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Existence and Multiplicity of Solutions for a p(x)-biharmonic Problem with Neumann Boundary Conditions

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ABSTRACT: In this paper, we study the following problem with Neumann boundary conditions

$$\begin{cases} \Delta^2_{p(x)} u + \alpha \, |u|^{p(x)-2} \, u = \beta f(x,u) & \text{in } \ \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial}{\partial \nu} \left(|\Delta u|^{p(x)-2} \, \Delta u \right) = 0 & \text{on } \ \partial\Omega. \end{cases}$$

Where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 1$, $\Delta^2_{p(x)} u := \Delta \left(|\Delta u|^{p(x)-2} \, \Delta u \right)$, is the p(x)-biharmonic operator, α and β are two positives reals numbers, p is a continuous function on $\overline{\Omega}$ with $\inf_{x \in \overline{\Omega}} p(x) > 1$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function such that $f(x, 0) = 0$. Using the three critical point Theorem, we establish the existence of at least three solutions of this problem.

Key Words: p(x)-biharmonic operator, Critical point, Nemytskii's operator.

Contents

1. Introduction

In recent years, the theory of p-Laplacian equations has been generalized to p(x)-Laplacian equations. Comparing with the p-Laplacian operator, the p(x)-Laplacian operator has more complicated nonlinear properties: it is not homogeneous and the infimum of its principle eigenvalue is zero (see [2,3,4,5,11,14]).

This paper is motivated by recent advances in mathematical modeling of non-Newtonien fluids and elastic mechanics, in particular, the electrorheological fluids (smart fluids). This important class of fluids is characterized by the change of viscosity which is not easy and which depends on the electric field. These fluids, which are known under the name ER fluids, have many applications in elastic mechanics, fluid dynamics etc... For more information, the reader can refer to [8,13].

These physical problems was facilitated by the development of Lebesgue and Sobolev spaces with variable exponent.

Neumann boundary value problems for p(x)-Laplacian operator and their existence of solutions was established in [2,11,14].

More recently, in [1], A. Ayoujil and A. R. El Amrouss interested to the spectrum of a fourth order elliptic equation with variable exponent. They proved the existence of infinitely many eigenvalue sequences and $\sup \Lambda = +\infty$, where Λ is the set of all eigenvalues. Moreover, they present some sufficient conditions for $\inf \Lambda = 0$.

Consider the following problem with Neumann boundary conditions

$$\begin{cases} \Delta^2_{p(x)} u + \alpha \, |u|^{p(x)-2} \, u = \beta f(x,u) & \text{in } \ \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial}{\partial \nu} \left(|\Delta u|^{p(x)-2} \, \Delta u \right) = 0 & \text{on } \ \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 1$, $\Delta^2_{p(x)}u := \Delta\left(|\Delta u|^{p(x)-2}\Delta u\right)$, is the p(x)-biharmonic operator, $p \in C\left(\overline{\Omega}\right)$ such that $2p(x) \geq N$ for all $x \in \overline{\Omega}$ and α, β are two positives reals numbers. We proves the existence of at least three weak solutions of the above problem, under the following assumptions

(f1) For all $(x, s) \in \overline{\Omega} \times \mathbb{R}$

$$|f(x,s)| \leq a(x) + b\,|s|^{\gamma(x)-1}$$

where $b \geq 0$, $a(x) \in L^{\frac{\gamma(x)}{\gamma(x)-1}}(\Omega)$, $a(x) \geq 0$ and $\gamma(x) \in C_+\left(\overline{\Omega}\right)$ with

$$\gamma(x) < \begin{cases} \frac{Np(x)}{N-2p(x)} & \text{if} \quad N > 2p(x) \\ +\infty & \text{if} \quad N \leq 2p(x), \end{cases}$$

(f2) There exists a constant $\eta > 0$ such that

$$\liminf_{t\to+\infty} \frac{f(x,t)}{t} \geq \eta$$

for all $x \in \overline{\Omega}$.

(f3) There exist a constant $c < 0$ and a positive odd constant δ such that

$$\limsup_{t\to 0} \frac{f(x,t)}{t^{\delta}} \leq c$$

for $x \in \overline{\Omega}$ uniformly.

Theorem 1.1. *Suppose that f satisfies the conditions (f1,f2,f3), $p^- > \gamma^+$ and $p(x) > \frac{N}{2}$. Then, there exist an open interval $\Lambda \subset]0, +\infty[$ and a positive real number q, such that for each $\beta \in \Lambda$, the problem (1.1) has at least three solutions in X whose norms are less than q.*

Theorem 1.2. *Suppose that f satisfies the conditions (f1) and*

(f4) *There exists an odd constant $\omega > p^+ - 1$ such that*

$$\liminf_{|t|\to+\infty} \frac{f(x,t)}{t^{\omega}} = +\infty.$$

If $\gamma^- > p^+ \geq p^- > \frac{N}{2}$, then there exist an open interval $\Lambda \subset]0, +\infty[$ and a positive real number q, such that for each $\beta \in \Lambda$, the problem (1.1) has at least three solutions in X whose norms are less than q.

This paper is divided into four sections, organized as follows: In section 2, we introduce some basic properties of the Lebesgue and Sobolev spaces with variable exponent. Moreover, we cite one lemma which is needed later. In the third section, we study boundary trace embedding theorems for variable exponent Sobolev space $W^{2,p(x)}(\Omega)$ and we present some important properties of the p(x)-biharmonic operator. In section 4, we give the proofs of our results.

2. Preliminaries

In order to deal with p(x)-Laplacian problems, we need some results on spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, and properties of p(x)-Laplacian, which we will use later.

