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On the Numerical Solutions for Nonlinear Volterra-Fredholm Integral Equations

Parviz Darania and Saeed Pishbin

ABSTRACT: In this note, we study a class of multistep collocation methods for the numerical integration of nonlinear Volterra-Fredholm Integral Equations (V-FIEs). The derived method is characterized by a lower triangular or diagonal coefficient matrix of the nonlinear system for the computation of the stages which, as it is known, can be exploited to get an efficient implementation. Convergence analysis and linear stability estimates are investigated. Finally numerical experiments are given, which confirm our theoretical results.

Key Words: Volterra-Fredholm integral equations, Multistep collocation method, Convergence analysis, Stability properties.

Contents

1. Introduction

This paper concerns the construction of both efficient and stable numerical method for nonlinear V-FIEs of the form

$$y(t) = f(t) + \lambda_1(\mathcal{V}y)(t) + \lambda_2(\mathcal{F}y)(t),\ t \in I := [t_0, T], \tag{1.1}$$

where

$$(\mathcal{V}y)(t) = \int_{t_0}^{t} k_1(t, s, y(s))ds,\ (\mathcal{F}y)(t) = \int_{t_0}^{T} k_2(t, s, y(s))ds,$$

and $y(t)$ is the unknown function to be determined, $f \in C(I)$ be a given function and $\lambda_i, i = 1, 2$ denotes real or complex parameters and $k_1 \in C(D \times \mathbb{R}),\ k_2 \in C(I \times I \times \mathbb{R})$ and $D = \{(t, s) :\ t_0 \le s \le t \le T\}$ ([3]).

The V-FIEs [7,14] arise from parabolic boundary value problems, from the mathematical modelling of the spatio-temporal development of an epidemic, and from various physical and biological models. In [1] the authors, analyze the linearization methods for V-FIEs under some verifiable conditions on the kernels and nonlinear functions. In [2], we study a class of collocation methods to the numerical solution of the nonlinear V-FIEs (1.1), such that for this class, we had uniform order m for any choice of collocation parameters. In [6], we developed the Taylor's series method for solving the Volterra-Fredholm integro-differential equations. Some numerical methods have been proposed to solution of V-FIEs; see, e.g., [8,9,11,12,15,16].

In this paper, we are interested in deriving higher order method with extensive stability region for solving V-FIEs. Next sections of this paper are organized as follows: In section 2, we review basic materials of the multistep collocation method and obtain existence and uniqueness results. The collocation

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method and construction of multistep collocation method are described in section 3 and in section 4, the convergence order of this method is determined and the paper is closed in section 5, by showing efficiency of the method on some numerical examples.

2. Preliminaries

For convenience of the reader, we will present a review of the multistep collocation method from [5].

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of $[t_0, T]$, such that $t_n = nh, n = 0, \ldots, N$ and let $\Omega_N := \{0 = t_0 < t_1 < \cdots < t_N = T\}$, $\sigma_0 := [t_0, t_1]$, $\sigma_n := (t_n, t_{n+1}]$ $(1 \le n \le N-1)$. With a given mesh Ω_N, we associate the set of its interior points, $Z_N := \{t_n : n = 1, \ldots, N-1\}$. For a fixed $N \ge 1$ and, for given $d = -1$ and $m \ge 1$, the piecewise polynomial space $S^{(-1)}_{m-1}(Z_N)$ is defined by

$$S^{(-1)}_{m-1}(Z_N) := \{u : \ u|_{\sigma_n} = u_h \in \pi_{m-1}, \ 0 \le n \le N-1\},$$

where π_{m-1} denotes the set of (real) polynomials of a degree not exceeding $m-1$.

Consider the set of collocation parameters $\{c_j\}_{j=1}^m$, where $0 \le c_1 < \cdots < c_m \le 1$, and define the set $X_N = \{t_{n,j} = t_n + c_j h, n = 0, 1, \ldots, N-1, j = 1, 2, \ldots, m\}$ of collocation points. The multistep collocation methods are obtained by introducing in the collocation polynomial the dependence from r previous time steps $y_{n-k}, k = 0, 1, \ldots, r-1$; namely we seek for a collocation polynomial u, whose restriction to the interval $[t_n, t_{n+1}]$ takes the form

$$u_h(t_n + sh) = \sum_{k=0}^{r-1} \varphi_k(s) y_{n-k} + \sum_{j=1}^{m} \psi_j(s) U_{n,j}, \ s \in [0,1], \ n = r, \ldots, N-1, \tag{2.1}$$

where $U_{n,j} = u_h(t_{n,j})$, and

$$\varphi_k(s) = \prod_{i=1}^{m} \frac{s - c_i}{-k - c_i} \cdot \prod_{\substack{i=0 \\ i \ne k}}^{r-1} \frac{s+i}{-k+i}, \qquad \psi_j(s) = \prod_{i=0}^{r-1} \frac{s+i}{c_j + i} \cdot \prod_{\substack{i=1 \\ i \ne j}}^{m} \frac{s - c_i}{c_j - c_i}. \tag{2.2}$$

More details can be found in [5].

