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# Characterizing Inequalities for Contact CR-warped Product Submanifolds of Generalized Sasakian Space Forms Admitting a Nearly Cosymplectic Structure

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ABSTRACT: This paper studies the contact CR-warped product submanifolds of a generalized Sasakian space form admitting a nearly cosymplectic structure. Some inequalities for existence of these types of warped product submanifolds are established, the obtained inequalities generalize the results that have acquired in [14]. Moreover, we also estimate another inequality for the second fundamental form in the expressions of the warping function, this inequality also generalize the inequalities that have obtained in [19]. In addition, we also explore the equality cases.

Key Words: Nearly cosymplectic manifold, Contact CR-submanifolds, Warped product.

## Contents



## 1. Introduction

It is illustrious that warped products of manifolds perform an significant role in differential geometry, the theory of relativity and mathematical physics. One of the most important examples of a warped product manifold is the excellent setting to the model spacetime close to black holes or bodies with high gravitational fields [20].

The concept of the warped products for the theory of submanifolds was first investigated by B. Y. Chen [2]. Basically, Chen worked out CR-warped product submanifolds in the background of Kaehler manifolds and established a sharp estimate for the length of the second fundamental form in the expressions of warping function. Motivated by Chen, I. Mihai ([10], [11]) studied the contact version of these warped products and acquired the similar estimate for the contact CR-warped product submanifolds of a Sasakian space form. In this line of research many articles have appeared in the setting of almost contact metric manifolds ([8], [12], [15], [18]). K. A. Khan et al. [13] deliberate the existence and nonexistence for the warped product submanifolds of the cosymplectic manifolds and a step forward was made by M. Atceken [14] who proved a characterizing inequality for the existence of the contact CR-warped product submanifolds of a cosymplectic space form.

In this paper, we achieve a characterization for the existence of the contact CR-warped product submanifolds isometrically immersed in a generalized Sasakian space form admitting a nearly cosymplectic structure and as a special case, we also discuss the existence of these warped products for the cosymplectic space forms. Moreover, we prove an estimate for the length of the second fundamental form in the expressions of warping function and some special conditions are also investigated.

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## 2. Preliminaries

Following [7] a $(2n+1)-$dimensional $C^\infty-$manifold $\bar{M}$ is said to have an almost contact structure if there exist on $\bar{M}$ a tensor field $\phi$ of type $(1,1)$, a vector field $\xi_1$ and a 1-form $\eta_1$ satisfying

$$\phi^2 = -I + \eta_1 \otimes \xi_1,\ \phi\xi_1 = 0,\ \eta_1 \circ \phi = 0,\ \eta_1(\xi_1) = 1. \tag{2.1}$$

There always exists a Riemannian metric $g$ on an almost contact metric manifold $\bar{M}$ satisfying the subsequent conditions

$$\eta_1(U_1) = g(U_1, \xi_1),\ g(\phi U_1, \phi V_1) = g(U_1, V_1) - \eta_1(U_1)\eta_1(V_1), \tag{2.2}$$

for all $U_1, V_1 \in T\bar{M}$.

An almost contact structure $(\phi, \xi_1, \eta_1)$ is said to be normal, if the almost complex structure $J$ on the product manifold $\bar{M} \times R$ is given by

$$J(U_1, \lambda\frac{d}{dx}) = (\phi U_1 - \lambda\xi_1, \eta_1(U_1)\frac{d}{dx}),$$

where $\lambda$ is a $C^\infty-$function on $\bar{M} \times R$ has no torsion that is $J$ is integrable and the condition for normality in terms of $\phi, \xi_1$ and $\eta_1$ is $[\phi, \phi] + 2d\eta_1 \otimes \xi_1 = 0$ on $\bar{M}$, where $[\phi, \phi]$ is the Nijenhuis tensor of $\phi$. The *fundamental 2-form* $\Phi_1$ is defined by $\Phi_1(U_1, V_1) = g(U_1, \phi V_1)$.

An almost contact metric structure $(\phi, \xi_1, \eta_1, g)$ on $\bar{M}$ is said to be cosymplectic if it is normal and both $\phi$ and $\eta_1$ are closed and the manifold $\bar{M}$ with the cosymplectic structure $(\phi, \xi_1, \eta_1, g)$ is said to be a cosymplectic manifold [5]. Moreover, the structure $(\phi, \xi_1, \eta_1, g)$ on $\bar{M}$ is said to be nearly cosymplectic if $\phi$ is of killing type. The manifold $\bar{M}$ equipped with a nearly cosymplectic structure is said to be a nearly cosymplectic manifold. The characteristic equation for the nearly cosymplectic manifolds is described by

$$(\bar{\nabla}_{U_1}\phi)U_1 = 0, \tag{2.3}$$

for any $U_1 \in T\bar{M}$, where $\bar{\nabla}$ is the Riemannian connection of the metric $g$ on $\bar{M}$.

