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# On a Characterization of Commutativity for Prime Rings via Endomorphisms

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ABSTRACT: Our aim in the present paper is to introduce new classes of endomorphisms and study their connection with commutativity of prime rings with involution of the second kind. Furthermore, we provide examples to show that the various restrictions imposed in the hypotheses of our theorems are not superfluous.

Key Words: Prime ring, Involution, Commutativity, Endomorphisms.

## Contents



## 1. Introduction

Throughout the present paper $R$ will denote an associative ring with center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$; while the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. Recall that $R$ is *prime* if $aRb = \{0\}$ implies $a = 0$ or $b = 0$, $R$ is called *semiprime* if, for $x \in R$, $xRx = \{0\}$ implies that $x = 0$ and $R$ is said to be 2-*torsion free* if $2x = 0$, $x \in R$, implies $x = 0$.

An additive map $* : R \longrightarrow R$ is called an *involution* if $*$ is an anti-automorphism of order 2. An element $x$ in a ring with involution $(R, *)$ is said to be *hermitian* if $x^* = x$ and *skew-hermitian* if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of $R$ will be denoted by $H(R)$ and $S(R)$, respectively. The involution is said to be of the *first kind* if $Z(R) \subseteq H(R)$, otherwise it is said to be of the *second kind*. In the later case it is straightforward to check that $S(R) \cap Z(R) \neq \{0\}$.
A mapping $F : R \longrightarrow R$ is called commutativity preserving on a subset $S$ of $R$ if $[x, y] = 0$ implies that $[F(x), F(y)] = 0$, for all $x, y \in S$. The mapping $F$ is called strong commutativity preserving (SCP) on $S$ if $[F(x), F(y)] = [x, y]$ for all $x, y \in S$.

Given the importance of some additive mappings and their relations with the global structure of the ring, a considerable amount of work has been done on this line, for example, derivations, homomorphisms, endomorphisms and related maps during the last decades ( see for example [4] and [6]). A well known result due to Posner [9] states that if $d$ is a derivation of a prime ring $R$ such that $[d(x), x] \in Z(R)$ for all $x \in R$, then either $d = 0$ or $R$ is commutative. In [5] Lanski generalizes the result of Posner by considering a derivation $d$ such that $[d(x), x] \in Z(R)$ for all $x$ in a nonzero Lie ideal $U$ of $R$.
Inspired by these works, Bell and Daif ( [1], [2]) investigated commutativity of prime and semiprime rings admitting derivations and endomorphisms, which are SCP on its certain subset.

Recently, the authors in [7] introduced new classes of endomorphisms and studied their connection with commutativity of prime rings with involution of the second kind. Moreover, they provide a complete description and classification for some of these endomorphisms. In fact, they introduced the concepts of $*$-SCP and $*$-Skew SCP mappings as follows:

Let $R$ be a ring and let $g : R \longrightarrow R$ be an endomorphism. Then $g$ is called strong anti-commutativity preserving (SACP) if $g(x) \circ g(y) = x \circ y$ for all $x, y \in R$. If $R$ is equipped with an involution $*$, then $g$ is called $*$-SCP (resp. $*$-Skew SCP) if $[g(x), g(x^*)] = [x, x^*]$ (resp. $[g(x), g(x^*)] = -[x, x^*]$) for all $x \in R$. Moreover, if $g$ satisfies $g(x) \circ g(x^*) = x \circ x^*$ for all $x \in R$, then $g$ is said to be $*$-SACP).

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Motivated by [7], the aim of the present paper is to introduce more general classes of endomorphisms and study their connection with commutativity of prime rings with involution of the second kind. Moreover, we will provide examples to prove that our results cannot be extended to semi-prime rings.

## 2. Endomorphisms with identity on commutator

We will use frequently the following facts which are very crucial for developing the proofs of our main results.

**Fact 1.** Let $(R,*)$ be a 2-torsion free prime ring with involution of the second kind.
If $[[x,x^*],a]\in Z(R)$ for all $x\in R$, then $a\in Z(R)$.

**Fact 2.** Let $(R,*)$ be a 2-torsion free prime ring with involution of the second kind.
If $[x\circ x^*,a]\in Z(R)$ for all $x\in R$, then $a\in Z(R)$.

