \prod_{i=1}^{m} l_i(u_{\sigma(i)}).$$

Proof. We put

$$q(u_1, u_2, \ldots, u_m) = \frac{1}{m!} \sum_{\sigma \in \mathcal{P}_m} \prod_{i=1}^{m} l_i(u_{\sigma(i)}).$$

q is multiaffine. Indeed, let $\lambda,\ \overline{\lambda} \in \mathbb{R}$ such that $\overline{\lambda} = 1 - \lambda$ and $k \in \{1, 2, \ldots, m\}$. Then

$$\begin{aligned}
q(u_1, \ldots, u_{k-1}, \lambda v_k + \overline{\lambda} w_k, u_{k+1}, \ldots, u_m) &= \sum_{\sigma \in \mathcal{P}_m} l_{\sigma^{-1}(k)}(\lambda v_k + \overline{\lambda} w_k) \prod_{\substack{i=1 \\ i \neq \sigma^{-1}(k)}}^{m} l_i(u_{\sigma(i)}) \\
&= \lambda \sum_{\sigma \in \mathcal{P}_m} l_{\sigma^{-1}(k)}(v_k) \prod_{\substack{i=1 \\ i \neq \sigma^{-1}(k)}}^{m} l_i(u_{\sigma(i)}) \\
&+ \overline{\lambda} \sum_{\sigma \in \mathcal{P}_m} l_{\sigma^{-1}(k)}(w_k) \prod_{\substack{i=1 \\ i \neq \sigma^{-1}(k)}}^{m} l_i(u_{\sigma(i)}) \\
&= \lambda \sum_{\sigma \in \mathcal{P}_m} \prod_{i=1}^{m} l_i(u_{\sigma(i)}) + \overline{\lambda} \sum_{\sigma \in \mathcal{P}_m} \prod_{i=1}^{m} l_i(u_{\sigma(i)}) \\
&= \lambda q(u_1, \ldots, u_{k-1}, v_k, u_{k+1}, \ldots, u_m) \\
&+ \overline{\lambda} q(u_1, \ldots, u_{k-1}, w_k, u_{k+1}, \ldots, u_m).
\end{aligned}$$

q is symmetric by construction.
If we suppose that $u_1 = u_2 = \ldots = u_n = u$, we obtain

$$q(u_1, u_2, \ldots, u_m) = \frac{1}{m!} \sum_{\sigma \in \mathcal{P}_m} \prod_{i=1}^{m} l_i(u) = \frac{1}{m!} \prod_{i=1}^{m} l_i(u) Card(\mathcal{P}_m) = p(u).$$

Then, q is diagonal.
Finally, q satisfies the properties of Definition 3.1 and by uniqueness of the blossom we deduce that

$$\hat{p}(u_1, u_2, \ldots, u_m) = q(u_1, u_2, \ldots, u_m).$$

Hence the result. □

3.2. Quadratic Hermite interpolation

By using the polar form approach, we give some result to represent the Hermite interpolant $\mathcal{H}f$ of any function f of class C^1 in $\mathbb{P}_2^1(I, \tau_1)$.

Let m be an integer greater than or equal to 2 and let A_i and B_i be two points such that

$$A_i := m a_i - (m-1) x_i, \ \ B_i := m b_i - (m-1) x_i. \tag{3.1}$$

It is well known, see [10] page 5, that every polynomial q of degree $\leq k$ defined on a segment $[a_i, b_i]$ can be written in the Bernstein basis of $\mathbb{P}_k$ as follows :

$$q(x) = \sum_{s=0}^{k} \widehat{q}\left(a_i^{k-s} b_i^{s}\right) \phi_{s,k}\left(\frac{x - a_i}{b_i - a_i}\right), \tag{3.2}$$

where $\phi_{s,k}$ are the Bernstein polynomials of degree k defined in subection 2.1.

Theorem 3.3. *For any $f \in C^1(I)$, the Hermite interpolant $\mathcal{H}f$ of f in the space $\mathbb{P}_2^1(I, \tau_1)$ is given by*

$$\mathcal{H}f(x) = \sum_{i=0}^{n} \sum_{s=0}^{1} \left(f(x_i) + \left(\frac{A_i^{1-s} B_i^{s} + (m-1) x_i}{m} - x_i \right) f'(x_i) \right) H_{i,s}(x) \ , \tag{3.3}$$

for all $x \in I$ and $m \geq 2$.

Proof. Define

$$\psi(x) := \sum_{i=0}^{n}\sum_{s=0}^{1}\left(f(x_i) + \left(\frac{A_i^{1-s}B_i^s + (m-1)x_i}{m} - x_i\right) f'(x_i)\right) H_{i,s}(x).$$

For $0 \leq l \leq n$, we have

$$\begin{aligned}
\psi(x_l) &= \sum_{i=0}^{n}\sum_{s=0}^{1}\left(f(x_i) + \left(\frac{A_i^{1-s}B_i^s + (m-1)x_i}{m} - x_i\right) f'(x_i)\right) H_{i,s}(x_l) \\
&= \sum_{s=0}^{1}\left(f(x_l) + \left(\frac{A_l^{1-s}B_l^s + (m-1)x_l}{m} - x_l\right) f'(x_l)\right) \phi_{s,1}\left(\frac{x_l - a_l}{b_l - a_l}\right) \\
&= \left(f(x_l) + \left(\frac{(m-1)x_l}{m} - x_l\right) f'(x_l)\right)\left(\phi_{0,1}\left(\frac{x_l - a_l}{b_l - a_l}\right) + \phi_{1,1}\left(\frac{x_l - a_l}{b_l - a_l}\right)\right) \\
&\quad + \frac{f'(x_l)}{m}\left(A_l\phi_{0,1}\left(\frac{x_l - a_l}{b_l - a_l}\right) + B_l\phi_{1,1}\left(\frac{x_l - a_l}{b_l - a_l}\right)\right).
\end{aligned}$$