In [16] Alegre et al. determined the concept of generalized Sasakian space form as that an almost contact metric manifold $(\bar{M}, \phi, \xi_1, \eta_1, g)$ whose curvature tensor $\bar{R}$ satisfies

$$\begin{aligned}\bar{R}(U_1, V_1)W_1 \quad = \quad & f_1\{g(V_1, W_1)U_1 - g(U_1, W_1)V_1\} \\ & + f_2\{g(U_1, \phi W_1)\phi V_1 - g(V_1, \phi W_1)\phi U_1 \\ & + 2g(U_1, \phi V_1)\phi W_1\} + f_3\{\eta_1(U_1)\eta_1(W_1)V_1 \\ & - \eta_1(V_1)\eta_1(W_1)U_1 + g(U_1, W_1)\eta_1(V_1)\xi_1 \\ & - g(V_1, W_1)\eta_1(U_1)\xi_1\},\end{aligned} \tag{2.4}$$

$\forall$ vector fields $U_1, V_1, W_1 \in T\bar{M}$, and three differentiable functions $f_1, f_2, f_3$ on $\bar{M}$. A generalized Sasakian space form with functions $f_1, f_2, f_3$ is denoted by $\bar{M}(f_1, f_2, f_3)$. If $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$, then $\bar{M}(f_1, f_2, f_3)$ is a Sasakian space form $\bar{M}(c)$ [7]. If $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$, then $\bar{M}(f_1, f_2, f_3)$ is a Kenmotsu space form $\bar{M}(c)$ [12], and if $f_1 = f_2 = f_3 = \frac{c}{4}$, then $\bar{M}(f_1, f_2, f_3)$ is a cosymplectic space form $\bar{M}(c)$ [16].

For a submanifold $M$ of a Riemannian manifold $\bar{M}$ with induced metric $g$, the Gauss and Weingarten formulae are governed by the following equations

$$\bar{\nabla}_{U_1}V_1 = \nabla_{U_1}V_1 + \sigma(U_1, V_1), \tag{2.5}$$

$$\bar{\nabla}_{U_1}N = -A_N U_1 + \nabla^\perp_{U_1}N, \tag{2.6}$$

where $\nabla$ and $\nabla^{\perp}$ are the induced connections on the tangent bundle $TM$ and normal bundle $T^{\perp}M$ of $M$, for any $U_1, V_1 \in TM, N \in T^{\perp}M$. The second fundamental form and the shape operator for the immersion of $M$ in $\bar{M}$ are denoted by $\sigma$ and $A_N$ respectively, and they are associated by

$$g(\sigma(U_1, V_1), N) = g(A_N U_1, V_1), \tag{2.7}$$

where $g$ represents the Riemannian metric on $\bar{M}$ as well as on $M$.

We have the following formula for squared norm of $\sigma$

$$\|\sigma\|^2 = \sum_{i,j=1}^{k} g(\sigma(u_i, u_j), \sigma(u_i, u_j)),$$

where $\{u_1, u_2, \dots, u_k\}$ is a local orthonormal set of vector fields on $M$.

A submanifold $M$ of $\bar{M}$ is said to be a totally geodesic submanifold if $\sigma(U_1, V_1) = 0, \forall\, U_1, V_1 \in TM$, and totally umbilical submanifold if $\sigma(U_1, V_1) = g(U_1, V_1)H$, where $H$ is the mean curvature vector.

For any $V_1 \in TM$, we write

$$\phi V_1 = PV_1 + FV_1, \tag{2.8}$$

where $PV_1$ and $FV_1$ are the tangential and normal parts of $\phi V_1$ correspondingly.

