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Cayley Approximation Operator with an Application to a System of Set-valued Cayley Type Inclusions

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ABSTRACT: In this paper, we introduce and study a system of set-valued Cayley type inclusions involving Cayley operator and (H, ψ)-monotone operator in real Banach spaces. We show that Cayley operator associated with the (H, ψ)-monotone operator is Lipschitz type continuous. Using the proximal point operator technique, we establish a fixed point formulation for the system of set-valued Cayley type inclusions. Further, the existence and uniqueness of the approximate solution is proved. Moreover, we suggest an iterative algorithm for the system of set-valued Cayley type inclusions and discuss the strong convergence of the sequences generated by the proposed algorithm. Some examples are constructed to illustrate some concepts used in this paper.

Key Words: Cayley type inclusion, Cayley operator, (H, ψ)-monotone operator, Proximal point operator, Covergence, Iterative algorithm.

Contents

1. Introduction

It is well known that variational inequalities, complementarity problems and equilibrium problems are among most important and interesting problems in mathematical analysis. Inclusion problems were introduced and studied as a generalization of equilibrium problems. Many nonlinear problems arising in applied sciences such as signal processing, image recovery and machine learning, etc., can be modelled as an inclusion problem. In recent past, variational inclusion problems have been studied extensively by number of researchers due to their wide ranging applications to convex analysis, partial differential equations, optimization, game theory, industry, transportation, mathematical finance, nonlinear programming, economics, ecology, engineering sciences, etc., see; for example, [4,5,6,8,10,11,12,14,19,20,21,22] and references cited therein. Recently, Luo and Huang [16] and Kim et al. [15] introduced a new class of $(H, \phi) - \eta$ and (H, ϕ, ψ)-η-monotone operators, respectively in Banach spaces. These operators provide a unified framework for class of maximal monotone operators, maximal η-monotone operators, H-monotone operators and (H, η)-monotone operators. Using proximal point operator technique, they studied the convergence analysis of the iterative algorithms for some classes of variational inclusions. Very recently, Ali et al. [2] studied a Cayley inclusion problem involving XOR-operation. They defined a Cayley operator associated with a resolvent operator of a rectangular multi-valued mapping and studied convergence analysis of Cayley inclusion problem.

On the other hand, iterative computation of zeros or fixed points of nonlinear operators have been studied extensively in the literature, see; for example, [1,3,7,13,23,24,25,27]. Zhang et al. [26] introduced an iterative procedure for approaching a solution of the inclusion problem and a fixed point of a non expansive mapping in Hilbert spaces. Peng et al. [18] presented a viscosity algorithm for finding a solution of a variational inclusion with set-valued maximal monotone mapping and inverse strongly monotone mappings, the set of solutions of an equilibrium problem and a fixed point of a non expansive mapping.

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Motivated by the facts mentioned above, in this paper, we introduce and study an interesting class of inclusions, namely, system of set-valued Cayley type inclusions involving Cayley operator and (H,ψ)-monotone operator in real Banach spaces. We show that Cayley operator associated with the (H,ψ)-monotone operator is Lipschitz type continuous. Using proximal point operator technique, we establish a fixed point formulation for the system of set-valued Cayley type inclusions. Further, existence and uniqueness of the approximate solution is proved. Moreover, an iterative algorithm for the system of set-valued Cayley type inclusions is suggested to discuss the strong convergence of the sequences generated by the proposed algorithm.

2. Preliminaries

Now, we mention some definitions, notations and conclusions which are needed in the sequel.

Let E be a real Banach space, E^* be the topological dual of E, with its norm $\|\cdot\|$ and d be a metric induced by the norm $\|\cdot\|$. Let $\langle\cdot,\cdot\rangle$ be the dual pair between E and E^* and $CB(E)$(respectively, 2^E) be the family of all nonempty closed and bounded subsets (respectively, all nonempty subsets) of E and $\mathcal{D}(\cdot,\cdot)$ be the Hausdorff metric on $CB(E)$ defined by

$$\mathcal{D}(P,Q)=\max\{\sup_{x\in P} d(x,Q),\sup_{y\in Q} d(P,y)\},$$

where $P,Q\in CB(E), d(x,Q)=\inf_{y\in Q} d(x,y)$ and $d(P,y)=\inf_{x\in P} d(x,y)$.