Define the generalized Lebesgue space by

$$L^{p(x)}(\Omega) := \left\{u : \Omega \to \mathbb{R} \quad \text{measurable and} \quad \int_{\Omega} |u(x)|^{p(x)}\,dx < \infty\right\},$$

where $p \in C_{+}\left(\overline{\Omega}\right)$ and

$$C_{+}\left(\overline{\Omega}\right) := \left\{p \in C\left(\overline{\Omega}\right) : p(x) > 1 \quad \forall x \in \overline{\Omega}\right\}.$$

Denote

$$p^{+} = \max_{x \in \overline{\Omega}} p(x), \quad p^{-} = \min_{x \in \overline{\Omega}} p(x),$$

and for all $x \in \overline{\Omega}$ and $k \geq 1$

$$p^{*}(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N \\ +\infty & \text{if } p(x) \geq N, \end{cases}$$

and

$$p_{k}^{*}(x) := \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if } kp(x) < N \\ +\infty & \text{if } kp(x) \geq N. \end{cases}$$

One introduces in $L^{p(x)}(\Omega)$ the following norm

$$|u|_{p(x)} = \inf\left\{\lambda > 0 / \int_{\Omega}\left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}$$

and the space $\left(L^{p(x)}(\Omega), |.|_{p(x)}\right)$ is a Banach.

Proposition 2.1. *[6]. The space $\left(L^{p(x)}(\Omega), |.|_{p(x)}\right)$ is separable, uniformly convex, reflexive and its conjugate space is $L^{q(x)}(\Omega)$ where $q(x)$ is the conjugate function of $p(x)$, i.e*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1, \forall x \in \Omega.$$

For all $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ we have

$$\left|\int_{\Omega} u(x)\, v(x) dx\right| \leq \left(\frac{1}{p^{-}} + \frac{1}{q^{-}}\right) |u|_{p(x)} |v|_{q(x)}.$$

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined by

$$W^{k,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : D^{\alpha}u \in L^{p(x)}(\Omega), |\alpha| \leq k\right\},$$

where $D^{\alpha}u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}} u$, (the derivation in distributions sense) with $\alpha = (\alpha_1, ..., \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^{N} \alpha_i$. The space $W^{k,p(x)}(\Omega)$, equipped with the norm

$$\|u\|_{k,p(x)} := \sum_{|\alpha| \leq k} |D^{\alpha}u|_{p(x)},$$

also becomes a Banach, separable and reflexive space. For more details, we can refer to [3,6,11,15]. We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ in $W^{k,p(x)}(\Omega)$.

Our results are based on the following lemma

Lemma 2.1. *Let X be a separable and reflexive real Banach space; $\Phi : X \to \mathbb{R}$ is a continuous Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^{*}; $\Psi; X \to \mathbb{R}$ is a continuous Gâteaux differentiable functional whose Gâteaux derivative is compact. Suppose the following assertions*

1. $\lim_{\|u\|_X \to \infty} (\Phi(u) + \lambda\Psi(u)) = \pm\infty$ *for all* $\lambda > 0$*;*

2. There exist $r \in \mathbb{R}$ and $u_0, u_1 \in X$ such that

$$\Phi(u_0) < r < \Phi(u_1);$$

3.

$$\inf_{u \in \Phi^{-1}]-\infty, r]} \Psi(u) > \frac{(\Phi(u_1) - r)\Psi(u_0) + (r - \Phi(u_0))\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)}.$$

Then there exist an open interval $\Lambda \subset]0, +\infty[$ and a positive real number q such that the equation

$$\Phi'(u) + \lambda\Psi'(u) = 0$$

admits at least three solutions in X whose norms are less than q, for all $\lambda \in \Lambda$.

3. Position of problem

We say that u is a weak solution for the problem (1.1) if

$$u \in X := \left\{ u \in W^{2,p(x)}(\Omega) / \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \right\}$$

and

$$\int_\Omega \left(|\Delta u|^{p(x)-2} \Delta u \Delta v + \alpha |u|^{p(x)-2} uv \right) dx = \beta \int_\Omega f(x, u(x)) v dx \quad \forall v \in X.$$

Put for $\alpha > 0$

$$\Phi_\alpha(u) = \int_\Omega \frac{1}{p(x)} \left(|\Delta u(x)|^{p(x)} + \alpha |u(x)|^{p(x)} \right) dx$$

and

$$\Psi(u) = -\int_\Omega F(x, u(x)) dx$$

where $F(x,s) = \int_0^s f(x,t)dt$.

The space $W^{2,p(x)}(\Omega)$ is equipped with the norm

$$\|u\|_\alpha = \inf \left\{ \lambda > 0 : \int_\Omega \left(\left| \frac{\Delta u(x)}{\lambda} \right|^{p(x)} + \alpha \left| \frac{u(x)}{\lambda} \right|^{p(x)} \right) dx \leq 1 \right\}.$$

Remark 3.1. *The norm $\|u\|_\alpha$ is equivalent to the norm*

$$|\Delta u|_{L^{p(x)}(\Omega)} + |u|_{L^{p(x)}(\Omega)}$$

Proposition 3.1. *If we put*

$$J_\alpha(u) := \int_\Omega \left(|\Delta u(x)|^{p(x)} + \alpha |u(x)|^{p(x)} \right) dx$$

then for all $u \in W^{2,p(x)}(\Omega)$ we have

(1) $\|u\|_\alpha < (=; > 1) \Leftrightarrow J_\alpha(u) < (=; > 1)$,

(2) $\|u\|_\alpha \leq 1 \Rightarrow \|u\|_\alpha^{p^+} \leq J_\alpha(u) \leq \|u\|_\alpha^{p^-}$,

(3) $\|u\|_\alpha \geq 1 \Rightarrow \|u\|_\alpha^{p^-} \leq J_\alpha(u) \leq \|u\|_\alpha^{p^+}$,
for all $u_n \in W^{2,p(x)}(\Omega)$ we have

(4) $\|u_n\|_\alpha \to 0 \Leftrightarrow J_\alpha(u_n) \to 0$,

(5) $\|u_n\|_\alpha \to \infty \Leftrightarrow J_\alpha(u_n) \to \infty$.

