



Fractional Calculus Operators Pertaining to Multivariable Aleph-function

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ABSTRACT: In this paper we study a pair of unified and extended fractional integral operator involving the multivariable Aleph-function, Aleph-function and general class of polynomials. During this study, we establish five theorems pertaining to Mellin transforms of these operators. Further, some properties of these operators have also been investigated. On account of the general nature of the functions involved herein, a large number of (known and new) fractional integral operators involving simpler functions can also be obtained. We will quote the particular case concerning the multivariable I -function defined by Sharma and Ahmad [27] and the I -function of one variable defined by Saxena [20].

Key Words: Multivariable Aleph-function, Aleph-function of one variable, fractional integral, general class of polynomials, Mellin transform, I -function, multivariable I -function.

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1. Introduction and Preliminaries

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. Recently, it has turned out many phenomena in physics, mechanics, chemistry, biology and other sciences can be described very successfully by models using mathematical tools from fractional calculus. Choi et al. [5], Daiya et al. [6], Kumar and Daiya [12], Kumar et al. [13] and others, have studied the fractional calculus pertaining to multivariable H -function defined by Srivastava and Panda [29]. Fractional calculus has many advance applications in different field of science and engineering, e.g. Quantum mechanics, Mathematical physics, Mathematical biology, Diffusion process, Mathematical modeling and many more (see, [3,4,10,14,17,18]).

The Aleph-function, introduced by Südländ et al. [30], however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral (see also, [1,2,11,15,26]):

$$\begin{aligned} \aleph(z) &= \aleph_{P_{i'}, Q_{i'}, c_{i'}; r'}^{M, N} \left[z \left| \begin{array}{l} (a_{j'}, A_{j'})_{1, n}, [c_{i'} (a_{j' i'}, A_{j' i'})]_{n+1, p_{i'}; r'} \\ (b_{j'}, B_{j'})_{1, m}, [c_{i'} (b_{j' i'}, B_{j' i'})]_{m+1, q_{i'}; r'} \end{array} \right. \right] \\ &= \frac{1}{2\pi\omega} \int_L \Omega_{P_{i'}, Q_{i'}, c_{i'}; r'}^{M, N}(s) z^{-s} ds, \end{aligned} \tag{1.1}$$

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for all z different to 0 and

$$\Omega_{P_{i'}, Q_{i'}, c_{i'}; r'}^{M, N}(s) = \frac{\prod_{j'=1}^M \Gamma(b_{j'} + B_{j'} s) \prod_{j'=1}^N \Gamma(1 - a_{j'} - A_{j'} s)}{\sum_{i'=1}^{r'} c_{i'} \left\{ \prod_{j'=N+1}^{P_{i'}} \Gamma(a_{j' i'} + A_{j' i'} s) \prod_{j'=M+1}^{Q_{i'}} \Gamma(1 - b_{j' i'} - B_{j' i'} s) \right\}}, \quad (1.2)$$

with $|\arg z| < \frac{1}{2}\pi\Omega$, where

$$\Omega = \sum_{j'=1}^M \beta_{j'} + \sum_{j'=1}^N \alpha_{j'} - c_{i'} \left(\sum_{j'=M+1}^{Q_{i'}} \beta_{j' i'} + \sum_{j'=N+1}^{P_{i'}} \alpha_{j' i'} \right) > 0 \quad (i = 1, \dots, r').$$

In 1972, Srivastava [28] introduced the general class of polynomials in the following manner:

$$S_N^M(x) = \sum_{k=0}^{[N/M]} \frac{(-N)_{Mk}}{k!} A_{N,k} x^k, \quad (N = 0, 1, 2, \dots), \quad (1.3)$$

where M is an arbitrary positive integer and the coefficient $A_{N,k}$ are arbitrary constants, real or complex. By suitably specialized the coefficient $A_{N,k}$ the polynomials $S_N^M(x)$ can be reduced to the classical orthogonal polynomials such as Jacobi, Hermite, Legendre and Laguerre polynomials etc.