The covariant differentiation of the tensors $\phi$, $P$, and $F$ are illustrated respectively, as follows

$$(\bar{\nabla}_{U_1}\phi)V_1 = \bar{\nabla}_{U_1}\phi V_1 - \phi\bar{\nabla}_{U_1}V_1, \tag{2.9}$$

$$(\bar{\nabla}_{U_1}P)V_1 = \nabla_{U_1}PV_1 - P\nabla_{U_1}V_1, \tag{2.10}$$

$$(\bar{\nabla}_{U_1}F)V_1 = \nabla^{\perp}_{U_1}FV_1 - F\nabla_{U_1}V_1 \tag{2.11}$$

Additionally, for any $U_1, V_1 \in TM$, the tangential and normal components of $(\bar{\nabla}_{U_1}\phi)V_1$ are symbolized by $P_{U_1}V_1$ and $Q_{U_1}V_1$ i.e.,

$$(\bar{\nabla}_{U_1}\phi)V_1 = P_{U_1}V_1 + Q_{U_1}V_1. \tag{2.12}$$

By (2.3) and (2.12), it is easy to see that

$$(a)\ P_{U_1}V_1 = -P_{V_1}U_1,\ (b)\ Q_{U_1}V_1 = -Q_{V_1}U_1, \tag{2.13}$$

for any $U_1, V_1 \in TM$.

**Definition 2.1.** *An $m$-dimensional Riemannian submanifold $M$ of an almost contact metric manifold $\bar{M}$ , such that $\xi_1$ is tangent to $M$ is said to be a contact CR-submanifold if there exist two orthogonal complementary distributions $D^T$ and $D^{\perp}$ such that $D^T$ is invariant i.e., $\phi D^T \subseteq D^T$ and $D^{\perp}$ is anti invariant i.e., $\phi D^{\perp} \subseteq T^{\perp}M$.*

Then for a contact CR-submanifold $\bar{M}$, the tangent bundle $TM$ of $M$ can be written as follows

$$TM = D^T \oplus D^{\perp} \oplus \langle\xi_1\rangle,$$

where $\langle\xi_1\rangle$ represents the 1-dimensional distribution spanned by $\xi_1$.

The normal bundle $T^{\perp}M$ can be decomposed as follows

$$T^{\perp}M = \mu \oplus \phi D^{\perp}, \tag{2.14}$$

where $\mu$ is the invariant subspace of $T^{\perp}M$.

A contact CR-submanifold $M$ of an almost contact metric manifold $\bar{M}$ is called contact CR-product if the distributions $D$ and $D^{\perp}$ are parallel on $M$. The warped product of Riemannian manifolds is the generalization of the product manifolds, which is defined as follows

**Definition 2.2.** *Let $(R, g_R)$ and $(S, g_S)$ be two Riemannian manifolds with Riemannian metrics $g_R$ and $g_S$ respectively and $\psi$ be a positive differentiable function on $R$. The warped product of $R$ and $S$ is the Riemannian manifold $(R \times S, g)$, where*

$$g = g_R + \psi^2 g_S \tag{2.15}$$

The warped product manifold $(R \times S, g)$ is denoted by $R \times_\psi S$. If $U_1$ is the tangent vector field to $M = R \times_\psi S$ at $(p, q)$ then

$$\|U_1\|^2 = \|d\pi_1 U_1\|^2 + \psi^2(p)\|d\pi_2 U_1\|^2. \tag{2.16}$$

We have the following theorem,

**Theorem 2.3.** *[17]. Let $M = R \times_\psi S$ be the warped product manifolds. If $X_1, Y_1 \in TR$ and $V_1, W_1 \in TS$ then*

*(i)* $\nabla_{X_1} Y_1 \in TR$,

*(ii)* $\nabla_{X_1} V_1 = \nabla_{V_1} X_1 = (\frac{X_1\psi}{\psi})V_1$,

*(iii)* $\nabla_{V_1} W_1 = \frac{-g(V_1, W_1)}{\psi}\nabla\psi$.

From the above theorem, for the warped product $M = R \times_\psi S$, it is easy to observe that

$$\nabla_{X_1} V_1 = \nabla_{V_1} X_1 = (X_1 ln\psi)V_1, \tag{2.17}$$

for any $X_1 \in TR$ and $V_1 \in TS$.

$\nabla\psi$ is the gradient of $\psi$ and is expressed by

$$g(\nabla\psi, U_1) = U_1\psi, \tag{2.18}$$

for all $U_1 \in TM$.

If the warping function $\psi$ is constant, then the warped product is said to be trivial warped product.

Let $M$ be a $m-$dimensional Riemannian manifold with Riemannian metric $g$ and let $\{u_1, \ldots, u_m\}$ be an orthonormal basis of $TM$. As a significance of (2.18), we have

$$\|\nabla\psi\|^2 = \sum_{i=1}^{m}(u_i(\psi))^2 \tag{2.19}$$

The Laplacian of $\psi$ is given by

$$\Delta\psi = \sum_{i=1}^{m}\{(\nabla_{u_i} u_i)\psi - u_i u_i \psi\}. \tag{2.20}$$

Now, we state the Hopf's Lemma.

