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(Jordan) Derivation on Amalgamated Duplication of a Ring Along an Ideal

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Abstract: Let A be a ring and I be an ideal of A. The amalgamated duplication of A along I is the subring of $A \times A$ defined by $A \bowtie I := \{(a, a+i) \mid a \in A, i \in I\}$. In this paper, we characterize $A \bowtie I$ over which any (resp. minimal) prime ideal is invariant under any derivation provided that A is semiprime. When A is noncommutative prime, then $A \bowtie I$ is noncommutative semiprime (but not prime except if $I = (0)$). In this case, we prove that any map of $A \bowtie I$ which is both Jordan and Jordan triple derivation is a derivation.

Key Words: (Jordan) derivation, Prime and semiprime rings, Extension of a ring by an ideal.

Contents

1. Introduction

Throughout, A will represent an associative ring with center $Z(A)$. By an ideal I in A, we shall always mean a two-sided ideal of A. An ideal P of A is said to be prime if $P \neq A$ and, for $a, b \in A$, $aAb \subseteq P$ implies that $a \in P$ or $b \in P$. The ring A is called a prime ring if (0) is a prime ideal of A. A ring A is called a semiprime ring if $aAa = (0)$ implies $a = 0$. A ring A is said to be 2-torsion free, if whenever $2a = 0$, with $a \in A$, then $a = 0$. The Jordan product of two elements x and y of A is $x \circ y = xy + yx$. By a derivation of A, we mean an additive map $d : A \to A$ satisfying $d(xy) = d(x)y + xd(y)$ for all pairs $x, y \in A$. Given a derivation d of A, an ideal I of A is said to be invariant under d (or d-invariant for short) if $d(I) \subseteq I$. It is well known that every minimal prime ideal of a torsion-free semiprime ring is invariant under all derivations [11]. Herstein raised the following problem:

Problem. Given a semiprime ring A, does $d(P) \subseteq P$ hold for any minimal prime ideal P of A and for any derivation d of A?

This problem has been often mentioned in the literature (see, for example, [3,13]). The best result of the conjecture is the following: A ring A is said to be of bounded index m if m is a positive integer such that $x^m = 0$ for all nilpotent elements $x \in A$. Beidar and Mikhalev proved the theorem: Let A be a ring of bounded index m such that the additive order of every nonzero torsion element of A, if any, is strictly larger than m. Then all minimal prime ideals of A are invariant under all derivations of A (see [1] or [2, Theorem 8.16]). As a special case of this, every minimal prime ideal of a reduced ring is invariant under derivations of the ring (See [7, p. 614]). Unfortunately, this problem turns out to be false in general. Chuang and Lee [7] constructed a semiprime ring A which possesses a minimal prime ideal not invariant under a derivation of the ring.
Let A be a ring and I be an ideal of A. The subset of $A \times A$ defined by:

$$A \bowtie I := \{(a, a+i) \mid a \in A, i \in I\}$$

is clearly a subring of $A \times A$, called the amalgamated duplication of A along I. The construction $A \bowtie I$ (in the commutative case) was introduced and its basic properties were studied by D'Anna and

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Fontana (2007) in [9,10], and then it was investigated by D'Anna in [8] with the aim of applying it to curve singularities (over algebraic closed fields) where he proved that the amalgamated duplication of an algebroid curve along a regular canonical ideal yields a Gorenstein algebroid curve [8, Theorem 14 and Corollary 17]. The aim of Section 2 of this paper is to characterize when the amalgamated duplication of a semiprime ring along an ideal satisfies the Herstein's Problem. Hence, Theorem 2.6 states that if A is a semiprime ring and I is an ideal of A. Then, the following are equivalent:

1. $d(P) \subseteq P$ holds for any (resp. minimal) prime ideal P of $A \bowtie I$ and for any derivation d of $A \bowtie I$.
2. $\delta(p) \subseteq p$ holds for any (resp. minimal) prime ideal p of A and for any derivation δ of A keeping I invariant.

An additive map $d : A \to A$ is called a Jordan derivation if $d(x^2) = d(x)x + xd(x)$ for all $x \in A$, and d is called a Jordan triple derivation if $d(xyx) = d(x)yx + xd(y)x + xyd(x)$ for all $x, y \in A$. If A is 2-torsion-free, then every Jordan derivation is a Jordan triple derivation ([4, Proposition 2]). Obviously, every derivation is a Jordan (resp. triple) derivation. The converse is in general not true. In [5, Theorem 4.3], Brešar proved that if A is 2-torsion free semiprime then every Jordan triple derivation is a derivation. Which means that derivations, Jordan derivations, and Jordan triple derivations of a 2-torsion-free semiprime ring are the same. The case when the ring is of characteristic 2 is due to Herstein who proved (in [12, Theorem 4.1]) that over a noncommutative ring any map which is both Jordan derivation and Jordan triple derivation becomes a derivation. In Section 3, we extend the Herstein's result to semiprime rings with form $A \bowtie I$ where A is a prime noncommutative ring.