The Aleph-function of several variables is an extension of the multivariable I -function defined by Sharma and Ahmad [27], itself is an a generalization of G - and H -functions of multiple variables. The multiple Mellin-Barnes integral occurring in this paper will be referred to as the multivariables Aleph-function throughout our present study and given by

$$\begin{aligned} & \aleph(z_1, \dots, z_r) \\ &= \aleph_{P_i, q_i, \tau_i; R; p_i(1), q_i(1), \tau_i(1); R^{(1)}; \dots; p_i(r), q_i(r), \tau_i(r); R^{(r)}}^{0, n; m_1, n_1, \dots, m_r, n_r} \left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_r \end{array} \middle| \begin{array}{c} \left[(a_j; \alpha_j^{(1)}, \dots, \alpha_j^{(r)})_{1, n} \right] \\ \dots \dots \dots \\ \dots \dots \dots \end{array} \right) \\ & \left[\tau_i \left(a_{ji}; \alpha_{ji}^{(1)}, \dots, \alpha_{ji}^{(r)} \right)_{n+1, p_i} \right] : \left[\left(c_j^{(1)}, \gamma_j^{(1)} \right)_{1, n_1} \right], \left[\tau_{i(1)} \left(c_{ji(1)}^{(1)}, \gamma_{ji(1)}^{(1)} \right)_{n_1+1, p_i^{(1)}} \right]; \\ & \left[\tau_i \left(b_{ji}; \beta_{ji}^{(1)}, \dots, \beta_{ji}^{(r)} \right)_{m+1, q_i} \right] : \left[\left(d_j^{(1)}, \delta_j^{(1)} \right)_{1, m_1} \right], \left[\tau_{i(1)} \left(d_{ji(1)}^{(1)}, \delta_{ji(1)}^{(1)} \right)_{m_1+1, q_i^{(1)}} \right]; \\ & \dots; \left[\left(c_j^{(r)}, \gamma_j^{(r)} \right)_{1, n_r} \right], \left[\tau_{i(r)} \left(c_{ji(r)}^{(r)}, \gamma_{ji(r)}^{(r)} \right)_{n_r+1, p_i^{(r)}} \right] \\ & \dots; \left[\left(d_j^{(r)}, \delta_j^{(r)} \right)_{1, m_r} \right], \left[\tau_{i(r)} \left(d_{ji(r)}^{(r)}, \delta_{ji(r)}^{(r)} \right)_{m_r+1, q_i^{(r)}} \right] \\ &= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \psi(s_1, \dots, s_r) \prod_{k=1}^r \theta_k(s_k) z_k^{s_k} ds_1 \dots ds_r, \quad (1.4) \end{aligned}$$

with $\omega = \sqrt{-1}$,

$$\psi(s) = \frac{\prod_{j=1}^n \Gamma(1 - a_j + \sum_{k=1}^r \alpha_j^{(k)} s_k)}{\sum_{i=1}^R \left\{ \tau_i \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \sum_{k=1}^r \alpha_{ji}^{(k)} s_k) \prod_{j=1}^{q_i} \Gamma(1 - b_{ji} + \sum_{k=1}^r \beta_{ji}^{(k)} s_k) \right\}} \quad (1.5)$$

where $s = (s_1, \dots, s_r)$ and

$$\begin{aligned} & \theta_k(s_k) \\ &= \frac{\prod_{j=1}^{m_k} \Gamma(d_j^{(k)} - \delta_j^{(k)} s_k) \prod_{j=1}^{n_k} \Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)}{\sum_{i(k)=1}^{R^{(k)}} \left\{ \tau_{i(k)} \prod_{j=m_k+1}^{p_{i(k)}} \Gamma(1 - d_{ji(k)}^{(k)} + \delta_{ji(k)}^{(k)} s_k) \prod_{j=n_k+1}^{q_{i(k)}} \Gamma(c_{ji(k)}^{(k)} - \gamma_{ji(k)}^{(k)} s_k) \right\}} \end{aligned}$$