3. Collocation and multistep collocation methods

3.1. Collocation method

We recall construction of the collocation method for V-FIEs from [2].

In order to approximate solution of (1.1) in $[t_n, t_{n+1}]$, we rewrite equation (1.1) in the form

$$\begin{aligned} u_h(t_{n,j}) = \quad & f(t_{n,j}) + \lambda_1 h \sum_{i=0}^{n-1} \int_0^1 k_1(t_{n,j}, t_i + sh, u_h(t_i + sh)) ds \\ & + \lambda_1 h \int_0^{c_j} k_1(t_{n,j}, t_n + sh, u_h(t_n + sh)) ds \\ & \lambda_2 h \sum_{i=0}^{N-1} \int_0^1 k_2(t_{n,j}, t_i + sh, u_h(t_i + sh)) ds. \end{aligned} \tag{3.1}$$

Where, we remember that the fixed collocation parameters are $0 \le c_1 < \cdots < c_m \le 1$ and collocation points are $t_{n,j} = t_n + c_j h, j = 1, \ldots, m$, the collocation polynomials restricted to subinterval $[t_n, t_{n+1}]$ are defined by

$$u_h(t_n + sh) = \sum_{j=1}^{m} L_j(s) U_{n,j}, \ s \in [0,1], \ n = 0, 1, ..., N-1, \tag{3.2}$$

where $U_{n,j} := u_h(t_{n,j})$ and $L_j(s)$ is the j-th Lagrange polynomial at the collocation points.

The following theorem states the convergence order of collocation method.

Theorem 3.1. *Let $\varepsilon(t) = y(t) - u_h(t)$ be error of the exact collocation method and assume that the given functions in*

$$y(t) = f(t) + \lambda_1 \int_0^t k_1(t,s)y(s)ds + \lambda_2 \int_0^T k_2(t,s)y(s)ds, \ t \in I := [0,T], \tag{3.3}$$

satisfy $f \in C^m$, $k_1 \in C^m(D)$, $k_2 \in C^m(I \times I)$. Then for all sufficiently small $h = \frac{T}{N}$ the constrained mesh collocation solution $u_h \in S_{m-1}^{(-1)}(Z_N)$ to (3.3), for all $n = 0, 1, \ldots, N-1$, satisfies

$$\|\varepsilon(t)\|_\infty = O(h^m).$$

Proof: For proof see [2]. □

3.2. Multistep collocation method

In this subsection, we describe construction of the multistep collocation method. Such methods compute the approximated solution of (1.1) in $[t_n, t_{n+1}]$ by using the approximated values of the solution in the r previous steps $y_{n-k}, k = 0, 1, \ldots, r-1$ and m collocation points in the subinterval $[t_n, t_{n+1}]$.

Let $u_h = u|_{\sigma_n}, \ u \in S_{m-1}^{(-1)}(Z_N)$, for all $t \in \sigma_n$, we have

$$u_h(t_n + sh) = \sum_{k=0}^{r-1} \varphi_k(s) y_{n-k} + \sum_{j=1}^{m} \psi_j(s) U_{n,j}, \ s \in [0,1], \ n = r, r+1, \ldots, N-1, \tag{3.4}$$

where $U_{n,j} = u_h(t_{n,j})$.

The collocation solution $u_h \in S_{m-1}^{(-1)}(Z_N)$ will be determined by imposing the condition that u_h satisfies the integral equation (1.1) on the finite set X_N

$$u_h(t) = f(t) + \lambda_1 \int_0^t k_1(t,s,u_h(s))ds + \lambda_2 \int_0^T k_2(t,s,u_h(s))ds. \tag{3.5}$$

After some computations, the exact multistep collocation method is obtained by collocating both sides of (3.5) at the points $t = t_{n,i}$ for $i = 1, 2, ..., m$ and computing $y_{n+1} = u_h(t_{n+1})$:

$$\begin{cases} U_{n,i} = F_{n,i} + \Phi_{n,i}, & i = 1, 2, \ldots, m, \\ y_{n+1} = \displaystyle\sum_{k=0}^{r-1} \varphi_k(1) y_{n-k} + \sum_{j=1}^{m} \psi_j(1) U_{n,j}, & n = r, r+1, \ldots, N-1, \end{cases} \tag{3.6}$$

where the lag-term $F_{n,i}$ and increment-term $\Phi_{n,i}$ are given by

$$\begin{aligned} F_{n,i} = \quad & f(t_{n,i}) + \lambda_1 h \sum_{q=0}^{n-1} \int_0^1 k_1(t_{n,i}, t_q + sh, u_h(t_q + sh))ds \\ & + \lambda_2 h \sum_{q=0}^{n-1} \int_0^1 k_2(t_{n,i}, t_q + sh, u_h(t_q + sh))ds, \\ \Phi_{n,i} = \quad & \lambda_1 h \int_0^{c_i} k_1(t_{n,i}, t_n + sh, u_h(t_n + sh))ds \\ & + \lambda_2 h \sum_{q=n}^{N-1} \int_0^1 k_2(t_{n,i}, t_q + sh, u_h(t_q + sh))ds. \end{aligned} \tag{3.7}$$

Generally, the integrals in lag-term $F_{n,i}$ and increment-term $\Phi_{n,i}$ cannot be evaluated analytically, but have to be approximate by suitable quadrature formulae.