It is necessary to show that X is a closed subspace of $W^{2,p(x)}(\Omega)$. In order to obtain our goal, we need the following boundary trace embedding theorem for variable exponent Sobolev spaces

Theorem 3.2. *Let Ω be a bounded domain in $\mathbb{R}^N$ with C^2 boundary. If $2p(x) \geq N \geq 2$ for all $x \in \overline{\Omega}$, then for all $q \in C_+\left(\overline{\Omega}\right)$ there is a continuous boundary trace embedding*

1. $$W^{2,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega),$$
   and
2. $$W^{2,p(x)}(\Omega) \hookrightarrow W^{1,p(x)}(\partial\Omega).$$

Proof: (1) Let $p, q \in C_+\left(\overline{\Omega}\right)$ such that for all $x \in \overline{\Omega}$, $2p(x) \geq N$. It's clear that there exists the following continuous embedding
$$W^{2,p(x)}(\Omega) \hookrightarrow W^{2,p^-}(\Omega) \tag{3.1}$$
and
$$L^{q^+}(\partial\Omega) \hookrightarrow L^{q(x)}(\partial\Omega). \tag{3.2}$$
Using the classical boundary trace embedding theorem, since $2p^- \geq N$ and $q^+ \geq 1$, there exists the continuous embedding
$$W^{2,p^-}(\Omega) \hookrightarrow L^{q^+}(\partial\Omega). \tag{3.3}$$
Hence, the space $W^{2,p(x)}(\Omega)$ is continuously embedded into $L^{q(x)}(\partial\Omega)$ by combining (3.1), (3.2) and (3.3).

(2) Since $2p^- \geq N$ and $p^+ > 1$, we have the continuous embedding (see [10])
$$W^{2,p^-}(\Omega) \hookrightarrow W^{1,p^+}(\partial\Omega). \tag{3.4}$$
Moreover, it's easy to see that
$$W^{1,p^+}(\partial\Omega) \hookrightarrow W^{1,p(x)}(\partial\Omega). \tag{3.5}$$
Then, the result is given from (3.1), (3.4) and (3.5). □

Proposition 3.2. *If $2p(x) \geq N$ for all $x \in \overline{\Omega}$ then, the set*
$$X = \left\{ u \in W^{2,p(x)}(\Omega) \,/\, \frac{\partial u}{\partial v}|_{\partial\Omega} = 0 \right\}$$
is a closed subspace of $W^{2,p(x)}(\Omega)$.

Proof: To prove the above proposition, we must show that the operator
$$\begin{array}{rccc} D: & W^{2,p(x)}(\Omega) & \to & L^{p(x)}(\partial\Omega) \\ & u & \longmapsto & \frac{\partial u}{\partial v}|_{\partial\Omega} \end{array}$$
is continuous from $\left(W^{2,p(x)}(\Omega), \|.\|_\alpha\right)$ to $\left(L^{p(x)}(\partial\Omega), |.|_{L^{p(x)}(\partial\Omega)}\right)$.

First, let us prove the continuity of the operator
$$\begin{array}{rccc} \nabla: & W^{2,p(x)}(\Omega) & \to & \left(L^{p(x)}(\partial\Omega)\right)^N \\ & u & \longmapsto & (\nabla u)|_{\partial\Omega}. \end{array}$$
from $\left(W^{2,p(x)}(\Omega), \|.\|_\alpha\right)$ to $\left(\left(L^{p(x)}(\partial\Omega)\right)^N, \|.\|_{p(x),N}\right)$ with
$$\|\overrightarrow{n}\|_{p(x),N} := \sum_{i=1}^{N} |n_i|_{p(x)}.$$

Let $(u_n)_n \subset W^{2,p(x)}(\Omega)$ be a sequence such that $u_n \to u$ in $W^{2,p(x)}(\Omega)$. From the second assertion of the theorem 3.2, we have

$$u_n \to u \quad \text{in} \quad W^{1,p(x)}(\partial\Omega)$$

what implies that

$$\nabla u_n \to \nabla u \quad \text{in} \quad \left(L^{p(x)}(\partial\Omega)\right)^N$$

and then ∇ is continuous.

Moreover, $D = T \circ \nabla$ with T is the linear function defined as

$$\begin{array}{cccc} T: & \left(L^{p(x)}(\partial\Omega)\right)^N & \to & L^{p(x)}(\partial\Omega) \\ & \overrightarrow{n} = (n_1, ..., n_N) & \longmapsto & \overrightarrow{n}.\overrightarrow{v} \end{array}$$

where $\overrightarrow{v}(x) = (\alpha_1(x), ..., \alpha_N(x))$ is the outer unit normal vector and

$$\sum_{i=1}^{N} |\alpha_i(x)|^2 = 1 \quad \text{for all} \quad x \in \partial\Omega. \tag{3.6}$$

The operator T is continuous, indeed, for $\overrightarrow{n} \in \left(L^{p(x)}(\partial\Omega)\right)^N$ we have

$$|\overrightarrow{n}.\overrightarrow{v}|_{p(x)} = \left|\sum_{i=1}^{N} n_i\alpha_i\right|_{p(x)} \leq \sum_{i=1}^{N} |n_i\alpha_i|_{p(x)}.$$

On the other hand, thanks to (3.6), we obtain

$$|\alpha_i(x)| \leq 1 \quad \text{for all} \quad x \in \partial\Omega, \ i \in \{1, ..., N\}.$$

Consequently, we deduct that

$$|\overrightarrow{n}.\overrightarrow{v}|_{L^{p(x)}(\partial\Omega)} \leq \sum_{i=1}^{N} |n_i|_{p(x)} = \|\overrightarrow{n}\|_{p(x),N} \tag{3.7}$$

which assert that T is continuous and then D is also continuous. Finally, since

$$X = D^{-1}\{0\},$$

it results that X is closed in $W^{2,p(x)}(\Omega)$. Hence, the proof of the proposition 3.2 is completed. □

In what follows, we have to need the following proposition which is an extension of Sobolev embedding theorems to the Sobolev spaces with variable exponent.

Proposition 3.3. *Let $p \in C_+\left(\overline{\Omega}\right)$ such that $2p(x) > N$ for all $x \in \overline{\Omega}$ then*

1. *there exists a continuous and compact embedding of $W^{2,p(x)}(\Omega)$ into $L^{q(x)}(\Omega)$ for all $q \in C_+\left(\overline{\Omega}\right)$.*
2. *there exists a continuous embedding of $W^{2,p(x)}(\Omega)$ into $C\left(\overline{\Omega}\right)$.*

Proof: (1) We can refer to [4].

