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Some Classes of 3-Dimensional Trans-Sasakian Manifolds with Respect to Semi-Symmetric Metric Connection

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ABSTRACT: The object of the present paper is to study semi-symmetric metric connection on a 3-dimensional trans-Sasakian manifold. We found the necessary condition under which a vector field on a 3-dimensional trans-Sasakian manifold will be a strict contact vector field. Then, we obtained extended generalized $\phi-$recurrent 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection. Next, a 3-dimensional trans-Sasakian manifold satisfies the condition $\tilde{L} \cdot \tilde{S} = 0$ with respect to semi-symmetric metric connection is studied.

Key Words: Semi-symmetric metric connection, Trans-Sasakian manifold, Extended generalized $\phi-$reccurent trans-Sasakian manifold, Conharmonically curvature tensor.

Contents

1. Introduction

Let (M, ϕ, ξ, η, g) be a $(2m+1)-$dimensional almost contact metric manifold. Then the product $\bar{M} = M \times R$ has a natural almost complex structure J with the product metric G being Hermitian metric. The geometry of the almost Hermitian manifold $(\bar{M}, J, G)$ gives the geometry of the almost contact metric manifold (M, ϕ, ξ, η, g). Sixteen different types of structures on M like Sasakian manifold, Kenmotsu manifold, and etc., are given by the almost Hermitian manifold $(\bar{M}, J, G)$. The notion of trans-Sasakian manifolds was introduced by Oubina [11] in 1985. Then, J. C. Marrero [6] have studied the local structure of trans-Sasakian manifolds. In general, a trans-Sasakian manifold $(M, \phi, \xi, \eta, g, \alpha, \beta)$ is called a trans-Sasakian manifold of type (α, β). Trans-Sasakian manifold of type $(0, 0)$, $(\alpha, 0)$, $(0, \beta)$ are called cosymplectic, $\alpha-$Sasakian and $\beta-$Kenmotsu manifold, respectively. Marrero has proved that trans-Sasakian structures are generalized quasi-Sasakian structure. He has also proved that a trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or $\alpha-$Sasakian or $\beta-$Kenmotsu manifold. So, we have considered here three dimensional trans-Sasakian manifold.

The notion of a semi-symmetric linear connection on a differential manifold has been first introduced

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by Friedmann and Schouten [4] in 1924. In 1932 Hayden has given the idea of metric connection with torsion on Riemannian manifold in [5]. Yano [18] has given a systematic study of semi-symmetric connection on Riemannian manifold in 1970. Later K. S. Amur and S. S. Pujar [1], C. S. Bagewadi [3], Sharafuddin and Hussian (1976) [15] and others have also studied semi-symmetric connection on Riemannian manifold.

The semi symmetric metric connection gives an important concept in the geometry of Riemannian manifold having physical applications i.e the displacement on the earth surface facing a fixed point is metric and semi-symmetric [13]. So the study of some classes of 3-dimensional trans-Sasakian manifold with respect to semi symmetric metric connection gives a geometric features in differential geometry. The object of this paper is to study 3-dimensional trans-Sasakian manifolds and their properties under certain conditions with respect to semi symmetric metric connection.

2. Preliminaries

An $n\ (= 2m+1)$ dimensional Riemannian manifold (M, g) is called an almost contact manifold if there exists a (1,1) tensor field ϕ, a vector field ξ and a 1-form η on M such that

$$\phi^2(X) = -X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = 1, \eta(\phi X) = 0, \tag{2.2}$$

$$\phi\xi = 0, \tag{2.3}$$

$$\eta(X) = g(X, \xi), \tag{2.4}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.5}$$

$$g(X, \phi Y) + g(Y, \phi X) = 0, \tag{2.6}$$

for any vector fields X, Y on M. An odd dimensional almost contact metric manifold M is called a trans-Sasakian manifold if it satisfies the following condition

$$(\nabla_X\phi)(Y) = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{2.7}$$

for some smooth functions α, β on M and we say that the trans-Sasakian structure is of type (α, β). For an n-dimensional trans-Sasakian manifold [9], from (2.7) we have

$$\nabla_X\xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.8}$$

$$(\nabla_X\eta)(Y) = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.9}$$

In an n-dimensional trans-Sasakian manifold, we have

$$R(\xi, X)\xi = (\alpha^2 - \beta^2 - \xi\beta)(\eta(X)\xi - X), \tag{2.10}$$

$$2\alpha\beta + \xi\alpha = 0, \tag{2.11}$$

$$\begin{aligned} S(X, \xi) = {} & [(n-1)(\alpha^2 - \beta^2) - (\xi\beta)]\eta(X) \\ & - ((\phi X)\alpha) - (n-2)(X\beta). \end{aligned} \tag{2.12}$$