4. Convergence

In this section, we will analyze the order of convergence of the multistep collocation method (3.6). In our convergence analysis, we examine the linear test equation

$$y(t) = f(t) + \lambda_1 \int_0^t k_1(t,s)y(s)ds + \lambda_2 \int_0^T k_2(t,s)y(s)ds, \ t \in I := [0,T], \tag{4.1}$$

with $y \in \mathbb{R}, \ k_1 \in C^{(m+r)}(D), \ k_2 \in C^{(m+r)}(I \times I), \ f \in C^{(m+r)}(I)$. We assume that, λ_2^{-1} is not in the spectrum $\sigma(\mathcal{F})$ of the Fredholm integral operator $\mathcal{F}$ and for any choice of distinct collocation abscissac $0 < c_1 < \cdots < c_m \le 1$, the exact solution $y(t)$ of the V-FIEs (4.1) satisfies

$$y(t_n + sh) = \sum_{k=0}^{r-1} \varphi_k(s) y(t_{n-k}) + \sum_{j=1}^{m} \psi_j(s) y(t_{n,j}) + h^{m+r} R_{m,r,n}(s), \ s \in [0,1], \tag{4.2}$$

where, the functions $\varphi_k(s)$ and $\psi_j(s)$ are given by (2.2) and

$$R_{m,r,n}(s) := \int_{1-r}^{1} K_{m,r}(s,z) y^{(m+r)}(t_n + zh) dz, \tag{4.3}$$

and

$$\begin{aligned} K_{m,r}(s,z) = \ & \tfrac{1}{(m+r-1)!} \Bigg\{ (s-z)_+^{m+r-1} - \sum_{k=0}^{r-1} \varphi_k(s)(-k-z)_+^{m+r-1} \\ & - \sum_{j=1}^{m} \psi_j(s)(c_j - z)_+^{m+r-1} \Bigg\}, \qquad z \in [0,1]. \end{aligned} \tag{4.4}$$

Theorem 4.1. *Let $\varepsilon(t) = y(t) - u_h(t)$ be the error of exact multistep collocation method and $p = m + r$. Suppose that*

1. *the given function in (4.1) satisfy $f \in C^p(I), \ k_1 \in C^p(D), \ k_2 \in C^p(I \times I)$,*
2. *the starting errors are $||\varepsilon||_{\infty,[0,t_r]} = O(h^p)$,*
3. *$\rho(\mathbf{A}) < 1$, for*

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{0}_{(r-1)\times 1} & \mathbf{I}_{r-1} \\ \hline \varphi_{r-1}(1) & \varphi_{r-2}(1), \ldots, \varphi_0(1) \end{array} \right], \tag{4.5}$$

where ρ denotes the spectral radius.

Then

$$||\varepsilon||_\infty = O(h^p).$$

Proof: By subtracting (3.4) from (4.2), the error of exact multistep collocation method, $\varepsilon(t)$, takes the local representation

$$\varepsilon(t_n + sh) = \sum_{k=0}^{r-1} \varphi_k(s) \varepsilon_{n-k} + \sum_{j=1}^{m} \psi_j(s) \varepsilon_{n,j} + h^{m+r} R_{m,r,n}(s), \ s \in [0,1], \tag{4.6}$$

with $n \ge r, \varepsilon_{n-k} = \varepsilon(t_{n-k}), \varepsilon_{n,j} = \varepsilon(t_{n,j})$.

On the other hand, by evaluating (3.5) and (4.1) for $t = t_{n,i}$, we have

$$\begin{aligned} y(t_{n,i}) = \ & f(t_{n,i}) + \lambda_1 h \sum_{l=0}^{n-1} \int_0^1 k_1(t_{n,i}, t_l + sh) y(t_l + sh) ds \\ & + \lambda_1 h \int_0^{c_i} k_1(t_{n,i}, t_n + sh) y(t_n + sh) ds \\ & + \lambda_2 h \sum_{l=0}^{n-1} \int_0^1 k_2(t_{n,i}, t_l + sh) y(t_l + sh) ds \\ & + \lambda_2 h \sum_{l=n}^{N-1} \int_0^1 k_2(t_{n,i}, t_l + sh) y(t_l + sh) ds, \end{aligned} \tag{4.7}$$

$$
\begin{aligned}
u_h(t_{n,i}) = \quad & f(t_{n,i}) + \lambda_1 h \sum_{l=0}^{n-1} \int_0^1 k_1(t_{n,i}, t_l + sh) u_h(t_l + sh) ds \\
& + \lambda_1 h \int_0^{c_i} k_1(t_{n,i}, t_n + sh) u_h(t_n + sh) ds \\
& + \lambda_2 h \sum_{l=0}^{n-1} \int_0^1 k_2(t_{n,i}, t_l + sh) u_h(t_l + sh) ds \\
& + \lambda_2 h \sum_{l=n}^{N-1} \int_0^1 k_2(t_{n,i}, t_l + sh) u_h(t_l + sh) ds.
\end{aligned} \tag{4.8}
$$