**Lemma 2.1.** *[3]. If $\psi$ is a differentiable function on a n-dimensional compact Riemannian manifold. If $\Delta\psi \geq 0$ or $\Delta\psi \leq 0$ everywhere on $M$, then $\psi$ is a constant function.*

## 3. Results and Discussion

This section deal with the study of the contact CR-warped product submanifolds of a nearly cosymplectic manifold and is divided into the two subsections.

### 3.1 Inequalities of the first kind

In this subsection, some characterizing inequalities are proved for existence of these types of warped product submanifolds.

**Theorem 3.1.** *[18]. A warped product submanifold $M = N_1 \times_\psi N_2$ of a nearly cosymplectic manifold $\bar{M}$ is simply a Riemannian product if the structure vector field $\xi_1$ is tangent to $N_2$, where $N_1$ and $N_2$ are the Riemannian manifolds.*

Throughout this paper, we consider the warped products of the type $M = N^T \times_\psi N^\perp$, such that $N^T$ is an invariant submanifold of $\bar{M}$ tangent to $\xi_1$ and $N^\perp$ is an anti-invariant submanifold of $\bar{M}$. These types of warped product submanifolds are called the contact CR-warped product submanifolds. Now we have the following basic results for further application.

**Lemma 3.1.** *[18]. Let $M = N^T \times_\psi N^\perp$ be a contact CR-warped product submanifold of a nearly cosymplectic manifold $\bar{M}$, then we have*

(i) $\xi_1 ln\psi = 0$,

(ii) $g(\sigma(U_1, V_1), \phi W_1) = 0$,

(iii) $g(\sigma(\phi U_1, W_1), \phi W_1) = U_1 ln\psi \|W_1\|^2$,

for any $U_1 \in TN^T$ and $W_1 \in TN^\perp$.

The following Lemma will be utilized for subsequent results.

**Lemma 3.2.** *Let $M = N^T \times_\psi N^\perp$ be a contact CR-warped product submanifold of a nearly cosymplectic manifold $\bar{M}$, then we have*

$$g(\sigma(\phi U_1, W_1), \phi\sigma(U_1, W_1)) = \|\sigma_\mu(U_1, W_1)\|^2 - g(\phi\sigma(U_1, W_1), Q_{U_1} W_1),$$

for any $U_1 \in TN^T$ and $W_1 \in TN^\perp$.

*Proof.* By (2.5) and (2.9)

$$\sigma(\phi U_1, W_1) = (\bar{\nabla}_{W_1}\phi)U_1 + \phi\nabla_{W_1} U_1 + \phi\sigma(U_1, W_1) - \nabla_{W_1}\phi U_1.$$

Thus by using (2.12) and (2.17)

$$\sigma(\phi U_1, W_1) = P_{W_1} U_1 + Q_{W_1} U_1 + U_1 ln\psi\phi W_1 + \phi\sigma(U_1, W_1) - \phi U_1 ln\psi W_1.$$

Comparing the normal parts

$$\sigma(\phi U_1, W_1) = Q_{W_1} U_1 + U_1 ln\psi\phi W_1 + \phi\sigma_\mu(U_1, W_1),$$

or

$$g(\sigma(\phi U_1, W_1), \phi\sigma(U_1, W_1)) = g(Q_{W_1} U_1, \phi\sigma(U_1, W_1)) + \|\sigma_\mu(U_1, W_1)\|^2.$$

By using (2.13)($b$), we get

$$g(\sigma(\phi U_1, W_1), \phi\sigma(U_1, W_1)) = \|\sigma_\mu(U_1, W_1)\|^2 - g(\phi\sigma(U_1, W_1), Q_{U_1} W_1).$$

□

Further, we have the following characterizing inequalities for the contact CR-warped product submanifolds of a generalized Sasakian space form admitting a nearly cosymplectic structure.

**Theorem 3.2.** *Let $M = N^T \times_\psi N^\perp$ be a contact CR-warped product submanifold of a generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting a nearly cosymplectic structure such that $M^T$ is a compact submanifold. Then, $M$ is a contact CR-product submanifold if either one of the following inequality holds*

*(i)* $\sum\limits_{i=1}^{2k_1}\sum\limits_{j=1}^{k_2} \|\sigma_\mu(u_i, u^j)\|^2 \geq 2.k_1.k_2.f_2 + \sum\limits_{i=1}^{2k_1}\sum\limits_{j=1}^{k_2} \|Q_{u_i} u^j\|^2,$