From (3.1) and the partition of the unity of the Bernstein basis, we have

$$\psi(x_l) = f(x_l) - x_l f'(x_l) + f'(x_l)\sum_{s=0}^{1}\mathcal{B}[x](a_l^{1-s}b_l^s)\phi_{s,1}\left(\frac{x_l - a_l}{b_l - a_l}\right).$$

Using (3.2), we obtain

$$x_l = \sum_{s=0}^{1}\mathcal{B}[x](a_l^{1-s}b_l^s)\phi_{s,1}\left(\frac{x_l - a_l}{b_l - a_l}\right).$$

Then,

$$\psi(x_l) = f(x_l),\ 0 \leq l \leq n.$$

On the other hand, we have

$$\psi'(x) = \sum_{i=0}^{n}\sum_{s=0}^{1}\left(f(x_i) + \left(\frac{A_l^{1-s}B_l^s + (m-1)x_i}{m} - x_i\right) f'(x_i)\right) H'_{i,s}(x).$$

Then, $0 \leq l \leq n$, we get

$$\begin{aligned}
\psi'(x_l) =& \sum_{i=0}^{n}\sum_{s=0}^{1}\left(f(x_i) + \left(\frac{A_l^{1-s}B_l^s + (m-1)x_i}{m} - x_i\right) f'(x_i)\right) H'_{i,s}(x_l) \\
=& \sum_{s=0}^{1}\left(f(x_l) + \left(\frac{A_l^{1-s}B_l^s + (m-1)x_l}{m} - x_l\right) f'(x_l)\right)\frac{\partial}{\partial x}\left(\phi_{s,1}\left(\frac{x - a_l}{b_l - a_l}\right)\right)(x_l) \\
=& \frac{f'(x_l)}{m(b_l - a_l)}(B_l - A_l).
\end{aligned}$$

By using (3.1), we get

$$\psi'(x_l) = f'(x_l),\ 0 \leq l \leq n.$$

Finally, by uniqueness of the Hermite interpolant in the space $\mathbb{P}_2^1(I, \tau_1)$, we deduce that $\psi = \mathcal{H}f$, which completes the proof. □

Let us denote by x_i^m the expression $\underbrace{x_i, x_i, \ldots, x_i}_{m\ times}$. The value m is called the multiplicity of x_i. In the following theorem we represent the Hermite interpolant of any spline in the space $\mathbb{P}_m^1(I, \tau_1)$ in terms of its polar form.

Theorem 3.4. *For any piecewise polynomial $\boldsymbol{S}$ of degree $m \geq 2$ and of class C^1 over the refinement τ_1, we have*

$$\mathcal{H}\boldsymbol{S}(x) = \sum_{i=0}^{n}\sum_{s=0}^{1} \widehat{\boldsymbol{S}}_i(A_i^{1-s}B_i^s, x_i^{m-1})H_{i,s}(x), \;\; \forall x \in I, \tag{3.4}$$

where $\boldsymbol{S}_i$ is the restriction of $\boldsymbol{S}$ on the interval $[x_{i-1,1}, x_i]$ or $[x_i, x_{i,1}]$.

Proof. Let $\mathbf{S}$ be a polynomial spline of degree $m \geq 2$ and of class C^1 on the interval I endowed with the refinement τ_1. From Theorem 3.3, the Hermite interpolant in $\mathbb{P}_2^1(I, \tau_1)$ of $\mathbf{S}$ can be written in the form

$$\mathcal{H}\mathbf{S}(x) = \sum_{i=0}^{n}\sum_{s=0}^{1}\left(\mathbf{S}(x_i) + \left(\frac{A_i^{1-s}B_i^s + (m-1)x_i}{m} - x_i\right)\mathbf{S}'(x_i)\right)H_{i,s}(x),$$

where $\mathbf{S}_i$ is the restriction of $\mathbf{S}$ on the interval $[x_{i-1,1}, x_i]$ or $[x_i, x_{i,1}]$. Then

$$\mathcal{H}\mathbf{S}(x) = \sum_{i=0}^{n}\sum_{s=0}^{1}\left(\mathbf{S}_i(x_i) + \left(\frac{A_i^{1-s}B_i^s + (m-1)x_i}{m} - x_i\right)\mathbf{S}_i'(x_i)\right)H_{i,s}(x).$$

From Taylor expansion, we have

$$\mathbf{S}_i(x) = \mathbf{S}_i(x_i) + (x - x_i)\mathbf{S}_i'(x_i) + \ldots. + \frac{(x-x_i)^m}{m!}\mathbf{S}_i^{(m)}(x_i).$$

For $k = 2, \ldots, m$, we put

$$l_j(x) = \begin{cases} x - x_i, & \text{if} \quad 1 \leq j \leq k \\ 1, & \text{if} \quad k+1 \leq j \leq m. \end{cases}$$

By using the Proposition 3.2, one can see that, for $m \geq 2$

$$\hat{p}(A_i^{1-s}B_i^s, x_i^{m-1}) \quad = \quad \frac{1}{m}\sum_{j=1}^{m} l_j(A_i^{1-s}B_i^s)\prod_{t=1, t\neq j}^{m} l_t(x_i) \quad = \quad 0.$$

where $p(x) = \prod_{j=0}^{m} l_j(x)$. Thus

$$\widehat{\mathbf{S}}_i(A_i^{1-s}B_i^s, x_i^{m-1}) = \mathbf{S}_i(x_i) + \left(\frac{A_i^{1-s}B_i^s + (m-1)x_i}{m} - x_i\right)\mathbf{S}_i'(x_i).$$