The normalized duality mapping $J_2:E\to 2^{E^*}$ is defined by

$$J_2(x)=\{f^*\in E^*:\langle x,f^*\rangle=\|x\|^2,\|f^*\|=\|x\|\},\forall x\in E.$$

If $E\equiv H$, a real Hilbert space, then J_2 becomes the identity mapping on E.
Let $\mathcal{S}=\{x\in E:\|x\|=1\}$. A Banach space E is called uniformly convex, if for each $\epsilon\in(0,2]$, there exists $\delta>0$ such that

$$\|x-y\|\geq\epsilon \text{ implies } \|\frac{x+y}{2}\|\leq 1-\delta,\forall x,y\in\mathcal{S}.$$

It is known that uniformly convex Banach spaces are reflexive and strictly convex.

A function $\rho_E:[0,\infty)\to[0,\infty)$ is called the modulus of smoothness of E and defined by

$$\rho_E(t)=\sup\Big\{\frac{\|x+y\|+\|x-y\|}{2}-1:\|x\|\leq 1,\|y\|\leq t\Big\}.$$

A Banach space E is called uniformly smooth, if

$$\lim_{t\to 0}\frac{\rho_E(t)}{t}=0.$$

A Banach space E is called q-uniformly smooth, if there exists a constant $c>0$ such that

$$\rho_E(t)\leq ct^q,\forall t>0,q>1.$$

Lemma 2.1. *[9] Let E be a uniformly smooth Banach space and $J:E\to 2^{E^*}$ be a normalized duality mapping. Then*

(i) $\|x+y\|^2\leq\|x\|^2+2\langle y,j(x+y)\rangle,\forall j(x+y)\in J(x+y),\forall x,y\in E;$

(ii) $\langle x-y,j(x)-j(y)\rangle\leq 2\tau^2\rho_E\Big(\frac{4\|x-y\|}{\tau}\Big)$, where $\tau=\sqrt{\frac{\|x\|^2+\|y\|^2}{2}},\forall x,y\in E.$

Lemma 2.2. *[17] Let E be a complete metric space with metric d and $T:E\to CB(E)$ be a multi-valued mapping. Then for any $\epsilon>0$ and for any $x,y\in E$, $u\in T(x)$; there exists $v\in T(y)$ such that*

$$d(u,v)\leq(1+\epsilon)\mathcal{D}(T(x),T(y)).$$

Definition 2.3. *A single-valued mapping $g : E \to E^*$ is said to be*

(i) monotone, if

$$\langle g(x) - g(y), x - y\rangle \geq 0, \forall x, y \in E;$$

(ii) strictly monotone, if

$$\langle g(x) - g(y), x - y\rangle > 0, \forall x, y \in E,$$

and the equality holds if and only if $x = y$;

(iii) δ_g-strongly monotone, if there exists a constant $\delta_g > 0$ such that

$$\langle g(x) - g(y), x - y\rangle \geq \delta_g \|x - y\|^2, \forall x, y \in E;$$

(iv) Lipschitz continuous, if there exists a constant $\lambda_g > 0$ such that

$$\|g(x) - g(y)\| \leq \lambda_g \|x - y\|, \forall x, y \in E;$$

(v) k-strongly accretive, if there exists a constant $k > 0$ such that

$$\langle g(x) - g(y), j(x - y)\rangle \geq k\|x - y\|^2, \forall x, y \in E,\ \ j(x - y) \in J(x - y).$$

Definition 2.4. *Let $H : E \to E^*, \psi : E^* \to E^*, N : E \times E \to E^*$ be the single-valued mappings and $M : E \rightrightarrows 2^{E^*}$ be a multi-valued mapping. Then*

(i) M is said to be monotone, if

$$\langle u - v, x - y\rangle \geq 0, \forall x, y \in E \text{ and } \forall u \in M(x), v \in M(y);$$

(ii) M is said to be H-monotone, if M is monotone and

$$(H + \lambda M)(E) = E^*, \forall \lambda > 0;$$

(iii) N is said to be Lipschitz continuous in the first argument, if there exists a constant $\alpha_1 > 0$ such that

$$\|N(x, \cdot) - N(y, \cdot)\| \leq \alpha_1 \|x - y\|, \forall x, y \in E;$$

(iv) N is said to be Lipschitz continuous in the second argument, if there exists a constant $\alpha_2 > 0$ such that

$$\|N(\cdot, x) - N(\cdot, y)\| \leq \alpha_2 \|x - y\|, \forall x, y \in E.$$

Definition 2.5. *Let E be a Banach space with its dual E^*. Let $H : E \to E^*, \psi : E^* \to E^*$ be the single-valued mappings. A multi-valued mapping $M : E \rightrightarrows 2^{E^*}$ is said to be (H, ψ)-monotone, if $(\psi \circ M)$ is monotone and*

$$[H + \lambda(\psi \circ M)](E) = E^*.$$

Theorem 2.6. *Let E be a Banach space with its dual E^*. Let $H : E \to E^*, \psi : E^* \to E^*$ be the single-valued mappings such that H is strictly monotone and $M : E \rightrightarrows 2^{E^*}$ be an (H, ψ)-monotone mapping. Then the mapping $[H + \lambda(\psi \circ M)]^{-1} : E^* \to E$ is single-valued.*