By subtracting (4.8) from (4.7), we obtain

$$
\begin{aligned}
\varepsilon_{n,i} = \quad & \lambda_1 h \sum_{l=0}^{n-1} \int_0^1 k_1(t_{n,i}, t_l + sh) \varepsilon(t_l + sh) ds \\
& + \lambda_1 h \int_0^{c_i} k_1(t_{n,i}, t_n + sh) \varepsilon(t_n + sh) ds \\
& + \lambda_2 h \sum_{l=0}^{n-1} \int_0^1 k_2(t_{n,i}, t_l + sh) \varepsilon(t_l + sh) ds \\
& + \lambda_2 h \sum_{l=n}^{N-1} \int_0^1 k_2(t_{n,i}, t_l + sh) \varepsilon(t_l + sh) ds.
\end{aligned} \tag{4.9}
$$

Now, by the hypothesis on the starting errors, it follows that

$$
\varepsilon(t_l + sh) = h^p q_l(s), \; l = 0, 1, \ldots, r-1, \tag{4.10}
$$

with $||q_l||_\infty \leq C_1$ independent of h.

Now, by substituting (4.6) and (4.10) in (4.9), we have

$$
\begin{aligned}
\varepsilon_{n,i} = \quad & \lambda_1 h^{p+1} \sum_{l=0}^{n} (\rho_{1n}^{(l)})_i + \lambda_2 h^{p+1} \sum_{l=0}^{N-1} (\rho_{2n}^{(l)})_i + \lambda_1 h \sum_{l=r}^{n} (\sum_{k=0}^{r-1} \varepsilon_{l-k} (B_{1n}^{(l)})_{ik}) \\
& + \lambda_2 h \sum_{l=r}^{N-1} (\sum_{k=0}^{r-1} \varepsilon_{l-k} (B_{2n}^{(l)})_{ik}) + \lambda_1 h \sum_{l=r}^{n-1} (\sum_{j=1}^{m} \varepsilon_{l,j} (D_{1n}^{(l)})_{ij}) \\
& + \lambda_1 h \sum_{j=1}^{m} \varepsilon_{n,j} (D_{1n})_{ij} + \lambda_2 h \sum_{l=r}^{N-1} (\sum_{j=1}^{m} \varepsilon_{l,j} (D_{2n}^{(l)})_{ij}),
\end{aligned} \tag{4.11}
$$

where the vectors$\rho_{1n}^{(l)} \in \mathbb{R}^m$, $\rho_{2n}^{(l)} \in \mathbb{R}^m$, and the matrices $B_{1n}^{(l)} \in \mathbb{R}^{m\times r}$, $B_{2n}^{(l)} \in \mathbb{R}^{m\times r}$, $D_{1n}^{(l)} \in \mathbb{R}^{m\times m}$, $D_{1n}^{(l)} \in \mathbb{R}^{m\times m}$ and $D_{1n} \in \mathbb{R}^{m\times m}$ are defined as

$$
(\boldsymbol{\rho}_{1n}^{(l)})_i = \begin{cases}
\displaystyle\int_0^1 k_1(t_{n,i}, t_l + sh) q_l(s) ds, & l = 0, ..., r-1, \\
\displaystyle\int_0^1 k_1(t_{n,i}, t_l + sh) R_{m,r,l}(s) ds, & l = r, ..., n-1, \\
\displaystyle\int_0^{c_i} k_1(t_{n,i}, t_n + sh) R_{m,r,n}(s) ds, & l = n, \\
0, & l = n+1, ..., N-1,
\end{cases} \tag{4.12}
$$

$$
(\boldsymbol{\rho}_{2n}^{(l)})_i = \begin{cases}
\displaystyle\int_0^1 k_2(t_{n,i}, t_l + sh) q_l(s) ds, & l = 0, \ldots, r-1, \\
\displaystyle\int_0^1 k_2(t_{n,i}, t_l + sh) R_{m,r,l}(s) ds, & l = r, \ldots, N-1,
\end{cases} \tag{4.13}
$$

$$(\mathbf{B}_{1n}^{(l)})_{ik} = \begin{cases} \int_0^1 k_1(t_{n,i}, t_l + sh)\varphi_k(s)ds, & l = r, \dots, n-1, \\ \\ \int_0^{c_i} k_1(t_{n,i}, t_n + sh)\varphi_k(s)ds, & l = n, \end{cases} \tag{4.14}$$