Let's adopt the following notations:

Notations. Let A be a ring and I be an ideal of A. By π_1 and π_2 we denote the naturel surjections of $A \bowtie I$ into A defined by

$$\pi_1(a, a+i) = a \quad \text{and} \quad \pi_2(a, a+i) = a + i \quad \text{for all } a \in A,\ i \in I.$$

For an additive map $d : A \bowtie I \to A \bowtie I$, we consider the maps $d_{i=1,2} : A \to A$ and $s_{i=1,2} : I \to A$ defined by

$$d_1(a) = \pi_1 \circ d(a, a),\ \ d_2(a) = \pi_2 \circ d(a, a),\ \ s_1(i) = \pi_1 \circ d(0, i),\ \ s_2(i) = \pi_2 \circ d(i, 0)$$

for all $a \in A$ and $i \in I$. It is clear that d_1, d_2, s_1, and s_2 are all additive.

2. Semiprime amalgamated duplication of a ring along an ideal with prime ideals invariant under derivations

In this section, we characterize the derivations of $A \bowtie I$, specially when A is a semiprime ring. Our aim is to see when every (minimal) prime ideal of $A \bowtie I$ is invariant under any derivation on $A \bowtie I$.

Let R and T be rings and let θ and ϕ be homomorphisms of T into R. Let X be an R-bimodule. Following [6], an additive mapping $d : T \to X$ is called a (θ, ϕ)-derivation (resp. a Jordan (θ, ϕ)-derivation) if $d(xy) = d(x)\phi(y) + \theta(x)d(y)$, for all $x, y \in T$ (resp. if $d(x^2) = d(x)\phi(x) + \theta(x)d(x)$, for all $x \in T$).
Suppose that $d : T \to T$ is a (resp. Jordan) derivation. Then, $\theta \circ d$ is a (resp. Jordan) (θ, θ)-derivation. Indeed, $\theta \circ d$ is clearly additive, and for all $x, y \in T$, we have

$$\theta \circ d(xy) = \theta(d(x)y + xd(y)) = \theta \circ d(x)\theta(y) + \theta(x)\theta \circ d(y)$$

$$\big(\text{resp.}\theta \circ d(x^2) = \theta(d(x)x + xd(x)) = \theta \circ d(x)\theta(x) + \theta(x)\theta \circ d(x)\big).$$

We start with the following lemma.

Lemma 2.1. *Let A be a ring and I be an ideal of A. A map $d : A \bowtie I \to A \bowtie I$ is a (resp. Jordan) derivation if and only if $\pi_1 \circ d$ is a (resp. Jordan) (π_1, π_1)-derivation and $\pi_2 \circ d$ is a (resp. Jordan) (π_2, π_2)-derivation.*

Proof: ($\Rightarrow$) Clear.
($\Leftarrow$) It is clear that, for all $a \in A$ and $i \in I$, we have

$$d(a, a+i) = \big(\pi_1 \circ d(a, a+i), \pi_2 \circ d(a, a+i)\big).$$

Hence, if $\pi_1 \circ d$ and $\pi_2 \circ d$ are additive then so is d.
Suppose that $\pi_1 \circ d$ is a (π_1, π_1)-derivation and $\pi_2 \circ d$ is a (π_2, π_2)-derivation. For all $a, b \in A$ and $i, j \in I$ we have

$$\begin{aligned} d\big((a, a+i)(b, b+j)\big) &= \big(\pi_1 \circ d\big((a, a+i)(b, b+j)\big), \pi_2 \circ d\big((a, a+i)(b, b+j)\big)\big) \\ &= \big(\pi_1 \circ d(a, a+i)\pi_1(b, b+j) + \pi_1(a, a+i)\pi_1 \circ d(b, b+j), \\ &\quad \pi_2 \circ d(a, a+i)\pi_2(b, b+j) + \pi_2(a, a+i)\pi_2 \circ d(b, b+j)\big) \\ &= \big(\pi_1 \circ d(a, a+i)b + a\pi_1 \circ d(b, b+j), \pi_2 \circ d(a, a+i)(b+j) \\ &\quad +(a+i)\pi_2 \circ d(b, b+j)\big) \\ &= \big(\pi_1 \circ d(a, a+i), \pi_2 \circ d(a, a+i)\big)(b, b+j) \\ &\quad +(a, a+i)\big(\pi_1 \circ d(b, b+j), \pi_2 \circ d(b, b+j)\big) \\ &= d(a, a+i)(b, b+j) + (a, a+i)d(b, b+j). \end{aligned}$$

Hence, d is a derivation.
By the same way, we show that if $\pi_1 \circ d$ is a Jordan (π_1, π_1)-derivation and $\pi_2 \circ d$ is a Jordan (π_2, π_2)-derivation then d is a Jordan derivation. □

The next result gives the necessary and sufficient conditions for an additive map d from $A \bowtie I$ into itself to be a derivation.