(2) For each $x \in \overline{\Omega}$, we have $2p(x) > N$. Then, there exists a neighborhood $U_x \subset \overline{\Omega}$ such that

$$2p^-(U_x) > N$$

where

$$p^-(U_x) = \inf_{y \in U_x} p(y).$$

Hence, we get a family open covering $\{U_x\}_{x\in\overline{\Omega}}$ for the compact set $\overline{\Omega}$. For a sub-covering $\{U_i\}_{i=1,\dots,r}$, one considers m_i such that

$$0 \leq m_i < 2 - \frac{N}{p_i^-} < m_i + 1.$$

Thanks to the theorem 7.26 [7], there exists a continuous embedding

$$W^{2,p_i^-}(U_i) \hookrightarrow C^{m_i,\alpha_i}\left(\overline{U_i}\right) \tag{3.8}$$

where $\alpha_i = 2 - \frac{N}{p_i^-} - m_i$. On the other hand, for all $i \in \{1,...,r\}$, it easy to see that

$$W^{2,p(x)}(U_i) \subset W^{2,p_i^-}(U_i) \tag{3.9}$$

and

$$C^{m_i,\alpha_i}\left(\overline{U_i}\right) \subset C\left(\overline{U_i}\right). \tag{3.10}$$

From (3.8), (3.9) and (3.10), it follows that

$$W^{2,p(x)}(U_i) \subset C\left(\overline{U_i}\right)$$

for all U_i ,$i = 1,...,r$. This assert that the embedding

$$W^{2,p(x)}(\Omega) \hookrightarrow C\left(\overline{\Omega}\right)$$

is continuous. The proof of proposition 3.3 is completed. □

Remark 3.3. *Note that the space* $(W^{2,p(x)}(\Omega), \|.\|_\alpha)$ *is a Banach, separable and reflexive.*

It's clear that $\Phi_\alpha \in C^1(X,\mathbb{R})$ with

$$\langle \Phi'_\alpha(u), v\rangle = \int_\Omega \left(|\Delta u(x)|^{p(x)-2}\Delta u\Delta v + \alpha|u(x)|^{p(x)-2}uv\right)dx.$$

For the operator Ψ, we cite the following theorem

Theorem 3.4. *If* $f:\Omega\times\mathbb{R}\to\mathbb{R}$ *is a Caratheodory function and satisfies the condition (f1), then the operator* Ψ *satisfies the following assertions*

(i) Ψ *is a* C^1 *operator and for all* u,v *in* X

$$\langle \Psi'(u), v\rangle = -\int_\Omega f(x,u)\,v dx.$$

(ii) The operator $\Psi' : X \to X'$ *is completely continuous.*

Proof: (i)From the condition (f1), we have

$$\begin{aligned} |F(x,s)| &\leq a(x)|s| + \frac{b}{\gamma(x)}|s|^{\gamma(x)} \\ &\leq A(x) + b|s|^{\gamma(x)} \end{aligned}$$

where $A(x)\geq 0, A\in L^1(\Omega)$ and $\gamma < p_2^*$. So, the Nemytskii's operator properties assert that Ψ is a C^1 function in $L^{\gamma(x)}(\Omega)$. Since X embedded continuously into $L^{\gamma(x)}(\Omega)$, Ψ is a C^1 function in X and

$$\langle \Psi'(u), v\rangle = -\int_\Omega f(x,u(x))\,v(x)\,dx.$$

(ii) Let $(u_n)_n \subset X$ be a sequence such that $u_n \rightharpoonup u$. By the compact embedding of X into $L^{\gamma(x)}(\Omega)$, there exists a subsequence, also noted $(u_n)_n$, such that $u_n \to u$ in $L^{\gamma(x)}(\Omega)$. Thanks to the Krasnoselki theorem, the Nemytskii's operator

$$\begin{array}{cccc} N_f: & L^{\gamma(x)}(\Omega) & \to & L^{\frac{\gamma(x)}{\gamma(x)-1}}(\Omega) \\ & u & \longmapsto & f(.,u) \end{array}$$

is continuous. It follows that $N_f(u_n) \to N_f(u)$ in $L^{\frac{\gamma(x)}{\gamma(x)-1}}(\Omega)$.

By Holder's inequality and the continuous embedding of X into $L^{\gamma(x)}(\Omega)$, we deduct

$$\begin{aligned} |\langle \Psi'(u_n) - \Psi'(u), v\rangle| &= \left| \int_\Omega (f(x,u_n) - f(x,u))\, v(x)\, dx \right| \\ &\leq 2\,|N_f(u_n) - N_f(u)|_{L^{\frac{\gamma(x)}{\gamma(x)-1}}(\Omega)} |v|_{L^{\gamma(x)}(\Omega)} \\ &\leq C\,|N_f(u_n) - N_f(u)|_{L^{\frac{\gamma(x)}{\gamma(x)-1}}(\Omega)} \|v\|_\alpha . \end{aligned}$$

Hence, $\Psi'(u_n) \to \Psi'(u)$ in X'. The proof of theorem 3.4 is completed. □

Consequently, the weak solutions of the problem (1.1) are the critical points of the operator $\Phi_\alpha(u) + \beta\Psi(u)$.

Remark 3.5. *The condition* $\liminf\limits_{t\to+\infty} \dfrac{f(x,t)}{t} = +\infty$ *implies (f2).*

Theorem 3.6. *The operator* $\Phi'_\alpha : X \to X'$ *satisfies the following assertions*

(1) Φ'_α *is a continuous, bounded and strictly monotone operator.*

(2) Φ'_α *is of* (S_+) *type.*

(3) Φ'_α *is homeomorphism.*

Proof: (1) Since Φ'_α is the Fréchet derivative of Φ_α, it follows that Φ'_α is continuous and bounded. Let's define the sets

$$U_p = \{x \in \Omega : p(x) \geq 2\}, \quad \text{and} \quad V_p = \{x \in \Omega : 1 < p(x) < 2\}.$$

Using the elementary inequalities

$$\begin{cases} |x-y|^\gamma \leq 2^\gamma \left(|x|^{\gamma-2} x - |y|^{\gamma-2} y \right).(x-y) \quad \text{if} \quad \gamma \geq 2, \\ |x-y|^2 \leq \frac{1}{(\gamma-1)} (|x| + |y|)^{2-\gamma} \left(|x|^{\gamma-2} x - |y|^{\gamma-2} y \right).(x-y) \quad \text{if} \quad 1 < \gamma < 2, \end{cases}$$

for all $(x,y) \in \left(\mathbb{R}^N\right)^2$, where $x.y$ denotes the usual inner product in $\mathbb{R}^N$, we obtain for all $u, v \in X$ such that $u \neq v$

$$\langle \Phi'_\alpha(u) - \Phi'_\alpha(v), u - v\rangle > 0,$$

which means that Φ'_α is strictly monotone.