$$(\mathbf{B}_{2n}^{(l)})_{ik} = \int_0^1 k_2(t_{n,i}, t_l + sh)\varphi_k(s)ds, \ l = r, \dots, N-1, \tag{4.15}$$

$$(\mathbf{D}_{1n}^{(l)})_{ij} = \int_0^1 k_1(t_{n,i}, t_l + sh)\psi_j(s)ds, \ l = r, ..., n-1, \tag{4.16}$$

$$(\mathbf{D}_{2n}^{(l)})_{ij} = \int_0^1 k_2(t_{n,i}, t_l + sh)\psi_j(s)ds, \ l = r, ..., N-1, \tag{4.17}$$

$$(\mathbf{D}_{1n})_{ij} = \int_0^{c_i} k_1(t_{n,i}, t_n + sh)\psi_j(s)ds, \tag{4.18}$$

From (4.11), we obtain

$$\begin{aligned} \varepsilon_n^{(2)} = \quad & \lambda_1 h^{p+1} \sum_{l=0}^{n} \boldsymbol{\rho}_{1n}^{(l)} + \lambda_2 h^{p+1} \sum_{l=0}^{N-1} \boldsymbol{\rho}_{2n}^{(l)} + \lambda_1 h \sum_{l=r}^{n} \mathbf{B}_{1n}^{(l)} \varepsilon_l^{(1)} \\ & + \lambda_2 h \sum_{l=r}^{N-1} \mathbf{B}_{2n}^{(l)} \varepsilon_l^{(1)} + \lambda_1 h \sum_{l=r}^{n-1} \mathbf{D}_{1n}^{(l)} \varepsilon_l^{(2)} + \lambda_1 h \mathbf{D}_{1n} \varepsilon_n^{(2)} \\ & + \lambda_2 h \sum_{l=r}^{N-1} \mathbf{D}_{2n}^{(l)} \varepsilon_l^{(2)}, \end{aligned} \tag{4.19}$$

where

$$\varepsilon_l^{(1)} = [\varepsilon_{l-r+1}, \dots, \varepsilon_l]^T, \tag{4.20}$$

$$\varepsilon_l^{(2)} = [\varepsilon_{l,1}, \dots, \varepsilon_{l,m}]^T. \tag{4.21}$$

From this equation and after some computations, we have

$$\mathbf{H}_n \mathbf{X}_{n,N-1} = h^{p+1} \sum_{l=0}^{N-1} \boldsymbol{\rho}_n^{*(l)} + 2h \mathbf{S}_{r,n-1} \mathbf{X}_{r,n-1}, \tag{4.22}$$

where $\boldsymbol{\rho}_n^{*(l)}$ and the matrices $\mathbf{H}_n, \ \mathbf{S}_{r,n-1}, \mathbf{X}_{i,j}$ are defined as

$$\boldsymbol{\rho}_n^{*(l)} = \begin{cases} \lambda_1 \boldsymbol{\rho}_{1n}^{(l)} + \lambda_2 \boldsymbol{\rho}_{2n}^{(l)} & l = 0, 1, \dots, n, \\ \\ \lambda_2 \boldsymbol{\rho}_{2n}^{(l)} & l = n+1, \dots, N-1, \end{cases} \tag{4.23}$$

$$\mathbf{H}_n = \left[\begin{array}{c|c} \mathbf{H}_n^1 & \mathbf{H}_n^2 \\ \hline \mathbf{E}_n^1 & \mathbf{E}_n^2 \end{array} \right], \tag{4.24}$$

$$\mathbf{H}_n^1 = \left[\begin{array}{cccc} \mathbf{I} - 2\lambda_1 h \mathbf{B}_{1n}^{(n)} & 0 & \dots & 0 \end{array} \right], \tag{4.25}$$

$$\mathbf{H}_n^2 = \left[\begin{array}{cccc} \mathbf{I} - 2\lambda_1 h \mathbf{D}_{1n} & 0 & \dots & 0 \end{array} \right], \tag{4.26}$$

$$\mathbf{E}_n^1 = \left[\begin{array}{cccc} \mathbf{I} + 2\lambda_2 h \mathbf{B}_{2n}^{(n)} & 2\lambda_2 h \mathbf{B}_{2n}^{(n+1)} & \dots & 2\lambda_2 h \mathbf{B}_{2n}^{(N-1)} \end{array} \right], \tag{4.27}$$

$$\mathbf{E}_n^2 = \left[\begin{array}{cccc} \mathbf{I} - 2\lambda_2 h \mathbf{D}_{2n}^{(n)} & -2\lambda_2 h \mathbf{D}_{2n}^{(n+1)} & \dots & -2\lambda_2 h \mathbf{D}_{2n}^{(N-1)} \end{array} \right], \tag{4.28}$$

$$\mathbf{X}_{i,j} = \left[\begin{array}{cccc|cccc} \boldsymbol{\varepsilon}_i^{(1)} & \boldsymbol{\varepsilon}_{i+1}^{(1)} & \dots & \boldsymbol{\varepsilon}_{j-1}^{(1)} & \boldsymbol{\varepsilon}_i^{(2)} & \boldsymbol{\varepsilon}_{i+1}^{(2)} & \dots & \boldsymbol{\varepsilon}_{j-1}^{(2)} \end{array} \right]^T, \tag{4.29}$$