*(ii)* $\sum\limits_{i=1}^{2k_1}\sum\limits_{j=1}^{k_2} \|\sigma_\mu(u_i, u^j)\|^2 \leq 2.k_1.k_2.f_2,$

*where $\sigma_\mu$ is the component of $\sigma$ in $\mu$, $2k_1 + 1$ and $k_2$ are the dimensions of $M^T$ and $M^\perp$ respectively.*

*Proof.* For any unit vector fields $U_1 \in TN^T - \langle\xi_1\rangle$ and $W_1 \in TN^\perp$. Then from (2.4) we have

$$\bar{R}(U_1, \phi U_1, W_1, \phi W_1) = -2.f_2.g(U_1, U_1)\ g(W_1, W_1). \tag{3.1}$$

By Codazzi equation, we calculate the curvature as follows

$$\begin{aligned}\bar{R}(U_1, \phi U_1, W_1, \phi W_1) &= g(\nabla^\perp_{U_1}\sigma(\phi U_1, W_1), \phi W_1) - g(\sigma(\nabla_{U_1}\phi U_1, W_1), \phi W_1) - \\ &\quad - g(\sigma(\phi U_1, \nabla_{U_1} W_1), \phi W_1) - g(\nabla^\perp_{\phi U_1}\sigma(U_1, W_1), \phi W_1) + \\ &\quad + g(\sigma(\nabla_{\phi U_1} U_1, W_1), \phi W_1) + g(\sigma(U_1, \nabla_{\phi U_1} W_1), \phi W_1).\end{aligned} \tag{3.2}$$

By using part (iii) of the Lemma 3.1, (2.9), (2.5) and (2.12), we get

$$\begin{aligned}g(\nabla^\perp_{U_1}\sigma(\phi U_1, W_1), \phi W_1) &= U_1 g(\sigma(\phi U_1, W_1), \phi W_1) - g(\sigma(\phi U_1, W_1), \bar{\nabla}_{U_1}\phi W_1) \\ &= U_1(U_1 ln\psi g(W_1, W_1)) - g(\sigma(\phi U_1, W_1), (\bar{\nabla}_{U_1}\phi)W_1 + \phi\bar{\nabla}_{U_1} W_1).\end{aligned}$$

On further simplification the above equation yields

$$\begin{aligned}g(\nabla^\perp_{U_1}\sigma(\phi U_1, W_1), \phi W_1) \quad = \quad & U_1^2 ln\psi g(W_1, W_1) + 2(U_1 ln\psi)^2 g(W_1, W_1) \\ & - g(\sigma(\phi U_1, W_1), Q_{U_1} W_1) \\ & - g(\sigma(\phi U_1, W_1), \phi\sigma(U_1, W_1)) \\ & - U_1 ln\psi g(\sigma(\phi U_1, W_1), \phi W_1).\end{aligned}$$

By using the Lemma 3.2, we have

$$\begin{aligned}g(\nabla^\perp_{U_1}\sigma(\phi U_1, W_1), \phi W_1) &= U_1^2 ln\psi g(W_1, W_1) + (U_1 ln\psi)^2 g(W_1, W_1) - \|\sigma_\mu(U_1, W_1)\|^2 \\ &\quad - g(\phi\sigma(U_1, W_1) - \sigma(\phi U_1, W_1), Q_{U_1} W_1).\end{aligned}$$

Further, using (2.5), (2.12), (2.13)($b$) and (2.17) in the last term of the above equation, we get

$$\begin{aligned}g(\nabla^\perp_{U_1}\sigma(\phi U_1, W_1), \phi W_1) &= U_1^2 ln\psi g(W_1, W_1) + (U_1 ln\psi)^2 g(W_1, W_1) \\ &\quad - \|\sigma_\mu(U_1, W_1)\|^2 + \|Q_{U_1} W_1\|^2.\end{aligned} \tag{3.3}$$

Similarly, we can calculate

$$-g(\nabla^{\perp}_{\phi U_1}\sigma(U_1, W_1), \phi W_1) = (\phi U_1)^2 ln\psi g(W_1, W_1) + (\phi U_1 ln\psi)^2 g(W_1, W_1) - \|\sigma_\mu(\phi U_1, W_1)\|^2 + \|Q_{\phi U_1} W_1\|^2. \tag{3.4}$$