(2) Let $(u_n)_n$ be a sequence of X such that

$$u_n \rightharpoonup u \quad \text{in } X \quad \text{and} \quad \limsup_{n\to+\infty} \langle \Phi'_\alpha(u_n), u_n - u\rangle \leq 0. \tag{3.11}$$

From the proposition 3.1, it suffices to shows that

$$\int_\Omega \left(|\Delta u_n - \Delta u|^{p(x)} + \alpha\, |u_n - u|^{p(x)} \right) dx \to 0. \tag{3.12}$$

In view of the monotonicity of Φ'_α, we have

$$\langle \Phi'_\alpha(u_n) - \Phi'_\alpha(u), u_n - u\rangle \geq 0.$$

On the other hand, (3.11) implies that

$$\limsup_{n\to+\infty} \langle \Phi'_\alpha(u_n) - \Phi'_\alpha(u), u_n - u\rangle \leq 0. \tag{3.13}$$

So,

$$\limsup_{n\to+\infty} \langle \Phi'_\alpha(u_n) - \Phi'_\alpha(u), u_n - u\rangle = 0. \tag{3.14}$$

Put

$$\begin{aligned}
\varphi_n(x) &= \left(|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u|^{p(x)-2}\Delta u\right).(\Delta u_n - \Delta u),\\
\xi_n(x) &= \left(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\right).(u_n - u).
\end{aligned}$$

By the compact embedding of X into $L^{p(x)}(\Omega)$, it follows that

$$u_n \to u \quad \text{in} \quad L^{p(x)}(\Omega)$$

and

$$|u_n|^{p(x)-2}u_n \to |u|^{p(x)-2}u \quad \text{in} \quad L^{q(x)}(\Omega)$$

where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$ for all $x \in \Omega$. It results that

$$\int_\Omega \xi_n(x)\,dx \to 0. \tag{3.15}$$

It follows by (3.14) and (3.15) that

$$\limsup_{n\to+\infty} \int_\Omega \varphi_n(x)\,dx = 0. \tag{3.16}$$

Thanks to above inequalities, we deduct

$$\int_{U_p} |\Delta u_n - \Delta u_k|^{p(x)}\,dx \leq 2^{p^+}\int_{U_p} \varphi_n(x)\,dx,$$

and

$$\int_{U_p} |u_n - u_k|^{p(x)}\,dx \leq 2^{p^+}\int_{U_p} \xi_n(x)\,dx.$$

Then

$$\int_{U_p}\left(|\Delta u_n - \Delta u|^{p(x)} + \alpha\,|u_n - u|^{p(x)}\right)dx \to 0 \quad \text{as} \quad n\to+\infty. \tag{3.17}$$

On the other hand, in V_p , setting $\delta_n = |\Delta u_n| + |\Delta u|$, we have

$$\int_{V_p} |\Delta u_n - \Delta u|^{p(x)}\,dx \leq \frac{1}{p^- - 1}\int_{V_p} (\varphi_n)^{\frac{p(x)}{2}}(\delta_n)^{\frac{p(x)}{2}(2-p(x))}\,dx,$$

and the Young's inequality gives that

$$\begin{aligned}
d\int_{V_p} |\Delta u_n - \Delta u|^{p(x)}\,dx &\leq \frac{1}{p^- - 1}\int_{V_p}\left[d(\varphi_n)^{\frac{p(x)}{2}}\right](\delta_n)^{\frac{p(x)}{2}(2-p(x))}\,dx,\\
&\leq \frac{1}{p^- - 1}\int_{V_p}\left(\varphi_n(d)^{\frac{2}{p(x)}} + (\delta_n)^{p(x)}\right)dx. \qquad (3.18)
\end{aligned}$$

From (3.16) and since $\varphi_n \geq 0$, one can consider that

$$0 \leq \int_{V_p} \varphi_n dx < 1.$$

If $\int_{V_p} \varphi_n dx = 0$ then $\int_{V_p} |\Delta u_n - \Delta u|^{p(x)}\, dx = 0$. If not, we take

$$d = \left(\int_{V_p} \varphi_n (x)\, dx \right)^{-\frac{1}{2}} > 1,$$

and the fact that $\frac{2}{p(x)} < 2$, the inequality (3.18) becomes

$$\begin{aligned}
(p^- - 1) \int_{V_p} |\Delta u_n - \Delta u|^{p(x)}\, dx &\leq \frac{1}{d} \left(\int_{V_p} \varphi_n d^2 dx + \int_\Omega \delta_n^{p(x)} dx \right), \\
&\leq \left(\int_{V_p} \varphi_n dx \right)^{\frac{1}{2}} \left(1 + \int_\Omega \delta_n^{p(x)} dx \right).
\end{aligned}$$

Note that, $\int_\Omega \delta_n^{p(x)} dx$ is bounded, this implies that

$$\int_{V_p} |\Delta u_n - \Delta u|^{p(x)}\, dx \to 0 \quad \text{as} \quad n \to +\infty.$$

A similar method gives

$$\int_{V_p} |u_n - u|^{p(x)}\, dx \to 0 \quad \text{as} \quad n \to +\infty.$$

Hence, it results that

$$\int_{V_p} \left(|\Delta u_n - \Delta u|^{p(x)} + \alpha\, |u_n - u|^{p(x)} \right) dx \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.19}$$

Finally, (3.12) is given by combining (3.17) and (3.19).