Proof. For any given $x^* \in E^*$, let $u, v \in (H + \lambda\psi \circ M)^{-1}(x^*)$. Then, we have

$$\frac{1}{\lambda}[x^* - H(u)] \in (\psi \circ M)(u)]$$

and

$$\frac{1}{\lambda}[x^* - H(v)] \in (\psi \circ M)(v)].$$

It follows from monotonicity of $(\psi \circ M)$ that

$$\frac{1}{\lambda}\langle x^* - H(u) - (x^* - H(v)), u - v\rangle \geq 0$$

which implies that

$$\frac{1}{\lambda}\langle -H(u) + H(v), u - v\rangle \geq 0. \tag{2.1}$$

Since H is strictly monotone, we have

$$\frac{1}{\lambda}\langle H(u) - H(v), u - v\rangle > 0. \tag{2.2}$$

It follows from (2.1) and (2.2) that $u = v$. Thus, $[H + \lambda(\psi \circ M)]^{-1}$ is single-valued. □

Definition 2.7. *Let E be a reflexive Banach space with its dual E^*. Let $H : E \to E^*, \psi : E^* \to E^*$ be the single-valued mappings such that H is γ-strongly monotone and $M : E \rightrightarrows 2^{E^*}$ be an (H, ψ)-monotone mapping. Then the operator $R^{H,\lambda}_{M,\psi} : E^* \to E$ defined by*

$$R^{H,\lambda}_{M,\psi}(x^*) = [H + \lambda(\psi \circ M)]^{-1}(x^*), \ \forall x^* \in E^* \tag{2.3}$$

is called proximal point operator associated with (H, ψ)-monotone mapping, where $\lambda > 0$ is a constant.

Theorem 2.8. *Let E be a reflexive Banach space with its dual E^*. Let $H : E \to E^*, \psi : E^* \to E^*$ be the single-valued mappings such that H is γ-strongly monotone and $M : E \rightrightarrows 2^{E^*}$ be an (H, ψ)-monotone mapping. Then the operator $R^{H,\lambda}_{M,\psi} : E^* \to E$ defined by (2.3) is $\frac{1}{\gamma}$-Lipschitz continuous, i.e.,*

$$\|R^{H,\lambda}_{M,\psi}(x^*) - R^{H,\lambda}_{M,\psi}(y^*)\| \leq \frac{1}{\gamma}\|x^* - y^*\|, \ \forall x^*, y^* \in E^*.$$

Proof. Let $x^*, y^* \in E^*$, then it follows from (2.3) that

$$R^{H,\lambda}_{M,\psi}(x^*) = [H + \lambda(\psi \circ M)]^{-1}(x^*)$$

and

$$R^{H,\lambda}_{M,\psi}(y^*) = [H + \lambda(\psi \circ M)]^{-1}(y^*),$$

which implies that

$$\frac{1}{\lambda}[x^* - H(R^{H,\lambda}_{M,\psi}(x^*))] \in (\psi \circ M)(R^{H,\lambda}_{M,\psi}(x^*))]$$

and

$$\frac{1}{\lambda}[y^* - H(R^{H,\lambda}_{M,\psi}(y^*))] \in (\psi \circ M)(R^{H,\lambda}_{M,\psi}(y^*))].$$

Since $(\psi \circ M)$ is monotone, we have

$$\frac{1}{\lambda}\langle x^* - H(R^{H,\lambda}_{M,\psi}(x^*)) - (y^* - H(R^{H,\lambda}_{M,\psi}(y^*))), R^{H,\lambda}_{M,\psi}(x^*) - R^{H,\lambda}_{M,\psi}(y^*)\rangle \geq 0,$$

which implies that

$$\langle x^* - y^*, R^{H,\lambda}_{M,\psi}(x^*) - R^{H,\lambda}_{M,\psi}(y^*)\rangle \geq \langle H(R^{H,\lambda}_{M,\psi}(x^*)) - H(R^{H,\lambda}_{M,\psi}(y^*)), R^{H,\lambda}_{M,\psi}(x^*) - R^{H,\lambda}_{M,\psi}(y^*)\rangle.$$