Proposition 2.2. *Let A be a ring, I be an ideal of A, and $d : A \bowtie I \to A \bowtie I$ be an additive map. Then, d is a derivation if and only if*

1. *d_1 and d_2 are derivations.*
2. *$s_k(ai) = as_k(i)$, $s_k(ia) = s_k(i)a$, and $s_k(ij) = 0$ for $k = 1, 2$ and for all $a \in A$ and $i, j \in I$.*

Proof: ($\Rightarrow$) From Lemma 2.1, $\pi_1 \circ d$ is a (π_1, π_1)-derivation and $\pi_2 \circ d$ is a (π_2, π_2)-derivation. Hence, for all $a \in A$, we have

$$\begin{aligned} d_1(ab) &= \pi_1 \circ d(ab, ab) \\ &= \pi_1 \circ d\big((a, a)(b, b)\big) \\ &= \pi_1 \circ d(a, a)b + a\pi_1 \circ d(b, b) \\ &= d_1(a)b + ad_1(b). \end{aligned}$$

Hence, d_1 is derivation. Similarly, we obtain that d_2 is a derivation.
Let $a \in A$ and $i \in I$. We have

$$\begin{aligned} s_1(ai) &= \pi_1 \circ d(0, ai) \\ &= \pi_1 \circ d\big((a, a)(0, i)\big) \\ &= \pi_1 \circ d(a, a)\pi_1(0, i) + \pi_1(a, a)\pi_1 \circ d(0, i) \\ &= as_1(i). \end{aligned}$$

Similarly, $s_1(ia) = s_1(i)a$. Now, for all $i, j \in I$, we have

$$\begin{aligned} s_1(ij) &= \pi_1 \circ d(0, ij) \\ &= \pi_1 \circ d\big((0, i)(0, j)\big) \\ &= \pi_1 \circ d(0, i)\pi_1(0, j) + \pi_1(0, i)\pi_1 \circ d(0, j) \\ &= 0. \end{aligned}$$

By the same argument, we prove that s_2 satisfies the same conditions.
($\Leftarrow$) For all $a \in A$ and $i \in I$ we have

$$\pi_1 \circ d(a, a+i) = \pi_1 \circ d(a, a) + \pi_1 \circ d(0, i) = d_1(a) + s_1(i).$$

and

$$\pi_2 \circ d(a, a+i) = \pi_2 \circ d(a+i, a+i) - \pi_2 \circ d(i, 0) = d_2(a+i) - s_2(i).$$

By Lemma 2.1, we have to prove that $\pi_1 \circ d$ is a (π_1, π_1)-derivation and $\pi_2 \circ d$ is a (π_2, π_2)-derivation. Let $a, b \in A$ and $i, j \in I$. We have

$$\begin{aligned}
\pi_1 \circ d\big((a, a+i)(b, b+j)\big) &= \pi_1 \circ d(ab, ab + aj + ib + ij) \\
&= d_1(ab) + s_1(aj + ib + ij) \\
&= d_1(a)b + ad_1(b) + as_1(j) + s_1(i)b \\
&= \big(d_1(a) + s_1(i)\big)b + a\big(d_1(b) + s_1(j)\big) \\
&= \pi_1 \circ d(a, a+i)\pi_1(b, b+j) \\
&\quad + \pi_1(a, a+i)\pi_1 \circ d(b, b+j)
\end{aligned}$$

and, since $0 = s_2(ij) = s_2(i)j = is_2(j)$, we get

$$\begin{aligned}
\pi_2 \circ d\big((a, a+i)(b, b+j)\big) &= \pi_2 \circ d(ab, ab + aj + ib + ij) \\
&= d_2\big((a+i)(b+j)\big) - s_2(aj + ib + ij) \\
&= d_2(a+i)(b+j) + (a+i)d_2(b+j) \\
&\quad - as_2(j) - s_2(i)b \\
&= \big(d_2(a+i) - s_2(i)\big)(b+j) \\
&\quad + (a+i)\big(d_2(b+j) - s_2(j)\big) \\
&= \pi_2 \circ d(a, a+i)\pi_2(b, b+j) \\
&\quad + \pi_2(a, a+i)\pi_2 \circ d(b, b+j)
\end{aligned}$$

Hence, we have the desired result. □

Next, we characterize the derivations of $A \bowtie I$ when A is a semiprime ring.

Proposition 2.3. *Let A be a semiprime ring, I be an ideal of A, and $d : A \bowtie I \to A \bowtie I$ be an additive map. Then, the following are equivalent:*

1. *d is a derivation*
2. *d_1 and d_2 are derivations and the ideals $0 \times I$ and $I \times 0$ of $A \bowtie I$ are d-invariant.*
3. *there exist a derivation $\delta_1 : A \to A$ keeping I invariant and a derivation $\delta_2 : A \to I$ such that*

$$d(a, a+i) = (\delta_1(a), \delta_1(a+i) + \delta_2(a+i)) \quad \textit{for all } a \in A,\ i \in I.$$

Proof: (1) $\Rightarrow$ (2) From Proposition 2.2, we have $s_k(ai) = as_k(i)$, $s_k(ia) = s_k(i)a$, and $s_k(ij) = 0$ for $k = 1, 2$ and for all $a \in A$ and $i, j \in I$. Then, for any $i \in I$, we have

$$s_k(i)as_k(i) = s_k(i)s_k(ai) = s_k\left(is_k(ai)\right) = s_k\left(s_k(iai)\right) = 0 \quad \text{for all } a \in A.$$

Thus, since A is semiprime, we have that $s_k(i) = 0$ for all $i \in I$. Hence, for all $i \in I$, we have $\pi_1 \circ d(0, i) = 0$ and $\pi_2 \circ d(i, 0) = 0$. So, $d(0, i) = (0, r) \in A \bowtie I$ and $d(i, 0) = (r', 0) \in A \bowtie I$. Consequently, $r, r' \in I$, $d(0, i) \in 0 \times I$, and $d(i, 0) \in I \times 0$.