Therefore

$$\mathcal{H}\mathbf{S}(x) = \sum_{i=0}^{n}\sum_{s=0}^{1}\widehat{\mathbf{S}}_i(A_i^{1-s}B_i^s, x_i^{m-1})H_{i,s}(x).$$

□

In particular if the spline $\mathbf{S}$ is a polynomial in $\mathbb{P}_m(\mathbb{R})$, we have the following results :

Corollary 3.5. *For each $p \in \mathbb{P}_m(\mathbb{R})$, the Hermite interpolant $\mathcal{H}p$ of p in the space $\mathbb{P}_2^1(I, \tau_1)$ is given by*

$$\mathcal{H}p(x) = \sum_{i=0}^{n}\sum_{s=0}^{1}\widehat{p}(A_i^{1-s}B_i^s, x_i^{m-1})H_{i,s}(x), \;\; \forall x \in I. \tag{3.5}$$

Remark 3.6. *(Marsden Identity). For $m = 2$, we have*

$$p(x) = \sum_{i=0}^{n}\sum_{s=0}^{1}\widehat{p}(A_i^{1-s}B_i^s, x_i)H_{i,s}(x), \;\; \forall x \in I.$$

4. Superconvergent discrete quasi-interpolants in $\mathbb{P}_2^1(I, \tau_1)$

In this section, we propose a method to build quasi-interpolants based on discrete values of a function f. We are interested on the quasi-interpolants of the form

$$Qf = \sum_{i=0}^{n} \sum_{s=0}^{1} \mu_{i,s}(f) H_{i,s}, \tag{4.1}$$

where, $\mu_{i,s}$ $i = 0, 1, \ldots, n$ and $s = 0, 1$ are linear functionals defined using values of f at some points in the neighbourhood of the supports of the B-splines $H_{i,s}$ and m is an integer greater than or equal to 2. The constructed quasi-interpolants is called discrete quasi-interpolants. Supported by these values, we construct in a neighbourhood of supp $H_{i,s}$ a local linear polynomial operator $\mathcal{I}_{i,s}$ that reproduces the space of polynomials of degree at most m, i.e. $\mathcal{I}_{i,s}(f) = f$ for each $f \in \mathbb{P}_m(\mathbb{R})$. Let us denote by Q_m the quasi-interpolant of degree m. Then we have the following result

Theorem 4.1. *Let f be a function defined on I such that the values of f are given at some discrete points in a neighbourhood of the support of $H_{i,s}$, $i = 0, \ldots, n$, $s = 0, 1$. If we denote $\mathcal{I}_{i,s}(f)$ by $p_{i,s}$, then the quasi-interpolant defined by (4.1) with*

$$\mu_{i,s}(f) = \widehat{p}_{i,s}(A_i^{1-s} B_i^s, x_i^{m-1}) \tag{4.2}$$

satisfies

$$Q_m p = \mathcal{H}p, \ \forall p \in \mathbb{P}_m.$$

Proof. Let $f \in \mathbb{P}_m$, then we have $p_{i,s} = \mathcal{I}_{i,s}(f) = f$ for $i = 0, .., n$, $s = 0, 1$. According to (4.1) and Corollary 3.5, we get

$$\begin{aligned} Q_m f &= \sum_{i=0}^{n} \sum_{s=0}^{1} \mu_{i,s}(f) H_{i,s} \\ &= \sum_{i=0}^{n} \sum_{s=0}^{1} \widehat{p}_{i,s}(A_i^{1-s} B_i^s, x_i^{m-1}) H_{i,s} \\ &= \sum_{i=0}^{n} \sum_{s=0}^{1} \widehat{f}(A_i^{1-s} B_i^s, x_i^{m-1}) H_{i,s}. \end{aligned}$$

Then,

$$Q_m f = \mathcal{H}f$$

□

To build a superconvergent discrete spline quasi-interpolant, it suffices to take m+1 distinct interpolation points in the support of $H_{i,s}$ for $i = 0, \ldots, n$ and $s = 0, 1$. Let $t_{i,s,k}$, $k = 0, \ldots, m$, be these points and consider the interpolation polynomial of f at $t_{i,s,k}$ i.e.,

$$p_{i,s} = \sum_{k=0}^{m} f(t_{i,s,k}) L_{i,s,k}, \tag{4.3}$$

where $L_{i,s,k}$ are the Lagrange basis functions of $\mathbb{P}_m$ associated with the points $t_{i,s,k}$. Then, the quasi-interpolant defined by (4.1) and (4.2) satisfies

$$Q_m p = \mathcal{H}p, \ \forall p \in \mathbb{P}_m.$$

In the following theorem, we give an explicit formula of the coefficients $\mu_{i,s}(f)$ in terms of the data values $f(t_{i,s,k})$ for $k = 0, \ldots, m$.