**Theorem 2.1.** *Let $(R,*)$ be a 2-torsion free prime ring with involution of the second kind and $T$ is a nontrivial endomorphism of $R$, then $R$ is a commutative integral domain if and only if $T[x,x^*]-[x,x^*]\in Z(R)$ for all $x\in R$.*

*Proof.* For the nontrivial implication, we are given that

$$T[x,x^*]-[x,x^*]\in Z(R)\quad\text{for all}\ \ x\in R.\tag{2.1}$$

Replacing $x$ by $x+y^*$ in (2.1), we find that

$$T[x,y]-[x,y]+T[y^*,x^*]-[y^*,x^*]\in Z(R)\quad\text{for all}\ \ x,y\in R.\tag{2.2}$$

Replacing $y$ by $yh$, where $h\in Z(R)\cap H(R)\backslash\{0\}$ and using the last equation, we obtain

$$(T[x,y]+T[y^*,x^*])T(h)-([x,y]+[y^*,x^*])h\in Z(R)\quad\text{for all}\ \ x,y\in R.\tag{2.3}$$

In view of (2.1), we can see that $\big[[x,x^*],T(z)\big]=0$ for all $z\in Z(R)$, that is $T(z)\in Z(R)$ for all $z\in Z(R)$ by Fact 1. Multiplying (2.2) by $T(h)$ and comparing it with (2.3), we arrive at

$$([x,y]+[y^*,x^*])(T(h)-h)\in Z(R)\quad\text{for all}\ \ x,y\in R.\tag{2.4}$$

In light of the primeness of $R$, we have either $[x,y]+[y^*,x^*]\in Z(R)$ or $T(h)=h$.
If $[x,y]+[y^*,x^*]\in Z(R)$ for all $x,y\in R$, putting $y=ys$, where $s$ is a nonzero element in $Z(R)\cap S(R)$, we get $[x,y]-[y^*,x^*]\in Z(R)$. Accordingly, $[x,y]\in Z(R)$ for all $x,y\in R$ which proves that $R$ is a commutative integral domain.
If $T(h)=h$ for all $h\in Z(R)\cap H(R)$, hence $T(s^2)=s^2$ for all $s\in S(R)\cap Z(R)$ therefore $(T(s)-s)(T(s)+s)=0$ and thus $(T(s)-s)R(T(s)+s)=\{0\}$ for all $s\in S(R)\cap Z(R)$. It follows that $T(s)=s$ or $T(s)=-s$. Using Brauer's trick we conclude that either $T(s)=s$ for all $s\in S(R)\cap Z(R)$ or $T(s)=-s$ for all $s\in S(R)\cap Z(R)$.
If $T(s)=s$ for all $s\in S(R)\cap Z(R)$; substituting $ys$ for $y$ in equation (2.2), where $s\in Z(R)\cap S(R)\backslash\{0\}$, we get

$$T[x,y]-T[y^*,x^*]-[x,y]+[y^*,x^*]\in Z(R)\quad\text{for all}\ \ x,y\in R.\tag{2.5}$$

Comparing Eqs (2.2) and (2.5), we may write

$$T[x,y]-[x,y]\in Z(R)\quad\text{for all}\ \ x,y\in R.\tag{2.6}$$

And thus $R$ is commutative by ([3], Theorem 3).
Assume that $T(s)=-s$ for all $s\in S(R)\cap Z(R)$. Replacing $y$ by $ys$ in equation (2.2), where $s\in Z(R)\cap S(R)\backslash\{0\}$, we find that

$$T[x,y]+[x,y]-T[y^*,x^*]-[y^*,x^*]\in Z(R)\quad\text{for all}\ \ x,y\in R.\tag{2.7}$$

Using equations (2.2) together with (2.7), we conclude that

$$T[x,y]-[x,y]^* \in Z(R) \quad \text{for all } \ x,y \in R. \tag{2.8}$$

In (2.8) we substitute $[y,r]^*$ for $y$, to get

$$\big[T(x)+x^*,[y,r]\big] \in Z(R) \quad \text{for all } \ x,y,r \in R.$$

In consequence of which $T(x)+x^* \in Z(R)$ for all $x \in R$, by Fact 1., this yields that

$$T[x,x^*]+[x,x^*] \in Z(R) \quad \text{for all } \ x \in R. \tag{2.9}$$

Thereby $[x,x^*] \in Z(R)$ for all $x \in R$, forces that $R$ is commutative by [ [8], Lemma 2.1.] $\square$

As an application of our theorem, the following corollary improves the result of ( [1], Corollary 2) for the case when the underlying identity belongs to the center of a prime ring with involution of the second kind.