(2) $\Rightarrow$ (3) Since $0 \times I$ and $I \times 0$ are d-invariant, we have clearly $s_1 = s_2 = 0$. Thus, for all $a \in A$ and $i \in I$, we have

$$d(a, a+i) = (d_1(a), d_2(a+i)).$$

Set $\delta_1 = d_1$ and $\delta_2 = d_2 - d_1$. For all $i \in I$, we have $\delta_1(i) = \pi_1 \circ d(i,i) = \pi_1 \circ d(i,0) + \pi_1 \circ d(0,i) = \pi_1 \circ d(i,0) \in I$. Hence, I is δ_1-invariant. Set $d(a,a) = (b, b+j)$ for some $b \in A$ and $j \in I$. We have

$$\delta_2(a) = d_2(a) - d_1(a) = \pi_2 \circ d(a,a) - \pi_1 \circ d(a,a) = (b+j) - b = j \in I.$$

Then, $\delta_2(A) \subseteq I$.
(3) $\Rightarrow$ (1) Suppose that

$$d(a, a+i) = (\delta_1(a), \delta_1(a+i) + \delta_2(a+i)) \quad \text{for all } a \in A,\ i \in I$$

with $\delta_1 : A \to A$ is a derivation keeping I invariant and $\delta_2 : A \to I$ is a derivation. Firstly, d is well defined. Indeed, for all $a \in A$ and $i \in I$, we have

$$(\delta_1(a+i) + \delta_2(a+i)) - \delta_1(a) = \delta_1(i) + \delta_2(a+i) \in I.$$

A simple check shows that such d is a derivation. □

We need the following lemmas.

Lemma 2.4. *Let p be a prime ideal of A. Then,*

$$p \bowtie I := \{(a, a+i) \mid a \in p,\ i \in I\}$$

and

$$\overline{p} := \{(a+i, a) \mid a \in p,\ i \in I\}$$

are prime ideals of $A \bowtie I$.

Proof: Clearly $p \bowtie I$ and $\overline{p}$ are ideals of $A \bowtie I$. Moreover, the mappings $\psi : \frac{A \bowtie I}{p \bowtie I} \to \frac{A}{p}$ and $\varphi : \frac{A \bowtie I}{\overline{p}} \to \frac{A}{p}$ defined by $\psi\left(\overline{(a,a+i)}\right) = \overline{a}$ and $\varphi\left(\overline{(a,a+i)}\right) = \overline{a+i}$ are a well defined isomorphisms of rings. Then, since p is prime, $\frac{A}{p}$ is a prime ring and so are $\frac{A \bowtie I}{p \bowtie I}$ and $\frac{A \bowtie I}{\overline{p}}$. Then, $p \bowtie I$ and $\overline{p}$ are prime ideals of $A \bowtie I$. □

Lemma 2.5. *Let P be a prime ideal of $A \bowtie I$. Then, $0 \times I \subseteq P$ or $I \times 0 \subseteq P$. Moreover,*

1. *If $0 \times I \subseteq P$ then there exists a prime ideal p of A such that*

$$P = p \bowtie I := \{(a, a+i) \mid a \in p,\ i \in I\}.$$

2. *If $I \times 0 \subseteq P$ then there exists a prime ideal p of A such that*

$$P = \overline{p} := \{(a+i, a) \mid a \in p,\ i \in I\}.$$

In the both cases, P is minimal if and only if p is minimal.

Proof: Suppose that $0 \times I \nsubseteq P$. Then, there exists $i_0 \in I$ such that $(0, i_0) \notin P$. However, for any $i, j \in I$ and $a \in A$, we have $(i,0)(a,a+j)(0,i_0) = (0,0) \in P$. Hence, $(i,0) \in P$ for all $i \in I$. Thus, $I \times 0 \subseteq P$.
(1) Set $p = \pi_1(P)$. It is clear that p is an ideal of A (since π_1 is surjective). Let $a, b \in A$ with $arb \in p$ for all $r \in A$. Then, for each r there exists $i_r \in I$ such that $(arb, arb + i_r) \in P$. Then, for all $j \in I$, $(arb, a(r+j)b) = (arb, arb + i_r) + (0, ajb - i_r) \in P$ since $0 \times I \subseteq P$. Thus, $(a,a)(r, r+j)(b,b) \in P$. Hence, $(a,a) \in P$ or $(b,b) \in P$. Then, $a \in p$ or $b \in p$. So, p is prime.
Clearly, we have $P \subseteq p \bowtie I$. For the reverse inclusion, let $a \in p$. There exists $i \in I$ such that $(a, a+i) \in P$.