Theorem 4.2. *Let $t_{i,s,k} := \beta_{i,s,k}x_i + (1-\beta_{i,s,k})A_i^{1-s}B_i^s$, for $k = 0,\dots,m$, be $m+1$ distinct points in a neighborhood of the support of $H_{i,s}$. If the quasi-interpolant defined by (4.1) with*

$$\mu_{i,s}(f) = \sum_{k=0}^{m} q_{i,s,k} f(t_{i,s,k}) \tag{4.4}$$

satisfies

$$Q_m p = \mathcal{H}p \ \forall \ p \in \mathbb{P}_m,$$

then

$$q_{i,s,k} = \frac{(-1)}{m} \frac{\sum\limits_{\substack{\alpha=0\\ \alpha\neq k}}^{m} \beta_{i,s,\alpha} \prod\limits_{\substack{\gamma=0\\ \gamma\neq\alpha,\gamma\neq k}}^{m} (1-\beta_{i,s,\gamma})}{\prod\limits_{\substack{\alpha=0\\ \alpha\neq k}}^{m} (\beta_{i,s,k}-\beta_{i,s,\alpha})} \tag{4.5}$$

Proof. Let $L_{i,s,k}$, $k = 0,\dots,m$, be the Lagrange basis corresponding respectively to $t_{i,s,k}$ $k = 0,\dots,m$. We have

$$L_{i,s,k}(x) = \prod_{\substack{\alpha=0\\ \alpha\neq k}}^{m} \left(\frac{x - t_{i,s,\alpha}}{t_{i,s,k} - t_{i,s,\alpha}}\right) = \frac{\prod\limits_{\substack{\alpha=0\\ \alpha\neq k}}^{m} l_{\alpha,k}(x)}{\prod\limits_{\substack{\alpha=0\\ \alpha\neq k}}^{m} l_{\alpha,k}(t_{i,s,k})},$$

where $l_{\alpha,k}(x) = x - t_{i,s,\alpha}$. We set

$$q_{i,s,k} = \widehat{L}_{i,s,k}(A_i^{1-s}B_i^s, x_i^{m-1}), \ \forall k = 0,\dots,m.$$

Then the quasi-interpolant

$$Q_m f = \sum_{i=0}^{n}\sum_{s=0}^{1} \mu_{i,s}(f) H_{i,s},$$

with

$$\mu_{i,s}(f) = \sum_{k=0}^{m} q_{i,s,k} f(t_{i,s,k}), \ i = 0,\dots,n, \ s = 0,1,$$

satisfies

$$Q_m p = \mathcal{H}p \ \forall \ p \in \mathbb{P}_m.$$

In order to compute the value of $q_{i,s,k}$ $k = 0,\dots,m$, we use Proposition 3.2, then

$$\widehat{L}_{i,s,k}(A_i^{1-s}B_i^s, x_i^{m-1}) = \frac{1}{m} \frac{\sum\limits_{\substack{\alpha=0\\ \alpha\neq k}}^{m} l_{\alpha,k}(A_i^{1-s}B_i^s) \prod\limits_{\substack{\gamma=0\\ \gamma\neq\alpha,\gamma\neq k}}^{m} l_{\gamma,k}(x_i)}{\prod\limits_{\substack{\alpha=0\\ \alpha\neq k}}^{m} l_{\alpha,k}(t_{i,s,k})}.$$

Since,

$$l_{\alpha,k}(A_i^{1-s}B_i^s) = A_i^{1-s}B_i^s - t_{i,s,\alpha} = -\beta_{i,s,\alpha}(x_i - A_i^{1-s}B_i^s),$$

and

$$l_{\gamma,k}(x_i) = x_i - t_{i,s,\gamma} = (1-\beta_{i,s,\gamma})(x_i - A_i^{1-s}B_i^s),$$

and

$$l_{\alpha,k}(t_{i,s,k}) = t_{i,s,k} - t_{i,s,\alpha} = (\beta_{i,s,k} - \beta_{i,s,\alpha})(x_i - A_i^{1-s}B_i^s),$$

it holds that

$$q_{i,s,k} = \frac{(-1)}{m} \frac{\displaystyle\sum_{\substack{\alpha=0\\ \alpha\neq k}}^{m} \beta_{i,s,\alpha} \prod_{\substack{\gamma=0\\ \gamma\neq\alpha,\gamma\neq k}}^{m} (1-\beta_{i,s,\gamma})}{\displaystyle\prod_{\substack{\alpha=0\\ \alpha\neq k}}^{m} (\beta_{i,s,k}-\beta_{i,s,\alpha})}$$

□

5. Error estimate of superconvegent discrete quasi interpolants

In this section, we will prove that the constructed quasi-interpolant is superconvergent at knots x_i, $i=0,\ldots,n$. Let f be a function in $C^3(I)$. Since the operators Q_m, $m\geq 2$ reproduce the space $\mathbb{P}_2$, there exists constants $C_k>0$, $k=0,1$, independent of m such that

$$\left\|(Q_m f)^{(k)} - f^{(k)}\right\|_{\infty,I} \leq C_k h^{3-k} \left\|f^{(3)}\right\|_{\infty,I},$$

where $\|.\|_{\infty,I}$ denotes the infinity norm on the interval I. In the following proposition we give the error estimates associated with Q_m and its first derivative at the knots.

Theorem 5.1. *For any function $f\in C^{m+1}(I)$, we have*

$$\left|(Q_m f)^{(k)}(x_i) - f^{(k)}(x_i)\right| = \mathcal{O}\left(h^{m+1-k}\right), \ \forall i=0,\ldots,n \ \ \textit{and} \ \ k=0,1.$$

Proof. Let $f\in C^{m+1}(I)$ the Taylor expansion of f around x_i, $i=1,\ldots,m$, is given by

$$f(x) = \sum_{j=0}^{m} \frac{f^{(j)}(x_i)}{j!}(x-x_i)^j + \mathcal{O}\left((x-x_i)^{m+1}\right).$$