(3) Note that the strictly monotonicity of Φ'_α implies this injectivity. Moreover, Φ'_α is a coercive operator. Indeed, since $p^- - 1 > 0$, for each $u \in X$ such that $\|u\| \geq 1$ we have

$$\frac{\langle \Phi'_\alpha(u), u \rangle}{\|u\|_\alpha} = \frac{J_\alpha (u)}{\|u\|_\alpha} \geq \|u\|_\alpha^{p^- - 1} \to \infty \quad \text{as} \quad \|u\|_\alpha \to \infty.$$

Consequently, thanks to a Minty-Browder theorem [16], the operator Φ'_α is an surjection and admits an inverse mapping. It suffices then to show the continuity of $(\Phi'_\alpha)^{-1}$. Let $(f_n)_n$ be a sequence of X' such that $f_n \to f$ in X'. Let u_n and u in X such that

$$(\Phi'_\alpha)^{-1} (f_n) = u_n \quad \text{and} \quad (\Phi'_\alpha)^{-1} (f) = u.$$

By the coercivity of Φ'_α, one deducts that the sequence (u_n) is bounded in the reflexive space X. For a subsequence, we have $u_n \rightharpoonup \widehat{u}$ in X, this implies that

$$\lim_{n \to +\infty} \langle \Phi'_\alpha(u_n) - \Phi'_\alpha(u), u_n - \widehat{u} \rangle = \lim_{n \to +\infty} \langle f_n - f, u_n - \widehat{u} \rangle = 0.$$

It follows by the second assertion and the continuity of Φ'_α that

$$u_n \to \widehat{u} \quad \text{in} \quad X \ \text{ and } \ \Phi'_\alpha(u_n) \to \Phi'_\alpha(\widehat{u}) = \Phi'_\alpha(u) \quad \text{in} \quad X'.$$

Moreover, since Φ'_α is an injection, we conclude that $u = \widehat{u}$. This ends the proof. $\square$

4. Existence Results

Proof: [Proof of theorem 1.1]
The function Φ_α is continuous and convex, so it is weakly lower semicontinuous. From the two theorems (theorem 3.4 and theorem 3.6), the functional Φ_α is continuous Gâteaux differentiable and sequentially weakly lower semicontinuous whose Gâteaux derivative admits a continuous inverse on X^*, and Ψ is a continuous Gâteaux differentiable functional whose Gâteaux derivative is compact. In what follows, we will prove that the conditions of lemma 2.1 are satisfied.

(1) Let $u \in X$ such that $\|u\|_\alpha \geq 1$. We have

$$\begin{aligned}
\Psi(u) &= -\int_\Omega F(x, u(x))\, dx = -\int_\Omega \left[\int_0^{u(x)} f(x,t)\, dt\right] dx \\
&\leq \int_{u(x)\geq 0} \left[\int_0^{u(x)} |f(x,t)|\, dt\right] dx + \int_{u(x)<0} \left[\int_{u(x)}^0 |f(x,t)|\, dt\right] dx \\
&\leq \int_\Omega \left[a(x)\,|u(x)| + \frac{b}{\gamma(x)}\,|u(x)|^{\gamma(x)}\right] dx \\
&\leq 2\,|a|_{L^{\frac{\gamma(x)}{\gamma(x)-1}}(\Omega)}\,|u|_{L^{\gamma(x)}(\Omega)} + \frac{b}{\gamma^-}\int_\Omega |u(x)|^{\gamma(x)}\, dx.
\end{aligned}$$

By the proposition 3.3, we have $u \in L^{\gamma(x)}(\Omega)$ and there exists $C > 0$ such that

$$|u|_{L^{\gamma(x)}(\Omega)} \leq C\,\|u\|_\alpha\,.$$

Moreover, it's clear that

$$\begin{aligned}
\int_\Omega |u(x)|^{\gamma(x)}\, dx &\leq \max\left\{|u|_{\gamma(x)}^{\gamma^-}, |u|_{\gamma(x)}^{\gamma^+}\right\} \\
&\leq C'\,\|u\|_\alpha^{\gamma^+},
\end{aligned}$$

so

$$|\Psi(u)| \leq 2C\,|a|_{L^{\frac{\gamma(x)}{\gamma(x)-1}}(\Omega)}\,\|u\|_\alpha + \frac{b}{\gamma^-}C'\,\|u\|_\alpha^{\gamma^+}\,. \tag{4.1}$$

Thanks to the proposition 3.1, we have

$$\begin{aligned}
\Phi_\alpha(u) &= \int_\Omega \frac{1}{p(x)}\left(|\Delta u(x)|^{p(x)} + \alpha\,|u(x)|^{p(x)}\right) dx \\
&\geq \frac{\|u\|_\alpha^{p^-}}{p^+}. \qquad (4.2)
\end{aligned}$$

Then, for all $\beta > 0$, from (4.1) and (4.2) we obtain

$$\Phi_\alpha(u) + \beta\Psi(u) \geq \frac{\|u\|_\alpha^{p^-}}{p^+} - 2\beta C\,|a|_{L^{\frac{\gamma(x)}{\gamma(x)-1}}(\Omega)}\,\|u\|_\alpha - \frac{\beta C' b}{\gamma^-}\,\|u\|_\alpha^{\gamma^+}\,.$$

Since $p^- > \gamma^+ > 1$, the right term of the inequality goes to $+\infty$ as $\|u\|_\alpha \to +\infty$. So,

$$\lim_{\|u\|_\alpha \to +\infty} \left[\Phi_\alpha(u) + \beta\Psi(u)\right] = +\infty,$$

for all $\alpha, \beta > 0$. Consequently, the assertion (1) of lemma 2.1 is satisfied. Let us prove the second assertion.