Hence, for all $j \in I$, we have $(a, a+j) = (a, a+i) + (0, j-i) \in P$. Then, $P = p \bowtie I$.
(2) Set $p = \pi_2(P)$. It is clear that p is an ideal of A (since π_2 is surjective). Let $a, b \in A$ with $arb \in p$ for all $r \in A$. Then, for each r there exists $i_r \in I$ such that $(arb + i_r, arb) \in P$. Then, for all $j \in I$, $(arb, a(r+j)b) = (a(r+j)b + i_{r+j}, a(r+j)b) - (i_{r+j} + ajb, 0) \in P$ since $I \times 0 \subseteq P$. Thus, $(a, a)(r, r+j)(b, b) \in P$. Then, $(a, a) \in P$ or $(b, b) \in P$. Hence, $a \in p$ or $b \in p$. So, p is prime.
Clearly, $P \subseteq \overline{p}$. Now, let $a \in p$. There exists $i \in i$ such that $(a+i, a) \in P$. Hence, for all $j \in I$, we have $(a+j, a) = (a+i, a) + (j-i, 0) \in P$. Then, $P = \overline{p}$.
For the last statement, Let p be a prime ideal of A.
Suppose that $P = p \bowtie I$ is minimal prime and let q be a prime ideal of A with $q \subseteq p$. Easily, we can see that $q \bowtie I \subseteq p \bowtie I = P$. Since $q \bowtie I$ is prime (by Lemma 2.4), we have $P = q \bowtie I$, and so $p = \pi_1(q \bowtie I) = q$.
Conversely, suppose that p is minimal prime, and let Q be a prime ideal of $A \bowtie I$ with $Q \subseteq P$. If $0 \times I \subseteq Q$ then $Q = q \bowtie I$ for some prime ideal q of A, and so we get $q \subseteq p$ which means that $q = p$, and then $Q = P$. Now, if $I \times 0 \subseteq Q$ then $Q = \overline{q}$ for some prime ideal q of A. Hence, $q \subseteq q + I = \pi_1(Q) \subseteq \pi_1(P) = p$, and then $I \subseteq q = p$. Hence,

$$Q = \overline{q} = \{(a+i, a) \mid a \in q,\, i \in I\} = \{(a, a+i) \mid a \in q,\, i \in I\} = q \bowtie I = P.$$

Now, suppose that $P = \overline{p}$ is minimal prime, and let q be a prime ideal of A such that $q \subseteq p$. Then, $\overline{q} \subseteq \overline{p} = P$. Then, $\overline{q} = \overline{p}$. Hence, $p = q$. Therefore, p is minimal.
Conversely, suppose that p is minimal and and let Q be a prime ideal of $A \bowtie I$ with $Q \subseteq P = \overline{p}$. If $0 \times I \subseteq Q$ then $Q = q \bowtie I$ for some prime ideal q of A. Then, $\pi_2(Q) \subseteq \pi_2(P)$ means that $q + I \subseteq p$. Hence, $I \subseteq q = p$. So,

$$Q = \{(a, a+i) \mid a \in q,\, i \in I\} = \{(a+i, a) \mid a \in q,\, i \in I\} = \overline{q} = P.$$

If $I \times 0 \subseteq Q$ then $Q = \overline{q}$ for some prime ideal q of A. Hence, $q \subseteq p$, and so $q = p$. Then, $Q = P$. □

The main result of this section is as follows:

Theorem 2.6. *Let A be a semiprime ring and I be an ideal of A. The following are equivalent:*

1. *$d(P) \subseteq P$ holds for any (resp. minimal) prime ideal P of $A \bowtie I$ and for any derivation d of $A \bowtie I$.*
2. *$\delta(p) \subseteq p$ holds for any (resp. minimal) prime ideal p of A and for any derivation δ of A keeping I invariant.*

Proof: ($\Rightarrow$) Let δ be a derivation on A with $\delta(I) \subseteq I$. Then, by Proposition 2.3, the additive map $d : A \bowtie I \to A \bowtie I$ defined by $d(a, a+i) = (\delta(a), \delta(a+i))$ is a derivation on $A \bowtie I$. Let p be a (resp. minimal) prime ideal of A. By Lemmas 2.4 and 2.5, $p \bowtie I$ is a (resp. minimal) prime ideal of $A \bowtie I$. Hence, $d(p \bowtie I) \subseteq p \bowtie I$. Let $a \in p$. Then, $(a, a) \in p \bowtie I$. Thus, $(\delta(a), \delta(a)) = d(a, a) \in p \bowtie I$, and so $\delta(a) \in p$. Hence, $\delta(p) \subseteq p$.
($\Leftarrow$) Let d be a derivation on $A \bowtie I$. Following Proposition 2.3, there exist a derivation $\delta_1 : A \to A$ keeping I invariant and a derivation $\delta_2 : A \to I$ such that

$$d(a, a+i) = (\delta_1(a), \delta_1(a+i) + \delta_2(a+i)) \quad \text{for all } a \in A,\, i \in I.$$