For $\alpha, \beta =$ constants then the above equations reduce to

$$R(\xi, X)Y = (\alpha^2 - \beta^2)(g(X, Y)\xi - \eta(Y)X), \tag{2.13}$$

$$R(X, Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y), \tag{2.14}$$

$$S(X, Y) = (\frac{r}{2} - (\alpha^2 - \beta^2))g(X, Y) - (\frac{r}{2} - 3(\alpha^2 - \beta^2))\eta(X)\eta(Y), \tag{2.15}$$

$$S(X, \xi) = 2(\alpha^2 - \beta^2)\eta(X), \tag{2.16}$$

$$S(\phi X, \phi Y) = (\frac{r}{2} - (\alpha^2 - \beta^2))g(X, Y), \tag{2.17}$$

$$S(\phi X, Y) = -S(X, \phi Y). \tag{2.18}$$

Definition 2.1. *A trans-Sasakian manifold M^n is said to be $\eta-$ Einstein manifold if its Ricci tensor S is of the form*

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a, b are smooth functions.

Definition 2.2. *A vector field X on a 3-dimensional trans-Sasakian manifold $(M^3, \phi, \eta, \xi, g)$ is said to be a contact vector field if*

$$(\mathcal{L}_X \eta)(Y) = \sigma\eta(Y), \tag{2.19}$$

where σ is scalar function on M^3 and $\mathcal{L}_X$ denotes the Lie derivative along X. X is called a strict contact vector field if $\sigma = 0$.

Let (M^n, g) be a Riemannian manifold with the Levi-Civita connection ∇. A linear connection $\tilde{\nabla}$ on (M^n, g) is said to be semi-symmetric ([15], [18]) if its torsion tensor T can be written as

$$T(X,Y) = \pi(Y)X - \pi(X)Y,$$

where π is an $1-$ form on M^n and the associated vector field ρ defined by $\pi(X) = g(X, \rho)$, for all vector fields $X \in \chi(M^n)$.

A semi-symmetric connection $\tilde{\nabla}$ is called semi-symmetric metric connection if $\tilde{\nabla} g = 0$.

In an almost contact manifold, semi-symmetric metric connection is defined by identifying the 1-form π of the above equation with the contact 1-form η, i.e., by setting [15]

$$T(X,Y) = \eta(Y)X - \eta(X)Y, \tag{2.20}$$

with

$$g(X, \rho) = \eta(X), \forall X \in \chi(M^n).$$

K. Yano has obtained the relation between semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ of M^n in [18] and it is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi, \tag{2.21}$$

where $g(X, \xi) = \eta(X)$.

Further, a relation between the curvature tensors R and $\tilde{R}$ of type (1,3) of the connections ∇ and $\tilde{\nabla}$, respectively is given by [18],

$$\tilde{R}(X,Y)Z = R(X,Y)Z - K(Y,Z)X + K(X,Z)Y - g(Y,Z)FX + g(X,Z)FX, \tag{2.22}$$

where K is a tensor field of type $(0,2)$ and F is a (1,1) tensor field defined by

$$K(Y,Z) = g(FY,Z) = (\nabla_Y \eta)(Z) - \eta(X)\eta(Z) + \frac{1}{2}\eta(\xi)g(Y,Z). \tag{2.23}$$

In this paper, we have considered that M^3 is 3-dimensional trans-Sasakian manifold. So, using (2.9), (2.19), (2.23) it follows that

$$K(Y,Z) = -\alpha g(\phi Y, Z) - (\beta + 1)\eta(Y)\eta(Z) + (\beta + \frac{1}{2})g(Y,Z). \tag{2.24}$$

Using (2.22), from above equation we get

$$FY = -\alpha\phi Y - (\beta+1)\eta(Y)\xi + (\beta+\frac{1}{2})Y. \tag{2.25}$$

Now, by using above two equations we get

$$\begin{aligned}\tilde{R}(X,Y)Z = R(X,Y)Z - \alpha(g(\phi X,Z)Y - g(\phi Y,Z)X) - \alpha(g(X,Z)\phi Y - g(Y,Z)\phi X)\\ -(\beta+1)(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X)\\ -(\beta+1)[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\xi\\ +(2\beta+1)(g(X,Z)Y - g(Y,Z)X).\end{aligned} \tag{2.26}$$

In the view of (2.26) we get

$$\tilde{S}(Y,Z) = S(Y,Z) + \alpha g(\phi Y,Z) + (\beta+1)\eta(Y)\eta(Z) - (3\beta+1)g(Y,Z), \tag{2.27}$$

where $\tilde{S}$ and S are Ricci tensors of M^3 with respect to semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇, respectively.