(2) Choose $u_0(x) = 0$ for all $x \in \Omega$. It is clear that $u_0 \in X$ and $\Phi_\alpha(u_0) = \Psi(u_0) = 0$. From the condition (f2), we can find a constant $M > \max\left\{1, \left(\frac{1}{\alpha|\Omega|}\right)^{\frac{1}{p^-}}\right\}$ such that $F(x, M) > 0$, it follows that

$$\Psi(M) = -\int_\Omega F(x, M)\,dx < 0,$$

this implies that

$$\frac{(\Phi_\alpha(M) - r)\Psi(u_0) + (r - \Phi_\alpha(u_0))\Psi(M)}{\Phi_\alpha(M) - \Phi_\alpha(u_0)} = \frac{r\Psi(M)}{\Phi_\alpha(M)} < 0.$$

Take for all $x \in \Omega$, $u_1(x) = M$ and $0 < r < \frac{1}{p^+}$, we have

$$\Phi_\alpha(M) = \int_\Omega \frac{\alpha}{p(x)} M^{p(x)} dx \geq \frac{\alpha}{p^+} M^{p^-} |\Omega| > \frac{1}{p^+},$$

it follows that

$$0 = \Phi_\alpha(u_0) < r < \frac{1}{p^+} < \Phi_\alpha(M).$$

(3) We must choose an $r > 0$ such that for all $u \in X$ satisfying $\Phi_\alpha(u) \leq r$, we have $\Psi(u) > 0$. This assert that

$$\inf_{u \in \Phi_\alpha^{-1}]-\infty, r]} \Psi(u) > 0 > \frac{r\Psi(M)}{\Phi_\alpha(M)}.$$

Let $u \in X$ such that $\Phi_\alpha(u) \leq r$. First, we give the following claim

Claim 1. *If $r < \frac{1}{p^+}$ then $\|u\|_\alpha \leq (p^+ r)^{\frac{1}{p^+}}$.*

Indeed, let us show that $\|u\|_\alpha < 1$. Suppose by contradiction that $\|u\|_\alpha \geq 1$. From the proposition 3.1, we have $J_\alpha(u) \geq 1$ and

$$\begin{aligned} \Phi_\alpha(u) &= \int_\Omega \frac{1}{p(x)} \left(|\Delta u(x)|^{p(x)} + \alpha |u(x)|^{p(x)}\right) dx \\ &\geq \frac{1}{p^+} J_\alpha(u) \geq \frac{1}{p^+}, \end{aligned}$$

this contradict the fact that $r < \frac{1}{p^+}$. Consequently, $\|u\|_\alpha < 1$, then

$$\Phi_\alpha(u) \geq \frac{1}{p^+} J_\alpha(u) \geq \frac{1}{p^+} \|u\|_\alpha^{p^+}.$$

Since $\Phi_\alpha(u) \leq r$, it follows that $\|u\|_\alpha \leq (p^+ r)^{\frac{1}{p^+}}$. By the second assertion of proposition 3.3 and the fact $p(x) > \frac{N}{2}$, there exists $C_3 > 0$ such that

$$|v| \leq C_3 \|v\|_\alpha \quad \text{for all} \quad v \in X$$

where $|v| = \max_{x \in \overline{\Omega}} |v(x)|$. Hence, we obtain

$$|u| \leq C_3 \left(p^+ r\right)^{\frac{1}{p^+}}. \tag{4.3}$$

On the other hand, the condition (f3) implies that there exists $\varepsilon > 0$ such that

$$\frac{f(x,t)}{t^\delta} \leq \frac{c}{2}, \quad \forall 0 < |t| < \varepsilon.$$

Integrating to respect t and since $\delta + 1$ is even, it follows that

$$F(x,t) \leq \frac{c}{2(\delta+1)} |t|^{\delta+1}, \quad \forall 0 < |t| < \varepsilon.$$

Therefore

$$\Psi(v) = -\int_\Omega F(x,v)\,dx \geq -\frac{c}{2(\delta+1)}\int_\Omega |v|^{\delta+1}\,dx$$

for all $v \in X$ satisfying $0 < |v| < \varepsilon$. From the proposition 3.3, it gives that $v \in L^{\delta+1}(\Omega)$. So

$$\Psi(v) \geq -\frac{c}{2(\delta+1)}|v|^{\delta+1}_{L^{\delta+1}(\Omega)} > 0 \quad \text{for all} \quad 0 < |v| < \varepsilon. \tag{4.4}$$

Then considering $r < \min\left\{\frac{1}{p^+}, \frac{1}{p^+}\left(\frac{\varepsilon}{2C_3}\right)^{p^+}\right\}$ in (4.3), one conclude that $|u| < \dfrac{\varepsilon}{2}$ and from (4.4), it follows that $\Psi(u) > 0$. Hence, the proof of theorem 3.4 is completed. □

Proof: [Proof of theorem 1.2]
The proof of this theorem is similar at the above ones. We start by the assertion (1) of lemma 2.1. Let $u \in X$, denote by $s = \|u\|_\alpha$ and $v = \dfrac{u}{s}$. It's clear that $\|v\|_\alpha = 1$ and $s \to +\infty$ as $\|u\|_\alpha \to +\infty$.

Replacing in Φ_α, we obtain

$$\begin{aligned}\Phi_\alpha(u) &\leq \frac{1}{p^-}\int_\Omega \left(|\Delta u(x)|^{p(x)} + \alpha |u(x)|^{p(x)}\right)dx\\ &\leq \frac{1}{p^-}\int_\Omega s^{p(x)}\left(|\Delta v(x)|^{p(x)} + \alpha |v(x)|^{p(x)}\right)dx.\end{aligned}$$

For $s > 1$ and using the relation (1) of proposition 3.1, we deduct that

$$\Phi_\alpha(u) \leq \frac{s^{p^+}}{p^-}J_\alpha(v) = \frac{s^{p^+}}{p^-}. \tag{4.5}$$

On the other hand, the condition (f4) implies that there exists $M > 1$ such that

$$\frac{f(x,t)}{t^\omega} > 1, \quad \text{for all} \quad |t| \geq M,$$

and

$$F(x,t) \geq \frac{1}{\omega+1}|t|^{\omega+1}, \quad \text{for all} \quad |t| \geq M. \tag{4.6}$$