$$\mathbf{S}_{r,n-1} = \left[\begin{array}{cccc|cccc} \mathbf{T}_{Bn}^{(r)} & \mathbf{T}_{Bn}^{(r+1)} & \dots & \mathbf{T}_{Bn}^{(n-1)} & & & \mathbf{0} & \\ \hline & & \mathbf{0} & & \mathbf{T}_{Dn}^{(r)} & \mathbf{T}_{Dn}^{(r+1)} & \dots & \mathbf{T}_{Dn}^{(n-1)} \end{array}\right], \tag{4.30}$$

$$\begin{aligned} \mathbf{T}_{Bn}^{(l)} &= \lambda_1 \mathbf{B}_{1n}^{(l)} + \lambda_2 \mathbf{B}_{2n}^{(l)}, \\ \mathbf{T}_{Dn}^{(l)} &= \lambda_1 \mathbf{D}_{1n}^{(l)} + \lambda_2 \mathbf{D}_{2n}^{(l)}. \end{aligned} \tag{4.31}$$

Since the kernel k_i are continuous on their domains, the elements of the matrixes $\mathbf{B}_{1n}^{(l)}, l = r, r+1, \dots, N-1$, are all bounded. By using the Neumann Lemma the inverse of the matrix $\mathbf{H}_n^1$ exists whenever $h|\lambda_1| \parallel \mathbf{B}_{1n}^{(l)} \parallel < \frac{1}{2}$, for some matrix norm. This clearly holds whenever h is sufficiently small. In other words, there is an $\bar{h} > 0$ so that for any mesh Ω_N with $h < \bar{h}$, each matrix $\mathbf{H}_n$ has a uniformly bounded inverse. Note that, from this assumptions, there exists a constant $P_0 < \infty$ so that for all mesh diameters $h \in (0, \bar{h})$, the uniform bound $\parallel \mathbf{H}_n^{-1} \parallel_1 \leq R_1$, holds.

Now, by setting

$$||\mathbf{B}_{1n}^{(l)}||_1 \leq P_0, \quad ||\mathbf{B}_{2n}^{(l)}||_1 \leq P_1, \quad ||\mathbf{D}_{1n}^{(l)}||_1 \leq Q_0, \quad ||\mathbf{D}_{2n}^{(l)}||_1 \leq Q_1, \tag{4.32}$$

$$\begin{aligned} ||\mathbf{T}_{Bn}^{(l)}||_1 &= ||\lambda_1 \mathbf{B}_{1n}^{(l)} + \lambda_2 \mathbf{B}_{2n}^{(l)}||_1 \leq |\lambda_1| P_0 + |\lambda_2| P_1 \leq P_2, \\ ||\mathbf{T}_{Dn}^{(l)}||_1 &= ||\lambda_1 \mathbf{D}_{1n}^{(l)} + \lambda_2 \mathbf{D}_{2n}^{(l)}||_1 \leq |\lambda_1| Q_0 + |\lambda_2| Q_1 \leq Q_2, \end{aligned} \tag{4.33}$$

$$||\boldsymbol{\rho}_n^{*(l)}||_1 \leq R_0 \ = \begin{cases} |\lambda_1|\gamma^1 + |\lambda_2|\gamma^2 & l = 0, 1, \dots, r-1, \\ |\lambda_1|\alpha_{m,r}^1 + |\lambda_2|\alpha_{m,r}^2 & l = r, r+1, \dots, n, \\ |\lambda_2|\alpha_{m,r}^2 & l = n+1, \dots, N-1, \end{cases} \tag{4.34}$$

$$||\mathbf{S}_{r,n-1}||_1 \leq \max\{P_2, Q_2\} \leq P_3, \qquad \parallel y^{(m+r)} \parallel_1 \leq M_{m,r}, \tag{4.35}$$

$$K_{m,r} = \max_{s\in[0,1]} \int_{1-r}^{1} |K_{m,r}(s,t)| dt, \quad \bar{K}_i = \max_{t\in I} \int_0^t |k_i(t,s)| ds, \ i = 1, 2, \tag{4.36}$$

$$\parallel \boldsymbol{\rho}_{1n}^{(l)} \parallel_1 \leq \begin{cases} \gamma^1 = m\bar{K}_1 C_1, & l = 0, 1, \dots, r-1, \\ \alpha_{m,r}^1 = m\bar{K}_1 K_{m,r} M_{m,r}, & l = r, \dots, n, \\ 0, & l = n+1, \dots, N-1, \end{cases} \tag{4.37}$$

$$\parallel \boldsymbol{\rho}_{2n}^{(l)} \parallel_1 \leq \begin{cases} \gamma^2 = m\bar{K}_2 C_1, & l = 0, 1, \dots, r-1, \\ \alpha_{m,r}^2 = m\bar{K}_2 K_{m,r} M_{m,r}, & l = r, \dots, N-1. \end{cases} \tag{4.38}$$