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.4) can be obtained by extension of the corresponding conditions for multivariable H -function given by

$$|\arg z_k| < \frac{1}{2} A_i^{(k)} \pi,$$

where

$$\begin{aligned} A_i^{(k)} &= \sum_{j=1}^n \alpha_j^{(k)} - \tau_i \sum_{j=n+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}} \\ &+ \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}} > 0, \end{aligned} \quad (1.6)$$

with $k = 1, \dots, r$; $i = 1, \dots, R$; $i^{(k)} = 1, \dots, R^{(k)}$.

The complex numbers z_k are not zero. Throughout the present paper, we assume the existence and absolute convergence conditions of the multivariable Aleph-function. We set up the asymptotic expansion in the following convenient form:

$$\begin{aligned} \aleph(z_1, \dots, z_r) &= 0 (|z_1|^{\alpha_1}, \dots, |z_r|^{\alpha_r}), \max(|z_1|, \dots, |z_r|) \rightarrow 0 \\ \aleph(z_1, \dots, z_r) &= 0 (|z_1|^{\beta_1}, \dots, |z_r|^{\beta_r}), \min(|z_1|, \dots, |z_r|) \rightarrow \infty \end{aligned}$$

here $k = 1, \dots, r$; $\alpha_k = \min \left[\Re \left(d_j^{(k)} \delta_j^{(k)} \right) \right]$, $j = 1, \dots, m_k$ and

$\beta_k = \max \left[\Re \left((c_j^{(k)} - 1) / \gamma_j^{(k)} \right) \right]$, $j = 1, \dots, n_k$.

We will use the following notations in this paper:

$$A = \left\{ \left(a_j; \alpha_j^{(1)}, \dots, \alpha_j^{(r)} \right)_{1,n} \right\}, \left\{ \tau_i \left(a_{ji}; \alpha_{ji}^{(1)}, \dots, \alpha_{ji}^{(r)} \right)_{n+1, p_i} \right\}, \quad (1.7)$$

$$B = \left\{ \tau_i (b_{ji}; \beta_{ji}^{(1)}, \dots, \beta_{ji}^{(r)})_{m+1, q_i} \right\}, \quad (1.8)$$

$$\begin{aligned} C &= \left\{ \left(c_j^{(1)}; \gamma_j^{(1)} \right)_{1, n_1} \right\}, \left\{ \tau_{i^{(1)}} \left(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)} \right)_{n_1+1, p_{i^{(1)}}} \right\}; \dots; \\ &\left\{ \left(c_j^{(r)}; \gamma_j^{(r)} \right)_{1, n_r} \right\}, \left\{ \tau_{i^{(r)}} \left(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)} \right)_{n_r+1, p_{i^{(r)}}} \right\}, \end{aligned} \quad (1.9)$$

$$\begin{aligned} D &= \left\{ \left(d_j^{(1)}; \delta_j^{(1)} \right)_{1, m_1} \right\}, \left\{ \tau_{i^{(1)}} \left(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)} \right)_{m_1+1, q_{i^{(1)}}} \right\}; \dots; \\ &\left\{ \left(d_j^{(r)}; \delta_j^{(r)} \right)_{1, m_r} \right\}, \left\{ \tau_{i^{(r)}} \left(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)} \right)_{m_r+1, q_{i^{(r)}}} \right\}. \end{aligned} \quad (1.10)$$

The Mellin transform of $f(x)$ will be denoted by $\mathcal{M}[f(x)]$ or $F(s)$. If p and y are real, we write $s = p^{-1} + iy$. If $p \geq 1$, $f(x) \in L_p(0, \infty)$, then for $p = 1$ we have

$$\mathcal{M}[f(x)] = F(s) = \int_0^\infty x^{s-1} f(x) dx, \quad \text{and} \quad f(x) = \frac{1}{2i\pi} \int_L F(s) x^{-s} ds. \quad (1.11)$$

For $p > 0$, we have

$$\mathcal{M}[f(x)] = F(s) = l.i.m. \int_{1/x}^x x^{s-1} f(x) dx, \quad (1.12)$$

where *l.i.m.* denotes the usual limit in the mean for L_p -spaces.