From part (iii) of the Lemma 3.1, we have

$$g(A_{\phi W_1} W_1, \phi U_1) = U_1 ln\psi,$$

replacing $U_1$ by $\nabla_{U_1} U_1$

$$g(A_{\phi W_1} W_1, \phi \nabla_{U_1} U_1) = \nabla_{U_1} U_1 ln\psi.$$

By utilsing (2.5) in the last equation, we acquire

$$g(A_{\phi W_1} W_1, \phi(\bar{\nabla}_{U_1} U_1 - \sigma(U_1, U_1)) = \nabla_{U_1} U_1 ln\psi. \tag{3.5}$$

By use of (2.5), (2.9), (2.3) and (2.17), it is easy to see that $\sigma(U_1, U_1) \in \mu$, applying this fact in (3.5), then we get

$$g(A_{\phi W_1} W_1, \bar{\nabla}_{U_1} \phi U_1 - (\bar{\nabla}_{U_1} \phi) U_1) = \nabla_{U_1} U_1 ln\psi.$$

By (2.3) and (2.5), the previous equation transformed to

$$g(A_{\phi W_1} W_1, \nabla_{U_1} \phi U_1) = \nabla_{U_1} U_1 ln\psi.$$

or

$$g(\sigma(\nabla_{U_1} \phi U_1, W_1), \phi W_1) = \nabla_{U_1} U_1 ln\psi \tag{3.6}$$

Similarly,

$$g(\sigma(\nabla_{\phi U_1} U_1, W_1), \phi W_1) = -\nabla_{\phi U_1} \phi U_1 ln\psi. \tag{3.7}$$

By use of (2.17) and the part (iii) of the Lemma 3.1, it is simple to see the following

$$g(\sigma(\phi U_1, \nabla_{U_1} W_1), \phi W_1) = (U_1 ln\psi)^2 g(W_1, W_1) \tag{3.8}$$

and

$$g(\sigma(U_1, \nabla_{\phi U_1} W_1), \phi W_1) = -(\phi U_1 ln\psi)^2 g(W_1, W_1). \tag{3.9}$$

Substituting (3.3), (3.4), (3.6), (3.7), (3.8) and (3.9) in (3.2), we find

$$\begin{aligned} \bar{R}(U_1, \phi U_1, W_1, \phi W_1) \quad = \quad & U_1^2 ln\psi g(W_1, W_1) + (\phi U_1)^2 ln\psi g(W_1, W_1) \\ & -\nabla_{U_1} U_1 ln\psi g(W_1, W_1) - \nabla_{\phi U_1} \phi U_1 g(W_1, W_1) \\ & -\|\sigma_\mu(U_1, W_1)\|^2 - \|\sigma_\mu(\phi U_1, W_1)\|^2 + \|Q_{U_1} W_1\|^2 \\ & +\|Q_{\phi U_1} W_1\|^2. \end{aligned} \tag{3.10}$$

Let

$$\begin{aligned} \{u_0 \quad = \quad & \xi_1, u_1, u_2, \ldots, u_{k_1}, u_{k_1+1} = \phi u_1, \\ & u_{k_1+2} = \phi u_2, \ldots, u_{2k_1} = \phi u_{k_1}, u^1, u^2, \ldots, u^{k_2}\} \end{aligned}$$

be an orthonormal frame of $TM$ such that the set of the vector fields

$$\{u_1, u_2, \ldots, u_{k_1}, \phi u_1, \phi u_2, \ldots, \phi u_{k_1}\}$$

is tangent to $M_T$ and $\{u^1, u^2, \ldots, u^{k_2}\}$ is tangent to $M_\perp$.

Using (3.1) and (2.20) in (3.10) and summing over $i = 1, 2, \dots, k_1$ and $j = 1, 2, \dots, k_2$, we get

$$k_2 \Delta ln\psi = 2.k_1.k_2.f_2 - \sum_{i=1}^{2k_1}\sum_{j=1}^{k_2} \|\sigma_\mu(u_i, u^j)\|^2 + \sum_{i=1}^{2k_1}\sum_{j=1}^{k_2} \|Q_{u_i} u^j\|^2. \tag{3.11}$$

From Hopf's Lemma and (3.11), if

$$\sum_{i=1}^{2k_1}\sum_{j=1}^{k_2} \|\sigma_\mu(u_i, u^j)\|^2 \geq 2.k_1.k_2.f_2 + \sum_{i=1}^{2k_1} \|Q_{u_i} u f^j\|^2,$$

or

$$\sum_{i=1}^{2k_1}\sum_{j=1}^{k_2} \|\sigma_\mu(u_i, u^j)\|^2 \leq 2.k_1.k_2.f_2,$$

then the warping function $\psi$ is constant on $M$ i.e., $M$ is simply a contact CR-product submanifold, which proves the theorem completely.
□

From the above observations, we have the following propositions, which can be confirmed easily.