Denote by R_m the polynomial part of the Taylor expansion. Then for each point x in the support of $H_{i,s}$, we have

$$f(x) = R_m(x) + \mathcal{O}\left((x-x_i)^{m+1}\right)$$

From Theorem 4.1, we have $\mathcal{H}R_m = Q_m R_m$, and by use the fact that $R_m(x_i) = \mathcal{H}R_m(x_i)$, we get

$$|Q_m f(x_i) - f(x_i)| = |Q_m f(x_i) - R_m(x_i)| = |Q_m(f-R_m)(x_i)|.$$

By (4.5), we assume that $q_{i,s,k}$ are bounded by a constant C. Then, from (4.4) we obtain

$$\begin{aligned} \left|\mu_{i,s}(f-R_m)\right| &= \left|\sum_{k=0}^{m} q_{i,s,k}\left(f(t_{i,s,k}) - R_m(t_{i,s,k})\right)\right| \\ &\leq C\sum_{k=0}^{m} \left|\left(f(t_{i,s,k}) - R_m(t_{i,s,k})\right)\right|. \end{aligned}$$

Then,

$$\left|\mu_{i,s}(f-R_m)\right| = \mathcal{O}\left((t_{i,s,k}-x_i)^{m+1}\right),$$

and therefore

$$|Q_m(f-R_m)(x_i)| = \mathcal{O}\left((t_{i,s,k}-x_i)^{m+1}\right).$$

Thus,

$$|Q_m f(x_i) - f(x_i)| = \mathcal{O}\left(h^{m+1}\right).$$

In a similar way, we prove that

$$\left|(Q_m f)'(x_i) - f'(x_i)\right| = \mathcal{O}(h^m),$$

which completes the proof. □

We present now the results of the numerical experiments for the classical case. To illustrate the superconvergence characteristic of the C^1 quadratic spline quasi-interpolants Q_m, we choose arbitary the interpolation points in the interval $[x_i, A_i^s B_i^{1-s}]$ for $s = 0, 1$ such that $\beta_{i,s,k} \in [0, 1],\ k = 0, \ldots, m$ shown in Table 3.

m	$\beta_{i,0,k}$	$\beta_{i,1,k}$
2	$\frac{1}{10}, \frac{1}{2}, \frac{9}{10}$	$\frac{8}{10}, \frac{6}{10}, \frac{2}{10}$
3	$\frac{1}{10}, \frac{1}{3}, \frac{2}{3}, \frac{9}{10}$	$\frac{8}{10}, \frac{6}{10}, \frac{3}{10}, \frac{2}{10}$
4	$\frac{1}{10}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{9}{10}$	$\frac{8}{10}, \frac{6}{10}, \frac{4}{10}, \frac{3}{10}, \frac{2}{10}$
5	$\frac{1}{10}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{9}{10}$	$\frac{8}{10}, \frac{7}{10}, \frac{5}{10}, \frac{4}{10}, \frac{3}{10}, \frac{2}{10}$

Table 3: The values of $t_{i,s,k}$ for $m = 2, 3, 4, 5$ and $k = 0, .., m$.

We consider the following test functions defined on $I = [0, 1]$ by

$$f_1(x) = \exp(-3x)\sin(\frac{\pi}{2}x) \text{ and } f_2(x) = \frac{x}{3}(\exp(x^2) - 1).$$

We define the local error between a function g and the quasi-interpolant $\mathcal{Q}_m g$ at the knots of τ by the following relation:

$$E_{m,n}^{(k)}(g) := \max_{0 \leqslant i \leqslant n} \left| Q_m^{(k)} g(x_i) - g^{(k)}(x_i) \right|,\ k = 0, 1,$$

and the numerical convergence order by

$$\mathcal{NCO}_m^{(k)} := \mathcal{NCO}_m^{(k)}(n_1 \to n_2) = \frac{\log\left(\frac{E_{m,n_1}^{(k)}(g)}{E_{m,n_2}^{(k)}(g)}\right)}{\log\left(\frac{n_2}{n_1}\right)},$$

where, $m = 2, 3, 4, 5$.

Approximating function values

To illustrate numerically the result, we give in Table 4, for different values of n, the maximum absolute errors at knots $E_{m,n}^{(0)}(f_1)$ associated with the operator Q_m for $m = 2, 3, 4, 5$. In the case of the function f_2 the same errors (i.e., $E_{m,n}^{(0)}(f_2),\ m = 2, 3, 4, 5$) are given in Table 6. Also, we list in Tables 5 and 7 respectively the numerical convergence orders $\mathcal{NCO}_m$ of the maximum absolute errors at the knots.

n	$E_{2,n}^{(0)}(f_1)$	$E_{3,n}^{(0)}(f_1)$	$E_{4,n}^{(0)}(f_1)$	$E_{5,n}^{(0)}(f_1)$
16	3.68420×10^{-6}	2.01571×10^{-6}	2.03558×10^{-8}	2.16091×10^{-9}
32	6.34560×10^{-7}	1.36744×10^{-7}	7.32423×10^{-10}	3.40381×10^{-11}
64	9.13294×10^{-8}	8.89603×10^{-9}	2.43058×10^{-11}	5.32296×10^{-13}
128	1.22035×10^{-9}	5.67123×10^{-10}	7.80653×10^{-13}	1.09357×10^{-14}

Table 4: The maximum absolute errors $E_{m,n}^{(0)}(f_1)$.