2. Definitions

The pair of new extended fractional integral operators are defined by the following equations:

$$\begin{aligned}
Q_{\gamma_n}^{\alpha,\beta} [f(x)] &= tx^{-\alpha-t\beta-1} \int_0^x y^\alpha (x-t)^{\beta} \aleph_{U:W}^{0,n;V} \left(\begin{array}{c|c} \gamma_1 v_1 & A : C \\ \vdots & \vdots \\ \gamma_n v_n & B : D \end{array} \right) \\
&\times \left\{ \prod_{i=1}^r S_{N_i}^{M_i} \left[z_i \left(\frac{y^t}{x^t} \right)^{g_i} \left(1 - \frac{y^t}{x^t} \right)^{h_i} \right] \right\} \\
&\times \left\{ \prod_{j=1}^k \aleph_{P_{i'j}, Q_{i'j}, c_{i'j}; r'}^{M_j, N_j} \left[z_j \left(\frac{y^t}{x^t} \right)^{a_j} \left(1 - \frac{y^t}{x^t} \right)^{b_j} \right] \right\} \\
&\times \psi \left(\frac{y^t}{x^t} \right) f(y) dy, \tag{2.1}
\end{aligned}$$

$$\begin{aligned}
R_{\gamma_n}^{\rho,\beta} [f(x)] &= tx^\rho \int_x^\infty y^\alpha (x-t)^{\beta} \aleph_{U:W}^{0,n;V} \left(\begin{array}{c|c} \gamma_1 v_1 & A : C \\ \vdots & \vdots \\ \gamma_n v_n & B : D \end{array} \right) \\
&\times \left\{ \prod_{i=1}^r S_{N_i}^{M_i} \left[z_i \left(\frac{y^t}{x^t} \right)^{g_i} \left(1 - \frac{y^t}{x^t} \right)^{h_i} \right] \right\} \\
&\times \left\{ \prod_{j=1}^k \aleph_{P_{i'j}, Q_{i'j}, c_{i'j}; r'}^{M_j, N_j} \left[z_j \left(\frac{y^t}{x^t} \right)^{a_j} \left(1 - \frac{y^t}{x^t} \right)^{b_j} \right] \right\} \\
&\times \psi \left(\frac{y^t}{x^t} \right) f(y) dy, \tag{2.2}
\end{aligned}$$

where $v_i = \left(\frac{y^t}{x^t} \right)^{u_i} \left(1 - \frac{y^t}{x^t} \right)^{v_i}$, $\mu_i = \left(\frac{x^t}{y^t} \right)^{u_i} \left(1 - \frac{x^t}{y^t} \right)^{v_i}$; t, u_i and v_i, g_i, h_i, a_j and b_j are positive numbers. The kernels $\psi \left(\frac{y^t}{x^t} \right)$ and $\psi \left(\frac{x^t}{y^t} \right)$ appearing in (2.1) and (2.2) respectively, are assumed to be continuous functions such the integrals make sense for wide classes of function $f(x)$.

The conditions for existence of these operators are as follows:

- (a) $f(x) \in L_p(0, \infty)$,
- (b) $1 \leq p, q < \infty$, $p^{-1} + q^{-1} = 1$,
- (c) $\Re \left(\alpha + ta_j \frac{b_{j'j}}{B_{j'j}} \right) + t \sum_{i=1}^n u_i \min_{1 \leq j \leq m_i} \Re \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) > -q^{-1}$,
- (d) $\Re \left(\beta + tb_j \frac{b_{j'j}}{B_{j'j}} \right) + t \sum_{i=1}^n v_i \min_{1 \leq j \leq m_i} \Re \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) > -q^{-1}$,
- (e) $\Re \left(\rho + ta_j \frac{b_{j'j}}{B_{j'j}} \right) + t \sum_{i=1}^n u_i \min_{1 \leq j \leq m_i} \Re \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) > -p^{-1}$, where $(j = 1, \dots, k, j' = 1, \dots, M_j)$. The operators (2.1) and (2.2) are extensions of fractional integral operators defined and studied by several authors like Erdélyi [8], Love [16], Saigo et al. [19], Saxena and Kiryakova [21], Saxena and Kumbhat [23,24,25], etc.