**Proposition 3.1.** *Let $M = N^T \times_\psi N^\perp$ be a contact CR-warped product submanifold of a generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting a nearly cosymplectic structure such that $M^T$ is a compact submanifold. Then $M$ is contact CR-product if and only if*

$$\sum_{i=1}^{2k_1}\sum_{j=1}^{k_2} \|\sigma_\mu(u_i, u^j)\|^2 = 2.k_1.k_2.f_2 + \sum_{i=1}^{2k_1}\sum_{j=1}^{k_2} \|Q_{u_i} u^j\|^2.$$

Moreover, as a special case, from Theorem 3.2, we derive the following result.

**Theorem 3.3.** *Let $M = N^T \times_\psi N^\perp$ be a contact CR-warped product submanifold of a cosymplectic space form $\bar{M}(c)$ such that $M^T$ is compact. Then $M$ is a contact CR-product submanifold if either the inequality*

$$\sum_{i=1}^{2k_1}\sum_{j=1}^{k_2} \|\sigma_\mu(u_i, u^j)\|^2 \geq \frac{c.k_1.k_2}{2},$$

*or*

$$\sum_{i=1}^{2k_1}\sum_{j=1}^{k_2} \|\sigma_\mu(u_i, u^j)\|^2 \leq \frac{c.k_1.k_2}{2}$$

*holds, where $\sigma_\mu$ denotes the component of $\sigma$ in $\mu$, $2k_1 + 1$ and $k_2$ are the dimensions of $M^T$ and $M^\perp$.*

**Corollary 3.4.** *Let $M = N^T \times_\psi N^\perp$ be a compact contact CR-warped product submanifold of a cosymplectic space form $\bar{M}(c)$. Then $M$ is a contact CR-product submanifold if and only if*

$$\sum_{i=1}^{2k_1}\sum_{j=1}^{k_2} \|\sigma_\mu(u_i, u^j)\|^2 = \frac{c.k_1.k_2}{2}.$$

### 3.2 Inequalities of the second kind

In the present subsection, we prove an approximation for the norm of the second fundamental form

**Theorem 3.5.** *Let $\bar{M}(f_1, f_2, f_3)$ be a $(2n+1)-$dimensional generalized Sasakian space form admitting a nearly cosymplectic structure and $M = N^T \times_\psi N^\perp$ be an $m$-dimensional contact CR-warped product submanifold, such that $M^T$ is a $(2k_1+1)-$dimensional invariant submanifold tangent to $\xi_1$ and $M^\perp$ be a $k_2-$dimensional anti-invariant submanifold of $\bar{M}(f_1, f_2, f_3)$. Then*

*(i) The squared norm of the second fundamental form $\sigma$ satisfies*

$$\|\sigma\|^2 \geq k_2\{\|\nabla ln\psi\|^2 - \Delta ln\psi\} + 2.k_1.k_2.f_2 + \|Q_D D^\perp\|^2, \tag{3.12}$$

*where $\Delta$ denotes the Laplace operator on $M^T$.*

*(ii) The equality sign of (3.12) holds identically if and only if we have*

*(a) $M^T$ is a totally geodesic invariant submanifold of $\bar{M}(f_1, f_2, f_3)$. Hence $N^T$ is a generalized Sasakian space form admitting a nearly cosymplectic structure.*

*(b) $M^\perp$ is a totally umbilical anti-invariant submanifold of $\bar{M}(f_1, f_2, f_3)$.*

*Proof.* For any $U_1 \in TN^T$ and $W_1 \in TN^\perp$, from (2.8), (2.6) and (2.3) we have

$$g(\sigma(\xi_1, W_1), \phi W_1) = 0$$

and from part (iii) of the Lemma 3.1

$$g(\sigma(\phi U_1, W_1), \phi W_1) = U_1 ln\psi \|W_1\|^2.$$

From above two equations one can get

$$\sum_{i=0}^{2k_1}\sum_{j=1}^{k_2} \|\sigma_{\phi D^\perp}(u_i, u^j)\|^2 = k_2\|\nabla ln\psi\|^2. \tag{3.13}$$

Again from equation (3.11)

$$\sum_{i=1}^{2k_1}\sum_{j=1}^{k_2} \|\sigma_\mu(u_i, u^j)\|^2 = 2.k_1.k_2.f_2 - k_2\Delta ln\psi + \sum_{i=1}^{2k_1}\sum_{j=1}^{k_2} \|Q_{u_i} u^j\|^2. \tag{3.14}$$

The following notation is assummed

$$\sum_{i=1}^{2k_1}\sum_{j=1}^{k_2} \|Q_{u_i} u^j\|^2 = \|Q_D D^\perp\|^2.$$

Utilizing the above notation in (3.14) and combining it with (3.13), we obtain the inequality (3.12).