Since H is γ-strongly monotone, we have

$$\begin{aligned} \|x^* - y^*\|\|R^{H,\lambda}_{M,\psi}(x^*) - R^{H,\lambda}_{M,\psi}(y^*)\| &\geq \langle H(R^{H,\lambda}_{M,\psi}(x^*)) - H(R^{H,\lambda}_{M,\psi}(y^*)), R^{H,\lambda}_{M,\psi}(x^*) - R^{H,\lambda}_{M,\psi}(y^*)\rangle \\ &\geq \gamma\|R^{H,\lambda}_{M,\psi}(x^*) - R^{H,\lambda}_{M,\psi}(y^*)\|^2. \end{aligned}$$

Thus, we have

$$\|R^{H,\lambda}_{M,\psi}(x^*) - R^{H,\lambda}_{M,\psi}(y^*)\| \leq \frac{1}{\gamma}\|x^* - y^*\|, \ \forall x^*, y^* \in E^*.$$

This completes the proof. □

Now, based on proximal point operator $R^{H,\lambda}_{M,\psi}$ defined by (2.3), we define a Cayley operator.

Definition 2.9. *Let E be a reflexive Banach space with its dual E^*. Let $H: E \to E^*, \psi: E^* \to E^*$ be the single-valued mappings such that H is γ-strongly monotone and $M: E \rightrightarrows 2^{E^*}$ be an (H,ψ)-monotone mapping. Then the operator $C^{H,\lambda}_{M,\psi}: E^* \to E$ defined by*

$$C^{H,\lambda}_{M,\psi}(x^*) = [2R^{H,\lambda}_{M,\psi} - I](x^*), \ \forall x^* \in E^*, \tag{2.4}$$

is called Cayley operator.

Lemma 2.10. *Let E be a reflexive Banach space with its dual E^*. Let $H: E \to E^*, \psi: E^* \to E^*$ be the single-valued mappings such that H is γ-strongly monotone and $M: E \rightrightarrows 2^{E^*}$ be an (H,ψ)-monotone mapping. Then, the Cayley operator $C^{H,\lambda}_{M,\psi}$ defined by (2.4) is $\left(\frac{2+\gamma}{\gamma}\right)$-Lipschitz continuous, i.e.,*

$$\|C^{H,\lambda}_{M,\psi}(x^*) - C^{H,\lambda}_{M,\psi}(y^*)\| \leq \left(\frac{2+\gamma}{\gamma}\right)\|x^* - y^*\|, \ \forall x^*, y^* \in E^*.$$

Proof. Let $x^*, y^* \in E^*$, then it follows from (2.4) and $\frac{1}{\gamma}$–Lipschitz continuity of proximal point operator $R^{H,\lambda}_{M,\psi}$ that

$$\begin{aligned}
\|C^{H,\lambda}_{M,\psi}(x^*) - C^{H,\lambda}_{M,\psi}(y^*)\| &= \|[2R^{H,\lambda}_{M,\psi} - I](x^*) - [2R^{H,\lambda}_{M,\psi} - I](y^*)\| \\
&\leq 2\|R^{H,\lambda}_{M,\psi}(x^*) - R^{H,\lambda}_{M,\psi}(y^*)\| + \|x^* - y^*\| \\
&\leq \frac{2}{\gamma}\|x^* - y^*\| + \|x^* - y^*\|,
\end{aligned}$$

which gives

$$\|C^{H,\lambda}_{M,\psi}(x^*) - C^{H,\lambda}_{M,\psi}(y^*)\| \leq \left(\frac{2+\gamma}{\gamma}\right)\|x^* - y^*\|.$$

Thus, the Cayley operator $C^{H,\lambda}_{M,\psi}$ is $\left(\frac{2+\gamma}{\gamma}\right)$-Lipschitz continuous. □

Example 2.11. *Let $E = \mathbb{R} = (-\infty, \infty), M(x) = 2x, H(x) = x + \frac{1}{2}, \psi(x) = x + 1, \forall x \in \mathbb{R}$. Then*

$$\begin{aligned}
\langle \psi \circ M(x) - \psi \circ M(y), x - y\rangle &= \langle (2x+1) - (2y+1), x - y\rangle \\
&= 2\langle x - y, x - y\rangle = 2(x-y)^2 \geq 0.
\end{aligned}$$

Thus, $\psi \circ M$ is a monotone mapping. It is easy to see that

$$(H + \psi \circ M)(x) = 3x + \frac{3}{2}, \forall x \in \mathbb{R},$$

i.e., $(H + \psi \circ M)$ is surjective. Hence, M is (H,ψ)-monotone.