3. Main Results

Theorem 3.1. *If $f(x) \in L_p(0, \infty)$, $1 \leq p \leq 2$; or $f(x) \in L_p(0, \infty)$, $p > 2$, also following conditions satisfied:*

$$p^{-1} + q^{-1} = 1, \Re \left(\alpha + ta_j \frac{b_{j'j}}{B_{j'j}} \right) + t \sum_{i=1}^n u_i \min_{1 \leq j \leq m_i} \Re \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) > -q^{-1},$$

$$\Re \left(\beta + tb_j \frac{b_{j'j}}{B_{j'j}} \right) + t \sum_{i=1}^n v_i \min_{1 \leq j \leq m_i} \Re \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) > -q^{-1},$$

and the integrals present are absolutely convergent, then

$$\mathcal{M} \left\{ Q_{\gamma_n}^{\alpha, \beta} [f(x)] \right\} = \mathcal{M} \{ f(x) \} R_{\gamma_n}^{\alpha-s+1, \beta} [1], \quad (3.1)$$

where $\mathcal{M}_p](0, \infty)$ stands for the class of all functions $f(x)$ of $L_p(0, \infty)$ with $p > 2$, which are inverse Mellin-transforms of the function of $L_p(-\infty, \infty)$.

Proof: By making use of the Mellin transform of (2.1), we have

$$\begin{aligned} \mathcal{M} \left\{ Q_{\gamma_n}^{\alpha, \beta} [f(x)] \right\} &= \int_0^\infty x^{s-1} \left\{ t x^{-\alpha-t\beta-1} \int_0^x y^\alpha (x-t)^{\beta} \aleph_{U;W}^{0, n; V} \left(\begin{array}{c|c} \gamma_1 \nu_1 & A : C \\ \vdots & \vdots \\ \gamma_n \nu_n & B : D \end{array} \right) \right. \\ &\times \left\{ \prod_{i=1}^r S_{N_i}^{M_i} \left[z_i \left(\frac{y^t}{x^t} \right)^{g_i} \left(1 - \frac{y^t}{x^t} \right)^{h_i} \right] \right\} \left\{ \prod_{j=1}^k \aleph_{P_{i'j}, Q_{i'j}, c_{i'j}; r'}^{M_j, N_j} \left[z_j \left(\frac{y^t}{x^t} \right)^{a_j} \left(1 - \frac{y^t}{x^t} \right)^{b_j} \right] \right\} \\ &\times \psi \left(\frac{y^t}{x^t} \right) f(y) dy \Big\} dx. \end{aligned} \quad (3.2)$$

On interchanging the order of integration, which is permissible under the conditions, the result (3.1) follows in view of (2.2). \square

Theorem 3.2. If $f(x) \in L_p(0, \infty)$, $1 \leq p \leq 2$; or $f(x) \in L_p(0, \infty)$, $p > 2$, also satisfied following conditions:

$$p^{-1} + q^{-1} = 1, \quad \Re \left(\beta + tb_j \frac{b_{j'j}}{B_{j'j}} \right) + t \sum_{i=1}^n v_i \min_{1 \leq j \leq m_i} \Re \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) > -q^{-1},$$

$$\Re \left(\rho + ta_j \frac{b_{j'j}}{B_{j'j}} \right) + t \sum_{i=1}^n u_i \min_{1 \leq j \leq m_i} \Re \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) > -p^{-1},$$

and the integrals present are absolutely convergent, then we have the following relation:

$$\mathcal{M} \left\{ R_{\gamma_n}^{\rho, \beta} [f(x)] \right\} = \mathcal{M} \{ f(x) \} Q_{\gamma_n}^{\rho+s-1, \beta} [1]. \quad (3.3)$$