From above, we have

$$\tilde{r} = r - 8\beta - 2, \tag{2.28}$$

where $\tilde{r}$ and r are scalar curvature of M^3 with respect to semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇, respectively.

We obtain from (2.15) and (2.27) that

$$\tilde{Q}\xi = 2(\alpha^2 - \beta^2 - \beta)\xi, \tag{2.29}$$

where $\tilde{Q}$ is the Ricci operator with respect to semi-symmetric metric connection $\tilde{\nabla}$.

3. Geometric Vector Fields on 3-Dimensional Trans-Sasakian Manifold with respect to Semi-Symmetric Metric Connection

Suppose that a contact vector field X on a 3-dimensional trans-Sasakian manifold leaves the Ricci tensor with respect to semi-symmetric metric connection invariant, i.e.

$$(\mathcal{L}_X\tilde{S})(Y,Z) = 0. \tag{3.1}$$

It follows from (3.1) we get

$$\mathcal{L}_X\tilde{S}(Y,Z) = \tilde{S}(\mathcal{L}_XY,Z) + \tilde{S}(Y,\mathcal{L}_XZ).$$

Putting $Z=\xi$ we have

$$\mathcal{L}_X\tilde{S}(Y,\xi) = \tilde{S}(\mathcal{L}_XY,\xi) + \tilde{S}(Y,\mathcal{L}_X\xi). \tag{3.2}$$

From the equations (2.15) and (2.27) we have

$$\tilde{S}(Y,\xi) = 2(\alpha^2 - \beta^2 - \beta)\eta(Y). \tag{3.3}$$

From the equation (3.3) we get

$$2(\alpha^2-\beta^2-\beta)(\mathcal{L}_X\eta)(Y)=\bar{S}(Y,\mathcal{L}_X\xi).$$

Hence we have

$$2(\alpha^2-\beta^2-\beta)\sigma\eta(Y)=\bar{S}(Y,\mathcal{L}_X\xi). \tag{3.4}$$

Taking $Y=\xi$ in (3.4) we obtain

$$\sigma=\eta(\mathcal{L}_X\xi). \tag{3.5}$$

Again, putting $Y=\xi$ in (2.19) one can get

$$\sigma=-\eta(\mathcal{L}_X\xi). \tag{3.6}$$

Hence, we have $\sigma=0$.

Therefore, we state the following theorem:

Theorem 3.1. *Every contact vector field on a 3-dimensional trans-Sasakian manifold leaving the Ricci tensor with respect to semi-symmetric connection invariant is a strict contact vector field.*

4. On Extended Generalized ϕ-Recurrent 3-Dimensional Trans-Sasakian Manifold with respect to Semi-Symmetric Connection

Definition 4.1. *A 3-dimensional trans-Sasakian manifold with respect to semi-symmetric connection is said to be a $\phi-$recurrent manifold if $\exists$ a nonzero 1-form B such that*

$$\phi^2((\tilde{\nabla}_W R)(X,Y)Z)=B(W)R(X,Y)Z,$$

for arbitrary vector fields X,Y,Z,W.

Definition 4.2. *A Riemannian manifold (M^3,g) is called $\phi-$generalized recurrent [2], if its curvature tensor R satisfies the condition*

$$\phi^2((\nabla_W R)(X,Y)Z)=A(W)R(X,Y)Z+B(W)[g(Y,Z)X-g(X,Z)Y],$$

where A and B are two 1-forms, B is non zero and these are defined by $g(W,\rho_1)=A(W)$ and $g(W,\rho_2)=B(W)$, $\forall W\in\chi(M)$.

ρ_1 and ρ_2 being the vector fields associated to the 1-form A and B respectively.