**Corollary 2.2.** *Let $(R,*)$ be a 2-torsion free prime ring with involution of the second kind and $T$ is a nontrivial endomorphism of $R$, then $R$ is a commutative integral domain if and only if $T[x,y]-[x,y] \in Z(R)$ for all $x,y \in R$.*

**Theorem 2.3.** *Let $(R,*)$ be a 2-torsion free prime ring with involution of the second kind and $T$ is an endomorphism of $R$, then $R$ is a commutative integral domain if and only if $T[x,x^*]+[x,x^*] \in Z(R)$ for all $x \in R$.*

*Proof.* For the nontrivial implication, suppose that

$$T[x,x^*]+[x,x^*] \in Z(R) \quad \text{for all } \ x \in R. \tag{2.10}$$

Linearizing (2.10), one can see that

$$T[x,y]+[x,y]+T[y^*,x^*]+[y^*,x^*] \in Z(R) \quad \text{for all } \ x,y \in R. \tag{2.11}$$

Putting $yh$ for $y$ in (2.11), where $h \in Z(R)\cap H(R)\backslash\{0\}$; it follows that

$$(T[x,y]+T[y^*,x^*])T(h)+([x,y]+[y^*,x^*])h \in Z(R) \quad \text{for all } \ x,y \in R. \tag{2.12}$$

Arguing as above we get $T(z) \in Z(R)$ for all $z \in Z(R)$. Multiplying (2.11) by $T(h)$, and subtracting the result from equation (2.12), we conclude that

$$([x,y]+[y^*,x^*])(T(h)-h) \in Z(R) \quad \text{for all } \ x,y \in R. \tag{2.13}$$

Using the same techniques, we get that $R$ is a commutative integral domain, $T(s)=s$ for all $s \in S(R)\cap Z(R)$ or $T(s)=-s$ for all $s \in S(R)\cap Z(R)$.
If $T(s)=s$ for all $s \in S(R)\cap Z(R)$, then substituting $ys$ for $y$ in equation (2.11), where $s \in Z(R)\cap S(R)\backslash\{0\}$, to obtain

$$T[x,y]+[x,y]-T[y^*,x^*]-[y^*,x^*] \in Z(R) \quad \text{for all } \ x,y \in R. \tag{2.14}$$

Using equations (2.11) together with (2.14), we conclude that

$$T[x,y]+[x,y] \in Z(R) \quad \text{for all } \ x,y \in R.$$

Replacing $x$ by $[x,x^*]$ and $y$ by $[y,y^*]$ in the last equation, we get

$$T[[x,x^*],[y,y^*]]+[[x,x^*],[y,y^*]] \in Z(R) \quad \text{for all } \ x,y \in R,$$

which reduces to

$$[[x,x^*],[y,y^*]] \in Z(R) \quad \text{for all } \ x,y \in R.$$

By virtue of Fact 1., the last equation implies that $[x,x^*] \in Z(R)$ for all $x \in R$ and thus $R$ is commutative by [ [8], Lemma 2.1].
If $T(s) = -s$ for all $s \in S(R) \cap Z(R)$, then Replacing $y$ by $ys$ in equation (2.11), where $s \in Z(R) \cap S(R) \backslash \{0\}$, we arrive at

$$T[x,y] - [x,y] - T[y^*,x^*] + [y^*,x^*] \in Z(R) \quad \text{for all } \ x,y \in R. \tag{2.15}$$

Combining (2.11) with (2.15), we may conclude that

$$T[x,y] + [x,y]^* \in Z(R) \quad \text{for all } \ x,y \in R. \tag{2.16}$$

In (2.16) we substitute $[y,y^*]$ for $y$, to get

$$[[y,y^*],T(x)+x^*] \in Z(R) \quad \text{for all } \ x,y \in R. \tag{2.17}$$

Then $T(x)+x^* \in Z(R)$; for all $x \in R$. So, we have

$$[T(x),y] = -[x^*,y] \quad \text{for all } \ x,y \in R$$

in such a way that

$$T[x,y] = [y,x]^* \text{ for all } \ x,y \in R. \tag{2.18}$$

Substituting $yx$ for $y$, one can see that

$$T([x,y]x) = x^*[y,x]^* \text{ for all } \ x,y \in R.$$

Using equation (2.18) we have

$$[y,x]^*T(x) = x^*[y,x]^* \text{ for all } \ x,y \in R.$$

In particular for $x = [u,v]$ where $u,v \in R$, one can see that

$$[y,[u,v]]^*[v,u]^* = [u,v]^*[y,[u,v]]^* \text{ for all } \ u,v,y \in R,$$

so that

$$[v,u][y,[u,v]] = [y,[u,v]][u,v] \text{ for all } \ u,v,y \in R$$

and thus

$$[[u,v],y] \circ [u,v] = 0 \text{ for all } \ u,v,y \in R.$$

Which implies that $[[u,v]^2,y] = 0$ for all $u,v,y \in R$ and thus

$$[u,v]^2 \in Z(R) \text{ for all } \ u,v \in R. \tag{2.19}$$

Linearizing the last equation, one can see that

$$[u,v][u,w] + [u,w][u,v] \in Z(R) \text{ for all } \ u,v,w \in R. \tag{2.20}$$

Let us fix an element $u \in R$ and consider $d_u$ the inner derivation induced by $u$, then using (2.20) we obtain