$n_1 \to n_2$	$\mathcal{NCO}_2^{(0)}$	$\mathcal{NCO}_3^{(0)}$	$\mathcal{NCO}_4^{(0)}$	$\mathcal{NCO}_5^{(0)}$
$16 \to 32$	2.53752	3.88174	4.79662	5.98834
$32 \to 64$	2.79660	3.94217	4.91330	5.99920
$64 \to 128$	2.90378	3.97143	4.96048	5.60511
Theoretical value	03	04	05	06

Table 5: The numerical convergence orders $\mathcal{NCO}_{m,n}^{(0)}$ for f_1.

n	$E^{(0)}_{2,n}(f_2)$	$E^{(0)}_{3,n}(f_2)$	$E^{(0)}_{4,n}(f_2)$	$E^{(0)}_{5,n}(f_2)$
16	6.41606×10^{-6}	2.09509×10^{-6}	6.06966×10^{-8}	1.02726×10^{-8}
32	7.83480×10^{-7}	1.48969×10^{-7}	1.84323×10^{-9}	1.89311×10^{-10}
64	9.58526×10^{-8}	9.94141×10^{-9}	5.61283×10^{-11}	3.22575×10^{-12}
128	1.18178×10^{-8}	6.42218×10^{-10}	1.72373×10^{-12}	5.97300×10^{-14}

Table 6: The maximum absolute errors $E^{(0)}_{m,n}(f_2)$.

$n_1 \to n_2$	$\mathcal{NCO}^{(0)}_2$	$\mathcal{NCO}^{(0)}_3$	$\mathcal{NCO}^{(0)}_4$	$\mathcal{NCO}^{(0)}_5$
$16 \to 32$	3.03372	3.81392	5.04131	5.76191
$32 \to 64$	3.03101	3.90542	5.03736	5.87498
$64 \to 128$	3.01986	3.95232	5.02512	5.75504
Theoretical value	03	04	05	06

Table 7: The numerical convergence orders $\mathcal{NCO}^{(0)}_{m,n}$ for f_2.

From the above examples, we remark that when we increase n or m, we get a quasi-interpolant with small errors and the numerical convergence order is in good agreement with the theoretical one.

Approximating derivative values

As in above, we illustrate numerically in Tables 8 and 10 the superconvergence phenomenon when derivative values are approximated. The same comments given previously are true in this case.

n	$E^{(1)}_{2,n}(f_1)$	$E^{(1)}_{3,n}(f_1)$	$E^{(1)}_{4,n}(f_1)$	$E^{(1)}_{5,n}(f_1)$
16	3.83988×10^{-3}	1.62514×10^{-4}	7.68511×10^{-6}	9.28639×10^{-8}
32	9.78455×10^{-4}	2.07757×10^{-5}	4.89128×10^{-7}	2.79467×10^{-9}
64	2.46955×10^{-4}	2.62611×10^{-6}	3.08382×10^{-8}	8.53984×10^{-11}
128	6.20331×10^{-5}	3.30093×10^{-7}	1.93561×10^{-9}	2.76401×10^{-12}

Table 8: The maximum absolute errors $E^{(1)}_{m,n}(f_1)$.

$n_1 \to n_2$	$\mathcal{NCO}^{(1)}_2$	$\mathcal{NCO}^{(1)}_3$	$\mathcal{NCO}^{(1)}_4$	$\mathcal{NCO}^{(1)}_5$
$16 \to 32$	1.97248	2.96759	3.97378	5.05437
$32 \to 64$	1.98626	2.98390	3.98742	5.03232
$64 \to 128$	1.99313	2.99198	3.99386	4.94937
Theoretical value	02	03	04	05

Table 9: The numerical convergence orders $\mathcal{NCO}^{(1)}_{m,n}$ for f_1.

n	$E^{(1)}_{2,n}(f_2)$	$E^{(1)}_{3,n}(f_2)$	$E^{(1)}_{4,n}(f_2)$	$E^{(1)}_{5,n}(f_2)$
16	2.67904×10^{-3}	1.41350×10^{-4}	1.03490×10^{-5}	9.86546×10^{-7}
32	7.03270×10^{-4}	1.83794×10^{-5}	7.30584×10^{-7}	3.34695×10^{-8}
64	1.86979×10^{-4}	2.34347×10^{-6}	4.86127×10^{-8}	1.08645×10^{-9}
128	4.82144×10^{-5}	2.95864×10^{-7}	3.13583×10^{-9}	3.59819×10^{-11}

Table 10: The maximum absolute errors $E^{(1)}_{m,n}(f_2)$.

$n_1 \to n_2$	$\mathcal{NCO}_2^{(1)}$	$\mathcal{NCO}_3^{(1)}$	$\mathcal{NCO}_4^{(1)}$	$\mathcal{NCO}_5^{(1)}$
$16 \to 32$	1.92957	2.94311	3.82429	4.88147
$32 \to 64$	1.91120	2.97137	3.90964	4.94515
$64 \to 128$	1.95534	2.98564	3.95441	4.91620
Theoretical value	02	03	04	05

Table 11: The numerical convergence orders $\mathcal{NCO}_{m,n}^{(1)}$ for f_2.

6. Super-superconvergence phenomenon

In this section, we add a concept called the super-superconvergence phenomenon. From a numerical observation , we have remarked that the approximation order is $\mathcal{O}(h^{m+2})$ at knots when the degree m of the local polynomials is even. This is what we call a super-convergence phenomenon, it leads to an improvement of approximation properties. Unfortunately, this phenomenon does not happen for an arbitrary data site. In the following result, we sum up how to choose quasi-interpolation points in order to achieve the super-superconvergence phenomenon.

For each $i = 1, \ldots, n-1$, let x_i be the midpoint of $[a_i, b_i]$.