Proof: Mellin transform of (2.2), we have

$$\begin{aligned} \mathcal{M} \left\{ R_{\gamma_n}^{\rho, \beta} [f(x)] \right\} &= \int_0^\infty x^{s-1} \left\{ t x^\rho \int_x^\infty y^{-\rho-t\beta-1} (y-t)^{\beta} \aleph_{U;W}^{0, n; V} \left(\begin{array}{c|c} \gamma_1 \mu_1 & A : C \\ \vdots & \vdots \\ \gamma_n \mu_n & B : D \end{array} \right) \right. \\ &\times \left\{ \prod_{i=1}^r S_{N_i}^{M_i} \left[z_i \left(\frac{x^t}{y^t} \right)^{g_i} \left(1 - \frac{x^t}{y^t} \right)^{h_i} \right] \right\} \left\{ \prod_{j=1}^k \aleph_{P_{i'j}, Q_{i'j}, c_{i'j}; r'}^{M_j, N_j} \left[z_j \left(\frac{x^t}{y^t} \right)^{a_j} \left(1 - \frac{x^t}{y^t} \right)^{b_j} \right] \right\} \\ &\times \psi \left(\frac{x^t}{y^t} \right) f(y) dy \Big\} dx. \end{aligned} \quad (3.4)$$

On interchanging the order of integration which is permissible under the conditions, then the result (3.3) follows in view of (2.1). \square

Theorem 3.3. *If $f(x) \in L_p(0, \infty)$, $v(x) \in L_p(0, \infty)$, also satisfies*

$$p^{-1} + q^{-1} = 1, \Re\left(\alpha + ta_j \frac{b_{j'j}}{B_{j'j}}\right) + t \sum_{i=1}^n u_i \min_{1 \leq j \leq m_i} \Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > \max\{p^{-1}, q^{-1}\},$$

$$\Re\left(\beta + tb_j \frac{b_{j'j}}{B_{j'j}}\right) + t \sum_{i=1}^n v_i \min_{1 \leq j \leq m_i} \Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0,$$

and the integrals present are absolutely convergent, then we have

$$\int_0^\infty v(x) Q_{\gamma_n}^{\alpha, \beta} [f(x)] dx = \int_0^\infty f(x) R_{\gamma_n}^{\alpha, \beta} [v(x)] dx. \quad (3.5)$$

Proof: The result (3.5) can be obtained in view of (2.1) and (2.2). \square

4. Inversion Formulas

Theorem 4.1. *If $f(x) \in L_p(0, \infty)$, $1 \leq p \leq 2$; or $f(x) \in L_p(0, \infty)$, $p > 2$, also satisfies the following conditions:*

$$p^{-1} + q^{-1} = 1, \Re\left(\alpha + ta_j \frac{b_{j'j}}{B_{j'j}}\right) + t \sum_{i=1}^n u_i \min_{1 \leq j \leq m_i} \Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -q^{-1},$$

$$\Re\left(\beta + tb_j \frac{b_{j'j}}{B_{j'j}}\right) + t \sum_{i=1}^n v_i \min_{1 \leq j \leq m_i} \Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -q^{-1},$$

and the presented integrals are absolutely convergent and

$$Q_{\gamma_n}^{\alpha, \beta} [f(x)] = v(x), \quad (4.1)$$

then we have

$$f(x) = \int_0^\infty y^{-1} [v(y)] \left[h\left(\frac{x}{y}\right) \right] dy, \quad (4.2)$$

where

$$h(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} y^{-1} \frac{x^{-s}}{R(s)} ds \quad (4.3)$$

and

$$R(s) = R_{\gamma_n}^{\alpha-s+1, \beta} [1]. \quad (4.4)$$

Proof: On taking Mellin transform of (4.1) and then applying Theorem 3.1, we get

$$\mathcal{M}\{f(x)\} = \frac{\mathcal{M}\{v(x)\}}{R(s)},$$

which on inverting leads to

$$f(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} x^{-s} \frac{\mathcal{M}\{v(x)\}}{R(s)} ds = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{x^{-s}}{R(s)} \left\{ \int_0^\infty [v(y)] dy \right\} ds.$$