Let P be a (resp. minimal) prime ideal of $A \bowtie I$. Then, using Lemma 2.5, $P = p \bowtie I$ or $P = \overline{p}$ for some (resp. minimal) prime ideal p of A. By hypothesis, $\delta_1(p) \subseteq p$ and $\delta_2(p) \subseteq p$ (see that I is also invariant under δ_2).
Suppose that $P = p \bowtie I$. Then, the elements of P have the form $(a, a+i)$ with $a \in p$ and $i \in I$, and we have

$$d(a, a+i) = (\delta_1(a), \delta_1(a+i) + \delta_2(a+i)) = \big(\delta_1(a), \delta_1(a) + (\delta_1(i) + \delta_2(a+i))\big) \in P$$

since $\delta_1(a) \in p$ and $\delta_1(i) + \delta_2(a+i) \in I$. Thus, $d(P) \subseteq P$.
Now, suppose that $P = \overline{p}$. The elements of P in this case have the form $(a+i, a)$ with $a \in p$ and $i \in I$,

and we have

$$\begin{aligned} d(a+i,a) &= (\delta_1(a+i), \delta_1(a)+\delta_2(a)) \\ &= \big(\delta_1(a)+\delta_2(a)+(\delta_1(i)-\delta_2(a)), \delta_1(a)+\delta_2(a)\big) \in P \end{aligned}$$

since $\delta_1(a)+\delta_2(a) \in p$ and $\delta_1(i)-\delta_2(a) \in I$. Again, $d(P) \subseteq P$. □

As consequences of the above theorem, we have the following corollaries.

Corollary 2.7. *Let A be a semiprime ring and I be a prime ideal of A. The following are equivalent:*

1. *$d(P) \subseteq P$ holds for any prime ideal P of $A \bowtie I$ and for any derivation d of $A \bowtie I$.*
2. *$\delta(p) \subseteq p$ holds for any prime ideal p of A and for any derivation δ.*

Corollary 2.8. *Let A be a semiprime ring and I be a minimal prime ideal of A. The following are equivalent:*

1. *$d(P) \subseteq P$ holds for any minimal prime ideal P of $A \bowtie I$ and for any derivation d of $A \bowtie I$.*
2. *$\delta(p) \subseteq p$ holds for any minimal prime ideal p of A and for any derivation δ.*

Corollary 2.9. *Let A be a prime ring and I an ideal of A. Then, $d(P) \subseteq P$ holds for any minimal prime ideal P of $A \bowtie I$ and for any derivation d of $A \bowtie I$.*

Proof: Follows immediately from Theorem 2.6 since the only minimal prime ideal of A is (0) which is always invariant under any derivation on A (in particular under those keeping I invariant). □

3. (Jordan) derivations on amalgamated duplication of a ring along an ideal

Proposition 3.1. *Let A be a ring, I be an ideal of A, and $d : A \bowtie I \to A \bowtie I$ be an additive map. Then, d is a Jordan derivation if and only if*

1. *d_1 and d_2 are Jordan derivations.*
2. *$s_k(a \circ i) = a \circ s_k(i)$ and $s_k(i^2) = 0$ for all $k = 1, 2$, $a \in A$ and $i, j \in I$.*

Proof: Let R and T be rings and let θ be a homomorphism of T into R. it's easy to check that if $d : T \to R$ is a Jordan (θ, θ)-derivation then for all $x, y \in T$ we have

$$d(x \circ y) = d(x) \circ \theta(y) + \theta(x) \circ d(y).$$

$(\Rightarrow)$ From Lemma 2.1, $\pi_1 \circ d$ is a Jordan (π_1, π_1)-derivation and $\pi_2 \circ d$ is a Jordan (π_2, π_2)-derivation. Hence, for all $a \in A$, we have

$$\begin{aligned} d_1(a^2) &= \pi_1 \circ d(a^2, a^2) \\ &= \pi_1 \circ d\big((a,a)(a,a)\big) \\ &= \pi_1 \circ d(a,a)a + a\pi_1 \circ d(a,a) \\ &= d_1(a)a + ad_1(a). \end{aligned}$$

Hence, d_1 is Jordan derivation. Similarly, we obtain that d_2 is a Jordan derivation.
Let $a \in A$ and $i \in I$. We have

$$\begin{aligned} s_1(a \circ i) &= \pi_1\left(d(0, a \circ i)\right) \\ &= \pi_1\left(d((a,a) \circ (0,i))\right) \\ &= \pi_1(d(a,a)) \circ \pi_1(0,i) + \pi_1(a,a) \circ \pi_1\left(d(0,i)\right) \\ &= a \circ s_1(i). \end{aligned}$$

Moreover, for all $i \in I$, we have

$$\begin{array}{rcl} s_1(i^2) & = & \pi_1 \circ d(0, i^2) \\ & = & \pi_1 \circ d\left((0,i)(0,i)\right) \\ & = & \pi_1 \circ d(0,i)\pi_1(0,i) + \pi_1(0,i)\pi_1 \circ d(0,i) \\ & = & 0. \end{array}$$

Similarly, s_2 satisfies the same conditions.
($\Leftarrow$) As in the proof of Proposition 2.2, for all $a \in A$ and $i \in I$, we have

$$\pi_1 \circ d(a, a+i) = d_1(a) + s_1(i) \quad \text{and} \quad \pi_2 \circ d(a, a+i) = d_2(a+i) - s_2(i).$$