Definition 4.3. *A three-dimensional trans-Sasakian manifold is said to be an extended generalized $\phi-$recurrent trans-Sasakian manifold if its curvature tensor R satisfies the relation*

$$\phi^2((\nabla_W R)(X,Y)Z)=A(W)\phi^2(R(X,Y)Z)+B(W)\phi^2([g(Y,Z)X-g(X,Z)Y]),$$

for all $X,Y,Z,W\in\chi(M)$, where A and B are two non-vanishing 1-forms such that $g(W,\rho_1)=A(W)$ and $g(W,\rho_2)=B(W)$, $\forall W\in\chi(M)$, with ρ_1 and ρ_2 being the vector fields associated to the 1-form A and B, respectively [16].

In this connection, we mention the works of Prakasha [7] on Sasakian manifolds.

First suppose that M^3 is an $\eta-$ Einstein trans-Sasakian manifold, i.e.,

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{4.1}$$

where, a, b are smooth function on M^3. Putting $X = Y = \xi$ in the above equation we get

$$a + b = 2(\alpha^2 - \beta^2). \tag{4.2}$$

In a local coordinate, (4.1) can be written as

$$R_{ij} = ag_{ij} + b\eta_i\eta_j, \tag{4.3}$$

which implies that

$$r = 3a + b. \tag{4.4}$$

Taking covariant derivative with respect to k from the equation (4.3), we have

$$R_{ij,k} = a_{,k}g_{ij} + b_{,k}\eta_i\eta_j + b\eta_{i,k}\eta_j + b\eta_i\eta_{j,k}. \tag{4.5}$$

Contracting (4.5) with g^{ik}, we get

$$R^k_{j,k} = a_{,j} + b_{,k}\xi^k\eta_j + b\eta_{i,k}g^{ik}\eta_j + b\eta_i\eta_{j,k}g^{ik}. \tag{4.6}$$

We also have

$$R^k_{j,k} = \frac{1}{2}r_{,j}. \tag{4.7}$$

Hence, we obtain

$$r_{,j} = 2[a_{,j} + b_{,k}\xi^k\eta_j + b\eta_{i,k}g^{ik}\eta_j + b\eta_i\eta_{j,k}g^{ik}]. \tag{4.8}$$

Since

$$\eta_{i,k} = -\alpha g_{ih}\phi^h_k + \beta(g_{ik} - \eta_i\eta_k),$$

we have

$$\eta_{i,k}g^{ik} = -\alpha g_{ih}\phi^h_k g^{ik} + 2\beta. \tag{4.9}$$

From the equation (4.8), we get

$$r_{,j} = 2[a_{,j} + b_{,k}\xi^k\eta_j + b(-\alpha\phi^h_h + 2\beta)\eta_j]. \tag{4.10}$$

Again,

$$a_{,k} + b_{,k} = 4[\alpha\alpha_{,k} - \beta\beta_{,k}]. \tag{4.11}$$

And also,

$$r_{,j} = 2a_{,j} + 4[\alpha\alpha_{,j} - \beta\beta_{,j}]. \tag{4.12}$$

From the equations (4.10) and (4.12) we get

$$2a_{,j} + 4[\alpha\alpha_{,j} - \beta\beta_{,j}] = 2[a_{,j} + b_{,k}\xi^k\eta_j + b(-\alpha\phi_h^h + 2\beta)\eta_j]. \tag{4.13}$$

Contracting (4.13) with ξ^j and using (4.2), we have

$$2b_{,k}\xi^k + 2b(-\alpha\phi_h^h + 2\beta) = a_{,k} + b_{,k}. \tag{4.14}$$

If b and a are constant functions, then the equation (4.14) implies that either $b = 0$ or α and β are related by

$$-\alpha\phi_h^h + 2\beta = 0. \tag{4.15}$$

Again, contracting (4.15) with ξ^h we get $\beta = 0$.