Theorem 6.1. *Let $f \in C^{m+2}(I)$ such that m is even. If the set of the local interpolation points corresponding to $H_{i,0}$ is symmetric to the one corresponding to $H_{i,1}$ with respect to x_i for $i = 1, \ldots, n-1$, and if the local polynomial approximant $p_{0,0}$ (resp. $p_{n,0}$) interpolates f at x_0 (resp. x_n), then the quasi-interpolant Q_m is super-superconvergent at x_i, and*

$$|Q_m f(x_i) - f(x_i)| = \mathcal{O}\left(h^{m+2}\right) \;\; \forall i = 0, \ldots, n.$$

Proof. Let $f \in C^{m+2}(I)$, the Taylor expansion of f around x_i for $i = 1, \ldots, n-1$ is given by

$$f(x) = \sum_{j=0}^{m+1} \frac{f^{(j)}(x_i)}{j!}(x - x_i)^j + \mathcal{O}\left((x - x_i)^{m+2}\right).$$

Denote by R_m the polynomial part of the Taylor expansion and by g_{m+1} its last term

$$g_{m+1}(x) = \frac{f^{(m+1)}(x_i)}{(m+1)!}(x - x_i)^{m+1}.$$

Using a similar way as in the proof of Proposition 5.1, we get

$$\begin{aligned} |Q_m f(x_i) - f(x_i)| &= |Q_m f(x_i) - Q_m R_m(x_i) + Q_m R_m(x_i) - f(x_i)| \\ &\leq |Q_m(f - R_m)(x_i)| + |Q_m g_{m+1}(x_i)| \end{aligned} \tag{6.1}$$

and

$$|\mu_{i,s}(f - R_m)| \leq C \sum_{k=0}^{m} |\left(f(t_{i,s,k}) - R_m(t_{i,s,k})\right)|.$$

This implies that

$$|\mu_{i,s}(f - R_m)| = \mathcal{O}\left((t_{i,s,k} - x_i)^{m+2}\right),$$

and therefore

$$|Q_m(f - R_m)(x_i)| = \mathcal{O}\left((t_{i,s,k} - x_i)^{m+2}\right). \tag{6.2}$$

By using the fact that $\sum_{k=0}^{m} q_{i,s,k} = 1$ and $\sum_{j=0}^{n}\sum_{s=0}^{1} H_{j,s}(x_i) = 1$, then for any function $g \in C^1(I)$ we have

$$Q_m g(x_i) = \sum_{s=0}^{1} \mu_{i,s}(g) H_{i,s}(x_i).$$

By considering that $a_i = x_i - d$ and $b_i = x_i + d$ with $d \geq 0$, we get

$$H_{i,0}(x_i) = \alpha_i = 1 - \frac{x_i - a_i}{b_i - a_i} = \frac{1}{2},\ H_{i,1}(x_i) = 1 - \alpha_i = \frac{1}{2}.$$

First of all, we show that the coefficients $q_{i,s,k}$, $s = 0, 1$ are the same for two symmetrical interpolation points with respect to x_i. i.e., $t_{i,0,k} + t_{i,1,k} = 2x_i$.
As

$$t_{i,s,k} = \beta_{i,s,k} x_i + (1 - \beta_{i,s,k}) A_i^{1-s} B_i^s, \quad k = 0, \dots, m,$$

we have

$$\frac{\beta_{i,0,k} x_i + (1 - \beta_{i,0,k}) A_i + \beta_{i,1,k} x_i + (1 - \beta_{i,1,k}) B_i}{2} = x_i,$$

and

$$\frac{x_i(\beta_{i,0,k} + \beta_{i,1,k}) + (1 - \beta_{i,0,k}) A_i + (1 - \beta_{i,1,k}) B_i}{2} = x_i.$$

Also, as A_i and B_i are symmetric with respect to x_i, we have $x_i = \frac{A_i + B_i}{2}$ and we deduce that $\beta_{i,0,k} = \beta_{i,1,k}$.
From Theorem 4.2, we easily obtain $q_{i,0,k} = q_{i,1,k} = q_{i,k}$, for $k = 0, \dots . m$
Then

$$Q_m g(x_i) = \frac{1}{2} \sum_{k=0}^{m} q_{i,k} \left(g(t_{i,0,k}) + g(t_{i,1,k}) \right).$$

Particularly, for $g = g_{m+1}$ we obtain

$$g_{m+1}(t_{i,0,k}) + g_{m+1}(t_{i,1,k}) = \left((t_{i,0,k} - x_i)^{m+1} + (t_{i,1,k} - x_i)^{m+1} \right) \frac{f^{(m+1)}(x_i)}{(m+1)!}.$$

Knowing that m is even and $t_{i,s,k}$, $s = 0, 1$ are symmetric with respect to x_i, then

$$g_{m+1}(t_{i,0,k}) + g_{m+1}(t_{i,1,k}) = 0,\ \forall i = 1, \dots, n-1.$$

Therefore, for $i = 1, \dots, n-1$,

$$Q_m g_{m+1}(x_i) = 0. \tag{6.3}$$

Using (6.1), (6.2) and (6.3) we get

$$|Q_m f(x_i) - f(x_i)| = \mathcal{O}(h^{m+2}).$$

For $i = 0$, we have

$$H_{0,0}(x_0) = 1,\ H_{0,1}(x_0) = 0,$$

then,

$$Q_m f(x_0) = \widehat{p}_{0,0}(x_0^m) H_{0,0}(x_0) + \widehat{p}_{0,1}(B_0, x_0^{m-1}) H_{0,1}(x_0) = p_{0,0}(x_0) = f(x_0).$$

Similarly, for $i = n$ we get

$$H_{n,0}(x_n) = 0,\ H_{n,1}(x_n) = 1,$$

then

$$Q_m f(x_n) = \widehat{p}_{n,0}(A_n, x_n^{m-1}) H_{n,0}(x_n) + \widehat{p}_{n,1}(x_n^m) H_{n,1}(x_n) = p_{n,1}(x_n) = f(x_n),$$

and the proof is complete. □

Remark 6.2. *If the data sites $t_{i,0,k}$, $k = 0, \dots, m$, are symmetric with respect to x_i, to achieve the super-superconvergence, it suffices to take $\{t_{i,0,k},\ k = 0, \dots, m\} = \{t_{i,1,k},\ k = 0, \dots, m\}$ for $i = 1, \dots, n-1$ i.e., we take the same interpolation points for $p_{i,0}$ and $p_{i,1}$.*