Then, from (4.22), we have

$$||\mathbf{X}_{n,N-1}||_1 \leq h^p G_1 + h G_2 ||\mathbf{X}_{r,n-1}||_1, \tag{4.39}$$

with $G_1 = 2R_0R_1T$, $G_2 = 2R_1P_3$. The above generalized discrete Gronwall inequality leads to the estimate

$$||\mathbf{X}_{n,N-1}||_1 \leq Ch^p, \tag{4.40}$$

with $C = G_1 \exp(hG_2)$. From (4.6) and (4.10), this is equivalent to the estimate $\parallel \varepsilon \parallel_\infty = O(h^p)$. □

5. Presentation of results

In this section, two examples will be investigated to show the reliability and efficiency of the proposed numerical method. We choose $c_1 = 0.7$ and $c_2 = 1$ as collocation parameters. The observed orders of convergence are computed from the maximum errors at the grid points. The starting values have been obtained from the known exact solutions. All computations are performed by the Mathematica® software.

Example 5.1. *Consider the nonlinear V-FIEs as:*

$$y(t) = f(t) + \int_0^t 2\cos(t-s)y^2(s)ds + \int_0^1 2\sin(t-s)y^2(s)ds, \ \ t \in [0,1],$$

where $f(t)$ *such that the exact solution is* $y(t) = e^t$.

Table 1: Maximum errors $\|y - u_h\|_\infty$ for $r = 2, 3$ and $m = 2$ in Example 5.1.

N	$\|\|y-u_h\|\|_\infty$ r=1 m=2	$\|\|y-u_h\|\|_\infty$ r=2 m=2	$\|\|y-u_h\|\|_\infty$ r=3 m=2
4	3.38×10^{-4}	2.58×10^{-5}	3.17×10^{-6}
8	3.30×10^{-5}	1.24×10^{-6}	6.71×10^{-8}
16	3.52×10^{-6}	6.77×10^{-8}	1.86×10^{-9}
32	4.08×10^{-7}	3.94×10^{-9}	5.47×10^{-11}
64	4.90×10^{-8}	2.38×10^{-10}	1.66×10^{-12}

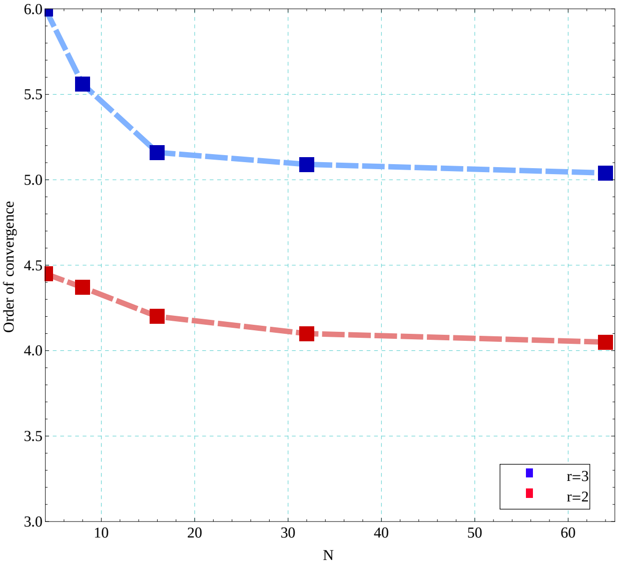

Figure 1: Orders of convergence of u_h for $r = 2, 3$ and $m = 2$ in Example 5.1.

The maximum errors have been shown for different values of r and N at the grid points in the Table 1. The orders of convergence of the multistep collocation method for $r = 2, 3$ and $m = 2$ are shown in Figure 1. which they confirm the theoretical results of the Theorem 4.1.

Example 5.2. *Consider the nonlinear V-FIEs as:*

$$y(t) = f(t) + \int_0^t (s+t+2)y^2(s)ds + \int_0^1 (s+t^3+1)y^3(s)ds, \;\; t \in [0,1],$$

where $f(t)$ such that the exact solution is $y(t) = \sin t$.

Table 2: Maximum errors $\|y - u_h\|_\infty$ for $r = 2, 3$ and $m = 2$ in Example 5.2

N	$\|\|y-u\|\|_\infty$ r=1 m=2	$\|\|y-u\|\|_\infty$ r=2 m=2	$\|\|y-u\|\|_\infty$ r=3 m=2
4	2.43×10^{-4}	1.22×10^{-5}	1.75×10^{-6}
8	2.90×10^{-5}	7.06×10^{-7}	6.26×10^{-8}
16	3.60×10^{-6}	4.36×10^{-8}	1.99×10^{-9}
32	4.49×10^{-7}	2.72×10^{-9}	6.20×10^{-11}
64	5.62×10^{-8}	1.70×10^{-10}	1.93×10^{-12}