So, we have the following theorem:

Theorem 4.4. *Suppose M^3 is an $\eta-$ Einstein trans-Sasakian manifold. If b and a are constant functions, then either M^3 is an Einstein manifold or M^3 is an $\alpha-$ Sasakian manifold.*

Now, we prove the following result:

Theorem 4.5. *An extended generalized ϕ-recurrent trans-Sasakian manifold (M^3, g) with respect to semi symmetric metric connection is an $\eta-$ Einstein manifold and more over, the 1-forms A and B are related as $A(W)[\alpha^2 - \beta^2 - \beta] + B(W) = 0$.*

Proof: Let us assume an extended generalized $\phi-$ recurrent trans-Sasakian manifold $(M^3, \phi, \eta, \xi, g)$ with respect to semi-symmetric connection. Then we have

$$\phi^2((\tilde{\nabla}_W\tilde{R})(X,Y)Z) = A(W)\phi^2(\tilde{R}(X,Y)Z) + B(W)\phi^2([g(Y,Z)X - g(X,Z)Y]).$$

From above we get

$$\begin{aligned} &-(\tilde{\nabla}_W\tilde{R})(X,Y)Z + \eta((\tilde{\nabla}_W\tilde{R})(X,Y)Z)\xi \\ &= A(W)[-\tilde{R}(X,Y)Z + \eta(\tilde{R}(X,Y)Z)\xi] + B(W)[-g(Y,Z)X + g(X,Z)Y \\ &\quad + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi]. \end{aligned} \tag{4.16}$$

From the equation (4.16), we have

$$\begin{aligned} &-g((\tilde{\nabla}_W\tilde{R})(X,Y)Z,U) + \eta((\tilde{\nabla}_W\tilde{R})(X,Y)Z)\eta(U) \\ &= A(W)[-g(\tilde{R}(X,Y)Z,U) + \eta(\tilde{R}(X,Y)Z)\eta(U)] + B(W)[-g(Y,Z)g(X,U) \\ &\quad + g(X,Z)g(Y,U) + g(Y,Z)\eta(X)\eta(U) - g(X,Z)\eta(Y)\eta(U)]. \end{aligned} \tag{4.17}$$

Let $\{e_1, e_2, e_3\}$ be an orthonormal basis for the tangent space of M^3 at a point $p \in M^3$. Putting $X = U = e_i$ in (4.17) and taking summation over i, we get

$$\begin{aligned} -(\tilde{\nabla}_W \tilde{S})(Y,Z) + \sum_{i=1}^{3} \eta((\tilde{\nabla}_W \tilde{R})(e_i,Y)Z)\eta(e_i) \\ = A(W)[-\tilde{S}(Y,Z) + \eta(\tilde{R}(\xi,Y)Z)] + B(W)[-g(Y,Z) - \eta(Y)\eta(Z)]. \end{aligned} \tag{4.18}$$

Putting $Z = \xi$, we have

$$\begin{aligned} -(\tilde{\nabla}_W \tilde{S})(Y,\xi) + \sum_{i=1}^{3} \eta((\tilde{\nabla}_W \tilde{R})(e_i,Y)\xi)\eta(e_i) \\ = A(W)[-\tilde{S}(Y,\xi) + \eta(\tilde{R}(\xi,Y)\xi)] - 2B(W)\eta(Y). \end{aligned} \tag{4.19}$$

Now,

$$\begin{aligned} g((\tilde{\nabla}_W \tilde{R})(e_i,Y)\xi,\xi) = g((\nabla_W \tilde{R})(e_i,Y)\xi,\xi) + \eta(\tilde{R}(e_i,Y)\xi)g(W,\xi) \\ - g(W,\tilde{R}(e_i,Y)\xi)g(\xi,\xi). \end{aligned} \tag{4.20}$$

We have

$$\begin{aligned} g((\nabla_W \tilde{R})(e_i,Y)\xi,\xi) = g(\nabla_W \tilde{R}(e_i,Y)\xi,\xi) - g(\tilde{R}(\nabla_W e_i,Y)\xi,\xi) \\ - g(\tilde{R}(e_i,\nabla_W Y)\xi,\xi) - g(\tilde{R}(e_i,Y)\nabla_W \xi,\xi). \end{aligned} \tag{4.21}$$

at $p \in M^3$. Since e_i is an orthonormal basis, so $\nabla_W e_i = 0$ at p.