Let $\sigma''$ be the second fundamental form of $M^\perp$ in $M$. Then, for any $U_1 \in TN^T$ and $Z_1, \bar{Z}_1 \in TN^\perp$, we have

$$g(\sigma''(Z_1, \bar{Z}_1), U_1) = g(\nabla_{Z_1}\bar{Z}_1, U_1) = -U_1 ln\psi g(Z_1, \bar{Z}_1),$$

By using (2.18), we get

$$\sigma''(Z_1, \bar{Z}_1) = -g(Z_1, \bar{Z}_1)\nabla ln\psi. \tag{3.15}$$

If the equality sign of (3.12) holds identically, then we get

$$\sigma(D, D) = 0, \ \sigma(D^\perp, D^\perp) = 0. \tag{3.16}$$

The first case of (3.16) implies that $M^T$ is totally geodesic in $M$. On the other hand, one has

$$g(\sigma(U_1, \phi V_1), \phi W_1) = g(\bar{\nabla}_{U_1}\phi V_1, \phi W_1) = -g(\phi V_1, (\bar{\nabla}_{U_1}\phi)W_1). \tag{3.17}$$

By use of (2.9) and (2.5) we get the following equation

$$g(\phi V_1, (\bar{\nabla}_{W_1}\phi)U_1) = g(\phi V_1, \nabla_{W_1}\phi U_1) - g(V_1, \nabla_{W_1}U_1),$$

in view of (2.17) the previous equation reduced to

$$g(\phi V_1, (\bar{\nabla}_{W_1}\phi)U_1) = 0. \tag{3.18}$$

From (3.17), (3.18) and (2.3) we have

$$g(\sigma(U_1, \phi V_1), \phi W_1) = -g(\phi V_1, (\bar{\nabla}_{U_1}\phi)W_1 + (\bar{\nabla}_{W_1}\phi)U_1) = 0. \tag{3.19}$$

From (3.19), it is evident that the submanifold $N^T$ is totally geodesic in $\bar{M}(f_1, f_2, f_3)$ and hence is a generalized Sasakian space form admitting a nearly cosymplectic structure.

The second case of (3.16) and (3.15) imply that the submanifold $M^\perp$ is totally umbilical in $\bar{M}(f_1, f_2, f_3)$.

□

In the last we have the following Corollary, which can be deduced from inequality (3.12)

**Corollary 3.6.** *Let $M = N^T \times_\psi N_\perp$ be a contact CR-warped product submanifold of a cosymplectic space form $\bar{M}(c)$, then the squared norm of the second fundamental form satisfies*

$$\|\sigma\|^2 \geq k_2\{\|\nabla ln\psi\|^2 - \Delta ln\psi\} + \frac{c.k_1.k_2}{2},$$

*where $\Delta$ is the Laplace operator on $M^T$, and $2k_1+1$ and $k_2$ are the dimensions of $M^T$ and $M^\perp$ respectively.*

**Note 3.1.** The inequality obtained in Corollary 2 is the improved version of the inequality obtained in [15].

## 4. Conclusion

The inequality $\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\|\sigma_\mu(u_i, u^j)\|^2 \geq \frac{c.k_1.k_2}{4}$ for the existence of a contact CR-warped product submanifold in a cosymplectic space form $\bar{M}(c)$ was proved by M. Atceken [14]. In the Theorem 3.3, the first inequality is equivalent to the inequality obtained by M. Atceken and we also prove the second inequality for the contact CR-warped product submanifolds of a cosymplectic space form. Moreover, the inequalities obtained in Theorem 3.2 are the generalization of the inequalities that have obtained by M. Atceken [14]. On the other hand various estimate for the squared norm of the second fundamental form for the contact CR-warped product submanifolds in the background of Sasakian and Kenmotsu space forms are studied (see [11], [12]), still these types of estimate are not studied in the setting of the cosymplctic space forms as well as in the generalized Sasakian space forms admitting a nearly cosymplectic structure. So the present author tries to fill this gap and in this connection proved an optimal inequality for the squared norm of the second fundamental form in the setting of generalized Sasakian space forms admitting a nearly cosymplectic structure and in particular for the cosymplectic space forms.

## 5. Declarations

**Competing Interests:** The author declares that he has no competing interests.

**Author Contributions:** The author read and approved the final manuscript.

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