$$d_u(v) \circ d_u(w) \in Z(R) \text{ for all } \ v,w \in R. \tag{2.21}$$

In view of ( [8], Corollary 3.6.) the last equation implies that $R$ is commutative or $d_u = 0$. Hence the latter case yields to $[u,v] = 0$ for all $u,v \in R$ and thus $R$ is commutative integral domain. This completes the proof of our theorem. □

**Corollary 2.4.** *Let $(R,*)$ be a 2-torsion free prime ring with involution of the second kind and $T$ is an endomorphism of $R$, then $R$ is a commutative integral domain if and only if $T[x,y]+[x,y]\in Z(R)$ for all $x,y\in R$.*

The following example shows that the **primeness** hypothesis in Theorems 2.1 and 2.3 is not superfluous. In particular, our theorems cannot be extended to semi-prime rings.

**Example 2.5.** *Let us consider $R=M_2(\mathbb{Z})$ and define*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

*Then $(R,*)$ is a prime ring with involution of the first kind such that $[x,x^*]=0\;\forall\, x\in R$.*
*Set $\mathcal{R}=R\times\mathbb{C}$, then it is obvious to verify that $(\mathcal{R},\sigma)$ is a semi-prime ring with involution of the second kind where $\sigma(r,z)=(r^*,\bar{z})$. Furthermore, all arbitrary endomorphism satisfied the conditions of Theorems 2.1 and 2.3. But $\mathcal{R}$ is not a commutative ring.*

The following example proves that the condition "* is of the **second kind**" is necessary in Theorems 2.1 and 2.3.

**Example 2.6.** *Let $R=M_2(\mathbb{Z})$ and let $*$ be the involution of the first kind defined in Example 1. It is straightforward to check that all homomorphisms fulfilled the conditions of Theorems 2.1 and 2.3. However $R$ is not a commutative ring.*

## 3. Endomorphisms with identity on anti-commutator

Our purpose in this section is to treat the commutativity of $R$ in case the commutator in the preceding theorems is replaced by anti-commutator.

**Theorem 3.1.** *Let $(R,*)$ be a 2-torsion free prime ring with involution of the second kind and $T$ is a nontrivial endomorphism of $R$, then $R$ is a commutative integral domain if and only if $T(x\circ x^*)-(x\circ x^*)\in Z(R)$ for all $x\in R$.*

*Proof.* For the nontrivial implication, assume that

$$T(x\circ x^*)-(x\circ x^*)\in Z(R)\quad\text{for all } x\in R. \tag{3.1}$$

Linearizing (3.1), we get

$$T(x\circ y)+T(y^*\circ x^*)-(x\circ y)-(y^*\circ x^*)\in Z(R)\quad\text{for all } x,y\in R. \tag{3.2}$$

Substituting $yh$ for $y$ in (3.2), where $h\in Z(R)\cap H(R)\backslash\{0\}$, we obtain

$$T(x\circ y)T(h)-(x\circ y)h+T(y^*\circ x^*)T(h)-(y^*\circ x^*)h\in Z(R) \tag{3.3}$$

Applying (3.1), we can see that $\big[x\circ x^*,T(z)\big]=0$ for all $x\in R$ and $z\in Z(R)$, then $T(z)\in Z(R)$ for all $z\in Z(R)$, by Fact 2. Using the same techniques as above, we get

$$\big(x\circ y+x^*\circ y^*\big)\big(T(h)-h\big)\in Z(R)\quad\text{for all } x,y\in R. \tag{3.4}$$