Also,

$$g(\tilde{R}(e_i,Y)\xi,\xi) = -g(\tilde{R}(\xi,\xi)Y,e_i) = 0. \tag{4.22}$$

Since $\nabla_W g = 0$, we obtain

$$g((\nabla_W \tilde{R})(e_i,Y)\xi,\xi) + g(\tilde{R}(e_i,Y)\xi,\nabla_W \xi) = 0, \tag{4.23}$$

which implies that

$$g((\nabla_W \tilde{R})(e_i,Y)\xi,\xi) = 0. \tag{4.24}$$

Since $\eta(\tilde{R}(e_i,Y)\xi) = 0$, we have from (4.20) that

$$g((\tilde{\nabla}_W \tilde{R})(e_i,Y)\xi,\xi) = -g(W,\tilde{R}(e_i,Y)\xi). \tag{4.25}$$

Therefore,

$$\sum_{i=1}^{3} \eta((\tilde{\nabla}_W \tilde{R})(e_i,Y)Z)\eta(e_i) = \alpha g(\phi Y,W) - (\alpha^2 - \beta^2 - \beta)g(\phi W,\phi Y). \tag{4.26}$$

Again, from (4.20) and (4.26) in (4.18) we have

$$\begin{aligned} -(\tilde{\nabla}_W \tilde{S})(Y,\xi) + \alpha g(\phi Y,W) - (\alpha^2 - \beta^2 - \beta)g(\phi W,\phi Y) \\ = A(W)[-\tilde{S}(Y,\xi) + \eta(\tilde{R}(\xi,Y)\xi)] - 2B(W)\eta(Y). \end{aligned} \tag{4.27}$$

Now,

$$(\tilde{\nabla}_W \tilde{S})(Y,\xi) = \tilde{\nabla}_W \tilde{S}(Y,\xi) - \tilde{S}(\tilde{\nabla}_W Y,\xi) - \tilde{S}(Y,\tilde{\nabla}_W \xi).$$

After brief calculations the equation (4.27) gives

$$\begin{aligned}(\tilde{\nabla}_W \tilde{S})(Y,\xi) &= 2(\alpha^2-\beta^2-\beta)\alpha g(\phi Y,W) + 2\beta(\alpha^2-\beta^2-\beta)g(\phi Y,\phi W)\\ &+\alpha S(\phi Y,W) + \alpha^2 g(\phi Y,\phi W) - \alpha(3\beta+1)g(\phi Y,W) - (\beta+1)S(Y,W)\\ &-\alpha(\beta+1)g(\phi Y,W) - (\beta+1)^2\eta(Y)\eta(W) + 2(\beta+1)(\alpha^2-\beta^2-\beta)\eta(Y)\eta(W)\\ &+ (3\beta+1)(\beta+1)g(Y,W) + 2(\alpha^2-\beta^2-\beta)g(\phi Y,\phi W). \end{aligned} \tag{4.28}$$

From the equation (4.28) we get

$$\begin{aligned}&\alpha g(\phi Y,W) - (\alpha^2-\beta^2-\beta)g(\phi W,\phi Y) - [2(\alpha^2-\beta^2-\beta)\alpha g(\phi Y,W)\\ &+2\beta(\alpha^2-\beta^2-\beta)g(\phi Y,\phi W) + \alpha S(\phi Y,W) + \alpha^2 g(\phi Y,\phi W)\\ &-\alpha(3\beta+1)g(\phi Y,W) - (\beta+1)S(Y,W) - \alpha(\beta+1)g(\phi Y,W) - (\beta+1)^2\eta(Y)\eta(W)\\ &+2(\beta+1)(\alpha^2-\beta^2-\beta)\eta(Y)\eta(W) + (3\beta+1)(\beta+1)g(Y,W) + 2(\alpha^2-\beta^2-\beta)g(\phi Y,\phi W)]\\ &= [-2A(W)(\alpha^2-\beta^2-\beta) - 2B(W)]\eta(Y). \end{aligned} \tag{4.29}$$

Replacing $Y=\xi$ in (4.29) we obtain

$$A(W)[\alpha^2-\beta^2-\beta] + B(W) = 0. \tag{4.30}$$

From, (4.29) we have

$$\begin{aligned}&\alpha g(\phi Y,W) - (\alpha^2-\beta^2-\beta)g(\phi W,\phi Y) - [2(\alpha^2-\beta^2-\beta)\alpha g(\phi Y,W)\\ &+2\beta(\alpha^2-\beta^2-\beta)g(\phi Y,\phi W) + \alpha S(\phi Y,W) + \alpha^2 g(\phi Y,\phi W) - \alpha(3\beta+1)g(\phi Y,W)\\ &-\alpha(\beta+1)g(\phi Y,W) - (\beta+1)^2\eta(Y)\eta(W) + 2(\beta+1)(\alpha^2-\beta^2-\beta)\eta(Y)\eta(W)\\ &- (\beta+1)S(Y,W) + (3\beta+1)(\beta+1)g(Y,W) + 2(\alpha^2-\beta^2-\beta)g(\phi Y,\phi W)] = 0. \end{aligned} \tag{4.31}$$