Using Lemma 2.1, we have to prove that $\pi_1 \circ d$ is a Jordan (π_1, π_1)-derivation and $\pi_2 \circ d$ is a Jordan (π_2, π_2)-derivation. Let $a \in A$ and $i \in I$. We have

$$\begin{array}{rcl} \pi_1 \circ d((a, a+i)^2) & = & \pi_1 \circ d(a^2, a^2 + ai + ia + i^2) \\ & = & d_1(a^2) + s_1(a \circ i + i^2) \\ & = & d_1(a) \circ a + a \circ s_1(i) \\ & = & (d_1(a) + s_1(i)) \circ a \\ & = & \pi_1 \circ d(a, a+i)\pi_1(a, a+i) + \pi_1(a, a+i)\pi_1 \circ d(a, a+i). \end{array}$$

and, since $0 = 2s_2(i^2) = s_2(i \circ i) = s_2(i) \circ i = is_2(i) + s_2(i)i$, we get

$$\begin{array}{rcl} \pi_2 \circ d((a, a+i)^2) & = & \pi_2 \circ d(a^2, (a+i)^2) \\ & = & d_2\left((a+i)^2\right) - s_2(a \circ i + i^2) \\ & = & d_2(a+i)(a+i) + (a+i)d_2(a+i) - a \circ s_2(i) \\ & = & (d_2(a+i) - s_2(i))(a+i) + (a+i)(d_2(a+i) - s_2(i)) \\ & = & \pi_2 \circ d(a, a+i)\pi_2(a, a+i) + \pi_2(a, a+i)\pi_2 \circ d(a, a+i). \end{array}$$

Hence, we have the desired result. □

Lemma 3.2. *Let A be a ring and I be an ideal of A. Then,*

1. *$A \bowtie I$ is prime if and only if $I = (0)$ and A is prime.*
2. *$A \bowtie I$ is semiprime if and only if A is semiprime.*
3. *$A \bowtie I$ is 2-torsion free if and only if A is 2-torsion free.*

Proof: (1) Suppose that $A \bowtie I$ is prime. Hence, $\{(0,0)\}$ is a prime ideal of $A \bowtie I$. Thus, by Lemma 2.5, $0 \times I \subseteq \{(0,0)\}$ or $I \times 0 \subseteq \{(0,0)\}$. In the both cases, $I = (0)$. By Lemma 2.5, $\{(0,0)\} = p \bowtie (0)$ for some prime ideal of A. Hence, $p = (0)$ is a prime ideal of A, and so A prime. Conversely, if $I = (0)$ and A is prime then $\{(0,0)\} = (0) \bowtie (0)$ is a prime ideal of $A \bowtie I$, and then $A \bowtie I$ is prime.
(2) Suppose that $A \bowtie I$ is semiprime and let $a \in A$ with $ara = 0$ for all $r \in A$. Then, $(a,a)(r, r+j)(a,a) = (0,0)$ for all $r \in A$ and all $j \in I$. Hence, $(a,a) = (0,0)$, and so $a = 0$. Thus, A is semiprime.
Conversely, suppose that A is semiprime and let $a \in A$ and $i \in I$ with $(a, a+i)(r, r+j)(a, a+i) = (0,0)$ for all $r \in A$ and $j \in I$. Then, $ara = 0$ for all $r \in A$, and then $a = 0$. Now, we have $i(r+j)i = 0$ for all $r \in A$ and $j \in I$. Which means that $iri = 0$ for all $r \in A$. Then, $i = 0$. Hence, $A \bowtie I$ is prime.
(3) Trivial.

□

Corollary 3.3. *Let A be a 2-torsion free semiprime ring, I be an ideal of A, and $s : I \to A$ be an additive map. Then,*

1. *if* $s(a \circ i) = a \circ s(i)$ *and* $s(i^2) = 0$ *for all* $a \in A$ *and* $i \in I$ *then* $s = 0$.
2. *if there exists a derivation* d *on* A *such that* $s(a \circ i) = d(a) \circ i + a \circ s(i)$ *and* $s(i^2) = s(i) \circ i$ *for all* $a \in A$ *and* $i \in I$ *then* d *and* s *coincide on* I.

Proof: (1) Consider the additive map $d : A \bowtie I \to A \bowtie I$ defined by $d(a, a + i) = (s(i), s(i))$. For all $a \in A$ and $i \in I$, we have

$$\begin{aligned} d((a, a+i)^2) &= (s(a \circ i + i^2), s(a \circ i + i^2)) \\ &= (a \circ s(i), a \circ s(i)) \\ &= (s(i), s(i))(a, a+i) + (a, a+i)(s(i), s(i)) \\ &= d(a, a+i)(a, a+i) + (a, a+i)d(a, a+i). \end{aligned}$$

since $s(i) \circ i = s(2i^2) = 0$. Hence, d is a Jordan derivation on $A \bowtie I$. But $A \bowtie I$ is a 2-torsion free semiprime ring (by Lemma 3.2). Thus, by [4, Theorem 1], d is a derivation. So, by Proposition 2.3, $0 \times I$ is d-invariant. Hence, for all $i \in I$, $d(0, i) = (s(i), s(i)) \in 0 \times I$, and so $s = 0$.
(2) Set $s' := s - d : I \to A$. For all $a \in A$ and $i \in I$, we have

$$\begin{aligned} s'(a \circ i) &= s(a \circ i) - d(a \circ i) = d(a) \circ i + a \circ s(i) - d(a) \circ i - a \circ d(i) \\ &= a \circ s(i) - a \circ d(i) = a \circ s'(i) \end{aligned}$$

and

$$s'(i^2) = s(i^2) - d(i^2) = s(2i^2) - s(i^2) - d(i^2) = s(i \circ i) - i \circ s(i) - d(i) \circ i = 0.$$