Interchanging the order of integration, we obtain

$$f(x) = \int_0^\infty \frac{v(y)}{y} \left\{ \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \left(\frac{x}{y}\right)^s \frac{1}{R(s)} ds \right\} dy.$$

Now in view of (4.3), we obtain the desired result (4.2). \square

Theorem 4.2. *If $f(x) \in L_p(0, \infty)$, $1 \leq p \leq 2$; or $f(x) \in L_p(0, \infty)$, $p > 2$, also satisfied*

$$p^{-1} + q^{-1} = 1, \quad \Re \left(\beta + tb_j \frac{b_{j'j}}{B_{j'j}} \right) + t \sum_{i=1}^n v_i \min_{1 \leq j \leq m_i} \Re \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) > -q^{-1},$$

$$\Re \left(\rho + ta_j \frac{b_{j'j}}{B_{j'j}} \right) + t \sum_{i=1}^n u_i \min_{1 \leq j \leq m_i} \Re \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) > -p^{-1};$$

and the integrals present are absolutely convergent and

$$R_{\gamma_n}^{\rho, \beta} [f(x)] = w(x), \quad (4.5)$$

then we have

$$f(x) = \int_0^\infty y^{-1} [w(y)] \left[g \left(\frac{x}{y} \right) \right] dy, \quad (4.6)$$

where

$$g(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} y^{-1} \frac{x^{-s}}{T(s)} ds, \quad (4.7)$$

$$T(s) = Q_{\gamma_n}^{\rho+s-1, \beta} [1]. \quad (4.8)$$

Proof: On taking Mellin transform of (4.5) and then applying Theorem 3.2, we get

$$f(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} x^{-s} \frac{M\{w(x)\}}{T(s)} ds = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{x^{-s}}{T(s)} \left\{ \int_0^\infty [w(y)] dy \right\} ds.$$

Interchanging the order of integration, we obtain

$$f(x) = \int_0^\infty \frac{w(y)}{y} \left\{ \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \left(\frac{x}{y} \right)^s \frac{1}{T(s)} ds \right\} dy.$$

Now in view of (4.7), we obtain the desired result (4.6). \square

5. General Properties

The properties given below are consequences of the definitions (2.1) and (2.2).

$$x^{-1} Q_{\gamma_n}^{\alpha, \beta} \left[\frac{1}{x} f \left(\frac{1}{x} \right) \right] = R_{\gamma_n}^{\alpha, \beta} [f(x)], \quad (5.1)$$

$$x^{-1} R_{\gamma_n}^{\rho, \beta} \left[\frac{1}{x} f \left(\frac{1}{x} \right) \right] = Q_{\gamma_n}^{\rho, \beta} [f(x)], \quad (5.2)$$

$$x^\mu Q_{\gamma_n}^{\alpha, \beta} [f(x)] = Q_{\gamma_n}^{\alpha-\mu, \beta} [x^\mu f(x)], \quad (5.3)$$

$$x^\mu R_{\gamma_n}^{\rho, \beta} [f(x)] = R_{\gamma_n}^{\rho+\mu, \beta} [x^\mu f(x)]. \quad (5.4)$$

The properties given below express the homogeneity of the operator Q and R respectively.

$$Q_{\gamma_n}^{\alpha, \beta} [f(x)] = v(x) \quad \text{then} \quad Q_{\gamma_n}^{\alpha, \beta} [f(cx)] = v(cx), \quad (5.5)$$

$$R_{\gamma_n}^{\rho, \beta} [f(x)] = w(x) \quad \text{then} \quad R_{\gamma_n}^{\rho, \beta} [f(cx)] = w(cx). \quad (5.6)$$