Interchanging Y and W and then adding we get

$$S(Y,W) = ag(Y,W) + b\eta(Y)\eta(W),$$

where $a = \frac{1}{(\beta+1)}[\alpha^2 + (3\beta+1)(\beta+1) + (\alpha^2-\beta^2-\beta)(2\beta+3)]$ and $b = \frac{1}{(\beta+1)}[2(\beta+1)(\alpha^2-\beta^2-\beta) - (\beta+1)^2 - (\alpha^2-\beta^2-\beta)(2\beta+3) - \alpha^2]$ with $\beta \neq -1$. Hence M^3 is an $\eta-$Einstein manifold with $\beta \neq -1$. □

5. Conharmonic Curvature Tensor on a 3-dimensional Trans-Sasakian Manifold with respect to Semi-Symmetric Metric Connection

A Conharmonic curvature tensor has been studied by Ozgur [12], Siddiqui and Ahsan [14], Tarafdar and Bhattacharyya [17] and many other authors. In almost contact manifold M of dimension $n \geq 3$, the conharmonic curvature tensor $\tilde{L}$ with respect to semi-symmetric connection $\tilde{\nabla}$ is given by

$$\tilde{L}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{n-2}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y + g(Y,Z)\tilde{Q}X - g(X,Z)\tilde{Q}Y], \tag{5.1}$$

for $X,Y,Z \in \chi(M)$, where $\tilde{R},\tilde{S},\tilde{Q}$ are the Riemannian curvature tensor, Ricci tensor and the Ricci operator with respect to semi-symmetric connection $\tilde{\nabla}$, respectively.

A conharmonic curvature tensor $\tilde{L}$ with respect to semi-symmetric connection $\tilde{\nabla}$ is said to flat if it vanishes identically with respect to the connection $\tilde{\nabla}$.

Theorem 5.1. *If a 3-dimensional trans-Sasakian manifold with semi-symmetric metric connection admitting a conharmonic curvature tensor and a non-zero Ricci tensor satisfies $\tilde{L}(X,Y)\cdot\tilde{S}=0$, then the non-zero eigen values of the endomorphism $\tilde{Q}$ of the tangent space corresponding to $\tilde{S}$ are $2(\alpha^2-\beta^2-\beta)$ and $(\beta^2-\alpha^2+\beta)$, where α,β are smooth functions on M^3.*

Proof: Consider, a 3-dimensional trans-Sasakian manifold with respect to a semi-symmetric metric connection satisfying the condition $\tilde{L}(X,Y)\cdot\tilde{S}=0$.

Thus, we have

$$\tilde{S}(\tilde{L}(\xi,X)Y,Z)+\tilde{S}(Y,\tilde{L}(\xi,X)Z)=0. \tag{5.2}$$

Hence, from the above equation we get

$$\begin{aligned}(\alpha^2-\beta^2-\beta)g(X,Y)\tilde{S}(\xi,Z)-(\alpha^2-\beta^2-\beta)\eta(Y)\tilde{S}(X,Z)+\alpha g(\phi X,Y)\tilde{S}(\xi,Z)\\ -\alpha\eta(Y)\tilde{S}(\phi X,Z)-\tilde{S}(X,Y)\tilde{S}(\xi,Z)+\tilde{S}(\xi,Y)\tilde{S}(X,Z)-g(X,Y)\tilde{S}(\tilde{Q}\xi,Z)\\ +g(\xi,Y)\tilde{S}(\tilde{Q}X,Z)-\alpha\eta(Z)\tilde{S}(Y,\phi X)+(\alpha^2-\beta^2-\beta)g(X,Z)\tilde{S}(Y,\xi)\\ -(\alpha^2-\beta^2-\beta)\eta(Z)\tilde{S}(Y,X)+\alpha g(\phi X,Z)\tilde{S}(Y,\xi)-\tilde{S}(X,Z)\tilde{S}(Y,\xi)+\tilde{S}(\xi,Z)\tilde{S}(Y,X)\\ g(X,Z)\tilde{S}(Y,\tilde{Q}\xi)+g(\xi,Z)\tilde{S}(Y,\tilde{Q}X)=0\end{aligned} \tag{5.3}$$