Hence, from (1), $s' = 0$, and so $s(i) = d(i)$ for all $i \in I$. □

Theorem 3.4. *Let* A *be a non commutative prime ring and* I *be a nonzero ideal. If* d *is both a Jordan derivation and a Jordan triple derivation of* $A \bowtie I$ *then* d *is a derivation.*

Proof: When the characteristic of A is different of 2, the result follows from by [4, Theorem 1] and Lemma 3.2. Hence, suppose that A is of characteristic two. Also, if $I = (0)$, then $A \bowtie (0) \cong A$ (following the isomorphism $(a, a) \mapsto a$). In this case, the result follows immediately from [12, Theorem 4.1]. Thus, we may suppose $I \neq (0)$. From Proposition 3.1, d_1 and d_2 are Jordan derivations and, for $k = 1, 2$ and for all $a \in A$ and $i \in I$, we have $s_k(a \circ i) = a \circ s_k(i)$ and $s_k(i^2) = 0$. Now, let $a, b \in A$, we have

$$\begin{aligned} d_1(aba) &= \pi_1 \circ d(aba, aba) \\ &= \pi_1 \circ d\left((a, a)(b, b)(a, a)\right) \\ &= \pi_1 \left(d(a, a)(ba, ba) + (a, a)d(b, b)(a, a) + (a, a)(b, b)d(a, a)\right) \\ &= \pi_1 \circ d(a, a)\, ba + a\pi_1 \circ d(b, b)a + ab\pi_1 \circ d(a, a) \\ &= d_1(a)ba + ad_1(b)a + abd_1(a). \end{aligned}$$

Hence, d_1 is also a Jordan triple derivation. Similarly, d_2 is a Jordan triple derivation. Hence, since A is non commutative prime, d_1 and d_2 are derivations (by [12, Theorem 4.1]).
Let $i, j \in I$. We have

$$\begin{aligned} s_1(iji) &= \pi_1 \circ d(0, iji) \\ &= \pi_1 \circ d((0, i)(0, j)(0, i)) \\ &= \pi_1 \left(d(0, i)(0, ji) + (0, i)d(0, j)(0, i) + (0, ij)d(0, i)\right) \\ &= 0 \end{aligned}$$

Also,

$$\begin{aligned} s_1(iji) &= \pi_1 \circ d(0, iji) \\ &= \pi_1 \circ d((i, i)(0, j)(i, i)) \\ &= \pi_1 \left(d(i, i)(0, ji) + (i, i)d(0, j)(i, i) + (0, ij)d(i, i)\right) \\ &= i\pi_1(d(0, j))i \\ &= is_1(j)i. \end{aligned}$$

Hence,

$$0 = s_1(iji) = is_1(j)i \quad \text{for all } i, j \in I. \tag{3.1}$$

By analogy s_2 satisfies the same condition.
For $k = 1, 2$, by linearizing the condition "$s_k(i^2) = 0$ for all $i \in I$", we obtain

$$s_k(i \circ j) = 0 \quad \text{for all } i, j \in I. \tag{3.2}$$

Hence,

$$ias_k(j) + s_k(j)ia = ia \circ s_k(j) = s_k(ia \circ j) = 0 \quad \text{for all } i, j \in I,\ a \in A. \tag{3.3}$$

Then,

$$i(a \circ s_k(j)) + (i \circ s_k(j))a = 0 \quad \text{for all } i, j \in I,\ a \in A. \tag{3.4}$$

But $i \circ s_k(j) = s_k(i \circ j) = 0$. Hence,

$$i(a \circ s_k(j)) = 0 \quad \text{for all } i, j \in I,\ a \in A. \tag{3.5}$$

So,

$$ir(a \circ s_k(j)) = 0 \quad \text{for all } i, j \in I,\ a, r \in A. \tag{3.6}$$

Since A is prime, we get that

$$i = 0 \quad \text{or} \quad a \circ s_k(j) = 0 \quad \text{for all } i, j \in I,\ a \in A. \tag{3.7}$$

But $I \neq (0)$, and then

$$a \circ s_k(j) = 0 \quad \text{for all } j \in I,\ a \in A. \tag{3.8}$$

which means that $s(j) \in Z(A)$ for all $j \in I$ since A is of characteristic two.
Thus, (3.1) means that

$$s_k(j)i^2 = 0 \quad \text{for all } i, j \in I. \tag{3.9}$$

Thus, since $s(j) \in Z(A)$ for all $j \in I$, we have

$$s_k(j) = 0 \quad \text{or} \quad i^2 = 0 \quad \text{for all } i, j \in I. \tag{3.10}$$

If $i^2 = 0$ for all $i \in I$, then $0 = ij + ji = ij - ji$ for all $i, j \in I$. Hence, for all $i \in I$ and $r \in A$, we have $iri = i(ri) = (ri)i = ri^2 = 0$. Thus, $i = 0$ for all $i \in I$ since A is prime. But $I \neq 0$, and so there exists $i \in I$ such that $i^2 \neq 0$. Consequently, by (3.10), $s_k(j) = 0$ for all $j \in I$. Seen Proposition 2.2, d is a derivation. □

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