Using the primeness hypothesis, it follows that $x\circ y+x^*\circ y^*\in Z(R)$ or $T(h)-h=0$.
Suppose that

$$x\circ y+x^*\circ y^*\in Z(R)\quad\text{for all } x,y\in R. \tag{3.5}$$

Replacing $y$ by $h$ in (3.5), where $h\in Z(R)\cap H(R)\backslash\{0\}$, it's obvious to verify that

$$(x+x^*)h\in Z(R)\quad\text{for all } x\in R \tag{3.6}$$

in consequence of which

$$x+x^*\in Z(R)\quad\text{for all } x\in R. \tag{3.7}$$

Replacing again $y$ by $s$ in (3.5), where $s \in Z(R) \cap S(R) \backslash \{0\}$, then

$$x - x^* \in Z(R) \quad \text{for all} \;\; x, y \in R. \tag{3.8}$$

Using equations (3.7) and (3.8) one can easily see that $R$ is commutative.
Now if $T(h) - h = 0$ for all $h \in Z(R) \cap H(R)$, the following two cases must be distinguished
$T(s) - s = 0 \quad$ for all $\; s \in Z(R) \cap S(R)$. Replacing $y$ by $ys$ in equation (3.2), where $s \in Z(R) \cap S(R) \backslash \{0\}$, we get

$$\big(T(x \circ y) - (x \circ y) - T(y^* \circ x^*) + (y^* \circ x^*)\big)s \in Z(R),$$

which reduces to

$$T(x \circ y) - (x \circ y) - T(y^* \circ x^*) + (y^* \circ x^*) \in Z(R) \quad \text{for all} \;\; x, y \in R. \tag{3.9}$$

Using equations (3.2) together with (3.9), we conclude that

$$T(x \circ y) - (x \circ y) \in Z(R) \quad \text{for all} \;\; x, y \in R.$$

Replacing $y$ in the last equation by a non zero element of $Z(R) \cap H(R)$ , we find that

$$T(x) - x \in Z(R) \quad \text{for all} \;\; x \in R. \tag{3.10}$$

thereby obtaining

$$T[x, y] - [x, y] \in Z(R) \quad \text{for all} \;\; x, y \in R. \tag{3.11}$$

and thus $R$ is a commutative integral domain by ( [3], Theorem 3).
$T(s) + s = 0 \quad$ for all $\; s \in Z(R) \cap S(R)$. Taking $y = ys$ in equation (3.2), where $s \in Z(R) \cap S(R) \backslash \{0\}$, we get

$$T(x \circ y) + (x \circ y) - T(y^* \circ x^*) - (y^* \circ x^*) \in Z(R) \quad \text{for all} \;\; x, y \in R. \tag{3.12}$$

Using equations (3.2) together with (3.12), we conclude that

$$T(x \circ y) - (x \circ y)^* \in Z(R) \quad \text{for all} \;\; x, y \in R.$$

Replacing $y$ by $h$ in the last equation, where $h \in Z(R) \cap H(R) \backslash \{0\}$, we obtain

$$T(x) - x^* \in Z(R) \quad \text{for all} \;\; x \in R. \tag{3.13}$$

Accordingly, we arrive at

$$[T(x), y] = [x^*, y] \quad \text{for all} \;\; x, y \in R$$

in such a way that

$$T[x, y] = [y, x]^* \text{ for all } \; x, y \in R. \tag{3.14}$$

Since (3.14) is the same as (2.18). So, we may argue as before that $R$ is a commutative integral domain. □

A slight modification in the proof of Theorem 3.1 yields the following result.

**Theorem 3.2.** *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind and $T$ is an endomorphism of $R$, then $R$ is a commutative integral domain if and only if $T(x \circ x^*) + (x \circ x^*) \in Z(R)$ for all $\; x \in R$.*

The following example proves that the condition "* is of the **second kind**" is necessary in Theorems 3.1 and 3.2.

**Example 3.3.** *Let us consider $R = M_2(\mathbb{Z})$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. It is straightforward to check that $(R, *)$ is a prime ring with involution of the first kind. Moreover, for all $x \in R$ we have*

$$x \circ x^* = \begin{pmatrix} 2(ad - bc) & 0 \\ 0 & 2(ad - bc) \end{pmatrix} \in Z(R)$$

*and thus $T = 0$ satisfies the conditions of Theorems 3.1 and 3.2. However, $R$ is a non commutative ring.*

The following example proves that the **primeness** hypothesis in Theorems 3.1 and 3.2 is not superfluous. In particular, our theorems cannot be extended to semi-prime rings.

**Example 3.4.** *Let us consider $(R, *)$ as in the preceding example. Let $(S, \sigma)$ be a commutative ring with involution of the second kind (for example the field of complex numbers with the conjugation involution). If we set $\mathcal{R} = R \times S$, then it is obvious to verify that $(\mathcal{R}, \tau)$ is a semi-prime ring with involution of the second kind where*

$$\tau(r, s) = (r^*, \sigma(s)) \quad \textit{for all} \quad (r, s) \in \mathcal{R}.$$

*Furthermore, the zero endomorphism satisfies conditions of Theorems 3.1 and 3.2. But $\mathcal{R}$ is a noncommutative ring.*

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