Let $\tilde{\lambda}$ be the eigenvalue of the endomorphism $\tilde{Q}$ corresponding to an eigenvector X. Then

$$\tilde{Q}X=\tilde{\lambda}X. \tag{5.4}$$

And also, we have

$$g(\tilde{Q}X,Y)=\tilde{S}(X,Y)=\tilde{\lambda}g(X,Y).$$

From the equation (2.27) we get

$$\tilde{S}(Z,Y)=\tilde{S}(Y,Z)+2\alpha g(\phi Z,Y). \tag{5.5}$$

Now, putting $Y=Z=\xi$ and using the equations (5.4) and (5.5) in the equation (5.3) we get

$$\tilde{\lambda}=2(\alpha^2-\beta^2-\beta),(\beta^2-\alpha^2+\beta). \tag{5.6}$$

Hence the theorem is proved. □

6. Example of 3-dimensional Trans-Sasakian Manifold with respect to Semi-Symmetric Metric Connection

Example 6.1: Let $M=\{(x,y,z)\in R^3: y,z\neq 0\}$ where (x,y,z) are the standard coordinates in R^3. The linearly independent vector fields are given by

$$e_1=\frac{\partial}{\partial y}, e_2=(\frac{\partial}{\partial z}+2y\frac{\partial}{\partial x}), e_3=\frac{\partial}{\partial x}$$

. Let g be the Riemannian metric defined by

$$g_{ij}=\begin{cases}1 & \text{for} \quad i=j,\\ 0 & \text{for} \quad i\neq j.\end{cases}$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M^3)$. Let ϕ be the $(1, 1)$ tensor field defined by $\phi(e_1) = e_2, \phi(e_2) = -e_1, \phi(e_3) = 0$. Then using the linearity property of ϕ and g we have,

$$\eta(e_3) = 1, \phi^2(Z) = -Z + \eta(Z)e_3, g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M^3)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M. Now, after some calculations we have,

$$[e_1, e_3] = 0, [e_1, e_2] = 2e_3, [e_2, e_3] = 0.$$

By Koszul's formula we get,

$$\nabla_{e_1}e_1 = 0, \nabla_{e_2}e_1 = -e_3, \nabla_{e_3}e_1 = -e_2, \nabla_{e_1}e_2 = e_3, \nabla_{e_2}e_2 = 0,$$

$$\nabla_{e_3}e_2 = e_1, \nabla_{e_1}e_3 = -e_2, \nabla_{e_2}e_3 = e_1, \nabla_{e_3}e_3 = 0.$$

From the above it can be easily shown that $M^3(\phi, \xi, \eta, g)$ is a trans-Sasakian manifold of type $(1, 0)$. Now we consider a linear connection $\tilde{\nabla}$ such that

$$\tilde{\nabla}_{e_i}e_j = \nabla_{e_i}e_j + \eta(e_j)e_i - g(e_i, e_j)e_3, \forall i, j = 1, 2, 3.$$

It is easily seen that

$$\tilde{\nabla}_{e_1}e_1 = -e_3, \tilde{\nabla}_{e_2}e_1 = -e_3, \tilde{\nabla}_{e_3}e_1 = -e_2, \tilde{\nabla}_{e_1}e_2 = e_3, \tilde{\nabla}_{e_2}e_2 = -e_3,$$

$$\tilde{\nabla}_{e_3}e_2 = e_1, \tilde{\nabla}_{e_1}e_3 = -e_2 + e_1, \tilde{\nabla}_{e_2}e_3 = e_1 + e_2, \tilde{\nabla}_{e_3}e_3 = 0.$$

If $\tilde{T}$ is the torsion tensor of the connection $\tilde{\nabla}$, then we have $\tilde{T}(X, Y) = \eta(Y)X - \eta(X)Y$ and $(\tilde{\nabla}_X, g)(Y, Z)$ $= 0$, which implies that $\tilde{\nabla}$ is a semi-symmetric metric connection on M.

7. Conclusion

The notion of curvatures play an important role in the differential geometry and also in physics. According to Newton's laws, the magnitude of a force required to move an object at constant speed along a curve path is a constant multiple of the curvature of the trajectory [10]. Here we study some curvature properties on 3-dimensional tran-Sasakian manifold with respect to semi-symmetric metric connection. We construct that an extended generalized ϕ-recurrent 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection becomes an η -Einstein manifold under some certain condition. We explain geometric properties of curvature tensors on 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection.

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