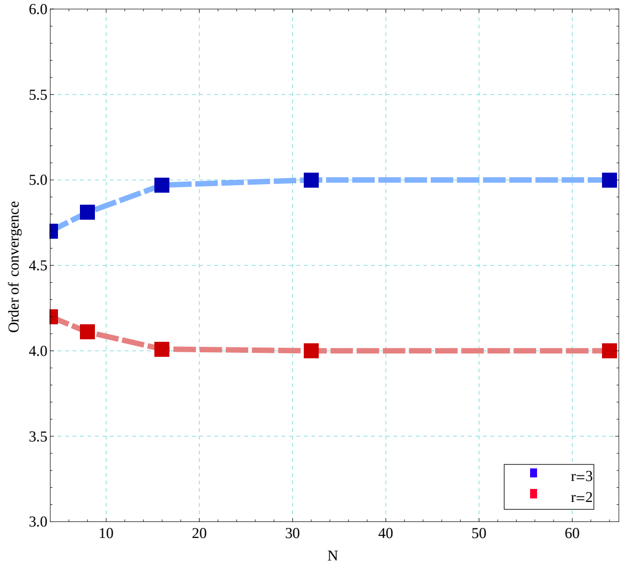

Figure 2: Orders of convergence of u_h for $r = 2, 3$ and $m = 2$ in Example 5.2.

The maximum errors have been shown for different values of r and N at the grid points in the Table 1. The orders of convergence of the multistep collocation method for $r = 2, 3$ and $m = 2$ are shown in Figure 2. which they confirm the theoretical results of the Theorem 4.1 (The order of convergence is $p = m + r$).

Example 5.3. *Consider the linear V-FIEs as:*

$$y(t) = f(t) + \int_0^t (t^2+s+2)y(s)ds + \int_0^1 (t^2+s^2+1)y(s)ds, \;\; t \in [0,1],$$

where $f(t)$ such that the exact solution is $y(t) = \arctan t$.

Table 3: Maximum errors $\|y - u_h\|_\infty$ for $r = 2, 3$ and $m = 2$ in Example 5.3

N	$\|\|y-u_h\|\|_\infty$ r=1 m=2	$\|\|y-u_h\|\|_\infty$ r=2 m=2	$\|\|y-u_h\|\|_\infty$ r=3 m=2
4	4.24×10^{-4}	6.96×10^{-5}	9.75×10^{-6}
8	4.61×10^{-5}	4.72×10^{-6}	1.84×10^{-7}
16	5.35×10^{-6}	2.95×10^{-7}	4.36×10^{-9}
32	6.43×10^{-7}	1.82×10^{-8}	2.62×10^{-10}
64	7.88×10^{-8}	1.13×10^{-9}	1.10×10^{-11}

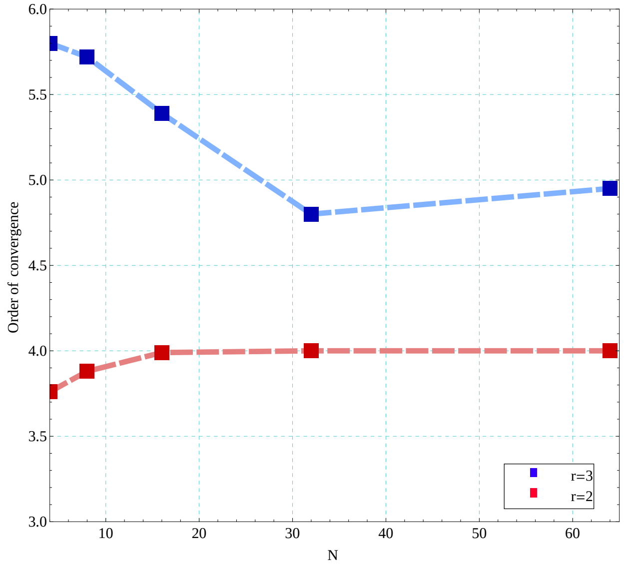

Figure 3: Orders of convergence of u for $r = 2, 3$ and $m = 2$ in Example 5.3.

The maximum errors have been shown for different values of r and N at the grid points in the Table 3. Also Figure 3. shows the orders of convergence of the numerical method for $r = 2, 3$ and $m = 2$. From this Figure, we can see that the numerical results are consistent with our theoretical analysis.

6. Conclusion

We have shown that the multistep collocation method yields an efficient and very accurate numerical method for the approximation of solutions to nonlinear V-FIEs. Numerical results show that this method is effective for nonlinear V-FIEs. Furthermore, the convergence, and stability properties of the multistep collocation method, it is more accurate than collocation method.

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Parviz Darania,
Department of Mathematics,
Faculty of Science, Urmia University, P.O.Box 165,
Urmia-Iran.
E-mail address: p.darania@urmia.ac.ir

and

Saeed Pishbin,
Department of Mathematics,
Faculty of Science, Urmia University, P.O.Box 165,
Urmia-Iran.
E-mail address: s.pishbin@urmia.ac.ir