6. Special cases of new extended fractional integral operators

If $\tau_i, \tau_{i(1)}, \dots, \tau_{i(n)} \rightarrow 1$, the multivariable Aleph-function reduces to multivariable I -function defined by Sharma and Ahmad [27] and if $c_{i'j} \rightarrow 1$, the Aleph-function reduces to I -function defined by Saxena [20]. We obtain two following operators:

$$\begin{aligned} Q_{\gamma_n}^{\alpha, \beta} [f(x)] &= tx^{-\alpha-t\beta-1} \int_0^x y^\alpha (x^t - y^t)^\beta I_{U:W}^{0, n; V} \left(\begin{array}{c|c} \gamma_1 \nu_1 & A : C \\ \vdots & \vdots \\ \gamma_n \nu_n & B : D \end{array} \right) \\ &\times \left\{ \prod_{i=1}^r S_{N_i}^{M_i} \left[z_i \left(\frac{y^t}{x^t} \right)^{g_i} \left(1 - \frac{y^t}{x^t} \right)^{h_i} \right] \right\} \left\{ \prod_{j=1}^k I_{P_{i'j}, Q_{i'j}; r'}^{M_j, N_j} \left[z_j \left(\frac{y^t}{x^t} \right)^{a_j} \left(1 - \frac{y^t}{x^t} \right)^{b_j} \right] \right\} \\ &\times \psi \left(\frac{y^t}{x^t} \right) f(y) dy, \end{aligned} \quad (6.1)$$

under the same notations and conditions that (2.1) with (1.9) $\tau_i, \tau_{i(1)}, \dots, \tau_{i(n)} \rightarrow 1$ and $c_{i'j} \rightarrow 1$.

$$\begin{aligned} R_{\gamma_n}^{\rho, \beta} [f(x)] &= tx^\rho \int_x^\infty y^{-\rho-t\beta-1} (y^t - x^t)^\beta I_{U:W}^{0, n; V} \left(\begin{array}{c|c} \gamma_1 \mu_1 & A : C \\ \vdots & \vdots \\ \gamma_n \mu_n & B : D \end{array} \right) \\ &\times \left\{ \prod_{i=1}^r S_{N_i}^{M_i} \left[z_i \left(\frac{x^t}{y^t} \right)^{g_i} \left(1 - \frac{x^t}{y^t} \right)^{h_i} \right] \right\} \\ &\times \left\{ \prod_{j=1}^k I_{P_{i'j}, Q_{i'j}; r'}^{M_j, N_j} \left[z_j \left(\frac{x^t}{y^t} \right)^{a_j} \left(1 - \frac{x^t}{y^t} \right)^{b_j} \right] \right\} \\ &\times \psi \left(\frac{x^t}{y^t} \right) f(y) dy, \end{aligned} \quad (6.2)$$

under the same notations and conditions that (2.2) with $\tau_i, \tau_{i(1)}, \dots, \tau_{i(n)} \rightarrow 1$ and $c_{i'j} \rightarrow 1$. We can obtain the similar theorems and properties concerning these operators as we have given for (2.1) and (2.2).

Remark 6.1. If $\tau_i, \tau_{i(1)}, \dots, \tau_{i(n)} \rightarrow 1$, $R = R^{(1)} = \dots = R^{(n)} = 1$ and $c_{i'j} \rightarrow 1$, the multivariable Aleph-functions reduce to multivariable H -function defined by Srivastava and Panda [29] and the Aleph-function reduces to I -function defined by Saxena [20], for more details, see work done by Kumar and Daiya [12].

7. Concluding Remarks

The functions involved in the established results in this paper are unified and general nature, hence a large number of known results lying in the literature follows as special cases. Further, on suitable specifications of the parameters involved, numerous new results involving simpler functions may also be obtained.

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