Approximating function values

To illustrate numerically the super-superconvergence phenomenon, we consider the same test functions taken in the previous section for the classical case. We choose the sets of interpolation points corresponding to $H_{i,0}$ and $H_{i,1}$ such that they are symmetric with respect to x_i. We take $\beta_{i,0,k} = \beta_{i,1,k} \in [0,1],\ k = 0,\dots,m$ as shown in the Table 12.

m	$\beta_{i,0,k}$	$\beta_{i,1,k}$
2	$\frac{1}{10}, \frac{1}{2}, \frac{9}{10}$	$\frac{9}{10}, \frac{1}{2}, \frac{1}{10}$
3	$\frac{1}{10}, \frac{1}{3}, \frac{2}{3}, \frac{9}{10}$	$\frac{9}{10}, \frac{2}{3}, \frac{1}{3}, \frac{1}{10}$
4	$\frac{1}{10}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{9}{10}$	$\frac{9}{10}, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{10}$
5	$\frac{1}{10}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{9}{10}$	$\frac{9}{10}, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{10}$

Table 12: The values of $t_{i,s,k}$ for $m = 2,3,4,5$ and $k = 0,..,m$.

We give in Table 13 (resp. Table 15), for different values of n, the maximum absolute errors at knots $E^{(0)}_{m,n}(f_1)$ (resp. $E^{(0)}_{m,n}(f_2)$) associated with the operator Q_m for $m = 2,3,4,5$. Also, we list in Tables 14 and 16 the numerical convergence orders $\mathcal{NCO}_m$.

n	$E^{(0)}_{2,n}(f_1)$	$E^{(0)}_{3,n}(f_1)$	$E^{(0)}_{4,n}(f_1)$	$E^{(0)}_{5,n}(f_1)$
16	1.47575×10^{-6}	1.78296×10^{-6}	2.72369×10^{-9}	1.59321×10^{-9}
32	9.98421×10^{-8}	1.20570×10^{-7}	4.27642×10^{-11}	2.51321×10^{-11}
64	6.48421×10^{-9}	7.83193×10^{-9}	6.68382×10^{-13}	3.92908×10^{-13}
128	4.13197×10^{-10}	4.98917×10^{-10}	1.04916×10^{-14}	6.10623×10^{-15}

Table 13: The maximum absolute errors $E^{(0)}_{m,n}(f_1)$.

$n_1 \to n_2$	$\mathcal{NCO}^{(0)}_2$	$\mathcal{NCO}^{(0)}_3$	$\mathcal{NCO}^{(0)}_4$	$\mathcal{NCO}^{(0)}_5$
$16 \to 32$	3.88565	3.88634	5.99301	5.98625
$32 \to 64$	3.94421	3.94436	5.99959	5.9992
$64 \to 128$	3.97246	3.97249	5.99337	6.000777
Theoretical value	04	04	06	06

Table 14: The numerical convergence orders $\mathcal{NCO}^{(0)}_{m,n}$ for f_1.

n	$E^{(0)}_{2,n}(f_2)$	$E^{(0)}_{3,n}(f_2)$	$E^{(0)}_{4,n}(f_2)$	$E^{(0)}_{5,n}(f_2)$
16	1.50503×10^{-6}	1.82277×10^{-6}	1.33319×10^{-8}	7.93178×10^{-9}
32	1.07808×10^{-7}	1.30273×10^{-7}	2.42292×10^{-10}	1.42970×10^{-10}
64	7.21751×10^{-9}	$8,71645 \times 10^{-9}$	4.08740×10^{-12}	2.40663×10^{-12}
128	4.66934×10^{-10}	5.63260×10^{-10}	6.57252×10^{-14}	4.06342×10^{-14}

Table 15: The maximum absolute errors $E^{(0)}_{m,n}(f_2)$.

$n_1 \to n_2$	$\mathcal{NCO}_2^{(0)}$	$\mathcal{NCO}_3^{(0)}$	$\mathcal{NCO}_4^{(0)}$	$\mathcal{NCO}_5^{(0)}$
$16 \to 32$	3.80325	3.80625	5.78199	5.79386
$32 \to 64$	3.90082	3.90166	5.88942	5.89255
$64 \to 128$	3.95021	3.95042	5.95859	5.88818
Theoretical value	04	04	06	06

Table 16: The numerical convergence orders $\mathcal{NCO}_{m,n}^{(0)}$ for f_2.

Through these examples, we remark that the numerical convergence order is in good agreement with the theoretical one. A comparison with the previous results allow us to see that when m is even, the associated errors of the super-superconvergence phenomenon are smaller than the ones of the superconvergence phenomenon.

7. Conclusion

In this paper, we have shown how to construct a new normalized B-spline basis of a C^1 continuous spline space of degree two. The basis functions have a local support, they are nonnegative, and they form a partition of unity. The classical C^1 quadratic B-splines are a particular case of our Hermite B-splines. Moreover, we used some results on blossoming to establish the B-spline representation of Hermite interpolant of any C^1 continuous spline of degree 2 in terms of its polar form. Hence we used this representation for constructing several superconvergent and super-superconvergent discrete quasi-interpolants. This new approach provides an interesting approximation and it can be used for solving some numerical analysis problems. In futur works, the generalization of the proposed results to higher degrees will be studied.

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