It follows that

$$\begin{aligned}\Psi(u) &= -\int_\Omega F(x,u(x))\,dx = -\int_{|u|\leq M} F(x,u(x))\,dx - \int_{|u|>M} F(x,u(x))\,dx\\ &\leq -\int_{|u|\leq M} F(x,u(x))\,dx - \frac{1}{\omega+1}\int_{|u|>M}|u|^{\omega+1}\,dx\\ &\leq -\int_{|u|\leq M} F(x,u(x))\,dx + \frac{1}{\omega+1}\int_{|u|\leq M}|u|^{\omega+1}\,dx - \frac{1}{\omega+1}\int_\Omega |u|^{\omega+1}\,dx.\end{aligned}$$

setting $M' = -\int_{|u|\leq M} F(x,u(x))\,dx + \frac{1}{\omega+1}\int_{|u|\leq M}|u|^{\omega+1}\,dx$ which is bounded, we have

$$\Psi(u) \leq M' - \frac{1}{\omega+1}\int_\Omega |u|^{\omega+1}\,dx \leq M' - \frac{s^{\omega+1}}{\omega+1}\int_\Omega |v|^{\omega+1}\,dx. \tag{4.7}$$

From the inequalities (4.5) and (4.7), for each $\beta > 0$

$$\Phi_\alpha(u) + \beta\Psi(u) \leq \frac{s^{p^+}}{p^-} - \beta\frac{s^{\omega+1}}{\omega+1}\int_\Omega |v|^{\omega+1}\,dx + \beta M'. \tag{4.8}$$

We must show that $\int_\Omega |v|^{\omega+1}\,dx \neq 0$. Assume par contradiction, suppose that $\int_\Omega |v|^{\omega+1}\,dx = 0$, so we have $v(x) = 0$ for a.e $x \in \Omega$. From the proposition 3.3, $v \in C\left(\overline{\Omega}\right)$ and then $v(x) = 0$ for all $x \in \Omega$. This implies that $\|v\|_\alpha = 0$, this contradicts the fact that $\|v\|_\alpha = 1$.

Let us return to the inequality (4.8). Since $\omega + 1 > p^+$, we deduct that the term on the right goes to $-\infty$ as $s \to +\infty$. Consequently, we have

$$\lim_{\|u\|_\alpha \to +\infty} \left[\Phi_\alpha(u) + \beta\Psi(u)\right] = -\infty,$$

and the assertion (1) is proved.

Let us show the assertion (2) of lemma 2.1. For $M > 1$ found by the condition (f4), we get

$$\Phi_\alpha(M) = \int_\Omega \frac{\alpha}{p(x)} M^{p(x)} dx \geq \frac{\alpha}{p^+} M^{p^-} |\Omega|. \tag{4.9}$$

Then for $M > \max\left\{1, \left(\frac{1}{\alpha|\Omega|}\right)^{\frac{1}{p^-}}\right\}$ and $0 < r < \frac{1}{p^+}$, we obtain

$$\Phi_\alpha(M) \geq \frac{1}{p^+} > r > 0 = \Phi_\alpha(0).$$

Finally, for the third assertion, replacing t by M in (4.6) it follows

$$-\Psi(M) = \int F(x, M)\,dx \geq \frac{1}{\omega+1} |M|^{\omega+1} |\Omega| \tag{4.10}$$

and

$$\Phi_\alpha(M) = \int_\Omega \frac{\alpha}{p(x)} M^{p(x)} dx \leq \frac{\alpha}{p^-} M^{p^+} |\Omega|. \tag{4.11}$$

Combining (4.10) and (4.11), we have

$$-r\frac{\Psi(M)}{\Phi_\alpha(M)} \geq \frac{rp^-}{\alpha(\omega+1)} M^{\omega+1-p^+}. \tag{4.12}$$

Let $r = \frac{\varepsilon}{p^+}$ with $0 < \varepsilon < 1$. From the preceding claim, for all $u \in X$ such that $\Phi_\alpha(u) \leq r < \frac{1}{p^+}$ we get

$$\|u\|_\alpha \leq \varepsilon^{\frac{1}{p^+}}.$$

The same method used in the preceding proof give

$$\begin{aligned} |\Psi(u)| &\leq 2|a|_{L^{\frac{\gamma(x)}{\gamma(x)-1}}(\Omega)} |u|_{L^{\gamma(x)}(\Omega)} + \frac{b}{\gamma^-} |u|^{\gamma^-}_{L^{\gamma(x)}(\Omega)} \\ &\leq 2C|a|_{L^{\frac{\gamma(x)}{\gamma(x)-1}}(\Omega)} (\varepsilon)^{\frac{1}{p^+}} + \frac{C''b}{\gamma^-} (\varepsilon)^{\frac{\gamma^-}{p^+}}. \end{aligned} \tag{4.13}$$

Define the function g as

$$g(t) := 2C|a|_{L^{\frac{\gamma(x)}{\gamma(x)-1}}(\Omega)} (t)^{\frac{1}{p^+}} + \frac{C''b}{\gamma^-} (t)^{\frac{\gamma^-}{p^+}}.$$

In what follows, we interest to find an $0 < \varepsilon < 1$ such that

$$g(\varepsilon) < \frac{p^- M^{\omega+1-p^+}}{\alpha p^+(\omega+1)} \varepsilon.$$

Hence, from (4.12) and (4.13), we obtain

$$-\Psi(u) \leq g(\varepsilon) < \frac{p^- M^{\omega+1-p^+}}{\alpha p^+(\omega+1)} \varepsilon \leq -r\frac{\Psi(M)}{\Phi_\alpha(M)},$$

for all $u \in X$ satisfying $\Phi_\alpha(u) \leq r = \frac{\varepsilon}{p^+}$. Which gives the assertion 3.

Let us search this ε. Define the function h as

$$h(t) := g(t) - \frac{p^- M^{\omega+1-p^+}}{\alpha p^+(\omega+1)} t.$$

This function is continuous on $[0,1]$ such that $h(0) = 0$, $h'(0) = +\infty$ and for $M > \left[\frac{p^+(\omega+1)}{p^-} g(1)\right]^{\frac{1}{\omega+1-p^+}}$ we have $h(1) < 0$. Then, there exists $\varepsilon \in]0,1[$ such that $h(\varepsilon) < 0$. This ends the proof of theorem 1.2. $\square$

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