For $\lambda = 1$, the proximal point operator associated to (H,ψ)-monotone mapping defined by (2.3) is given by

$$R^{H,\lambda}_{M,\psi}(x) = (H + \lambda\psi \circ M)^{-1}(x) = \frac{x}{3} - \frac{1}{2}, \forall x \in \mathbb{R}.$$

Now,

$$\|R^{H,\lambda}_{M,\psi}(x) - R^{H,\lambda}_{M,\psi}(y)\| = \frac{1}{3}\|x - y\| \leq \frac{1}{n}\|x - y\|, \forall x \in \mathbb{R}, n = 1, 2, 3.$$

Thus, the proximal point operator $R^{H,\lambda}_{M,\psi}$ is $\frac{1}{n}$–Lipschitz continuous, for $n = 1, 2, 3$.
The Cayley operator $C^{H,\lambda}_{M,\psi}$ defined by (2.4) is given by

$$C^{H,\lambda}_{M,\psi}(x) = \frac{-x-1}{3}, \forall x \in \mathbb{R}.$$

$$\text{Also, } \|C^{H,\lambda}_{M,\psi}(x) - C^{H,\lambda}_{M,\psi}(y)\| = \frac{1}{3}\|x-y\| \leq \Big(\frac{2}{n}+1\Big)\|x-y\|, \forall x \in \mathbb{R}, n = 1,2,3.$$

Thus, the Cayley operator $C^{H,\lambda}_{M,\psi}$ *is* $\Big(\frac{2}{n}+1\Big)$*–Lipschitz continuous.*

Example 2.12. *Let* $E = M^2_S$*, the space of all* 2×2 *symmetric matrices. The inner product is defined as* $\langle M^2_{S_1}, M^2_{S_2}\rangle = \mathrm{trace}(M^2_{S_1}M^2_{S_2}),\ \forall M^2_{S_1}, M^2_{S_2} \in E$.

Let $H\Big(\begin{bmatrix} x_1 & a \\ a & x_2\end{bmatrix}\Big) = \begin{bmatrix} x_1 & -\frac{a}{2} \\ -\frac{a}{2} & x_2\end{bmatrix}$, $M\Big(\begin{bmatrix} x_1 & a \\ a & x_2\end{bmatrix}\Big) = \begin{bmatrix} 6x_1 & \frac{a}{6} \\ \frac{a}{6} & 6x_2\end{bmatrix}$ *and* $\psi\Big(\begin{bmatrix} x_1 & a \\ a & x_2\end{bmatrix}\Big) = \begin{bmatrix} \frac{x_1}{2} & 3a \\ 3a & \frac{x_2}{2}\end{bmatrix}$. *Then*

$$\begin{aligned}\langle \psi\circ M(x) - \psi\circ M(y), x-y\rangle &= \mathrm{trace}[(\psi\circ M(x) - \psi\circ M(y))(x-y)] \\ &= \mathrm{trace}\Big(\begin{bmatrix} 3(x_1-y_1) & \frac{a-b}{2} \\ \frac{a-b}{2} & 3(x_2-y_2)\end{bmatrix}\begin{bmatrix} x_1-y_1 & a-b \\ a-b & x_2-y_2\end{bmatrix}\Big) \\ &= 3(x_1-y_1)^2 + (a-b)^2 + 3(x_2-y_2)^2 \geq 0.\end{aligned}$$

Thus, $(\psi\circ M)$ *is a monotone mapping. It is easy to see that*

$$(H+\psi\circ M)(x) = \begin{bmatrix} x_1 & -\frac{a}{2} \\ -\frac{a}{2} & x_2\end{bmatrix} + \begin{bmatrix} 3x_1 & \frac{a}{2} \\ \frac{a}{2} & 3x_2\end{bmatrix} = \begin{bmatrix} 4x_1 & 0 \\ 0 & 4x_2\end{bmatrix},$$

i.e., $(H+\psi\circ M)(E) \neq E$*. Thus, it is clear that* $(\psi\circ M)$*-monotone mapping need not be* (H,ψ)*-monotone.*

3. Formulation of the System of Set-valued Cayley Type Inclusions and Convergence Result

This section begins with the formulation of a system of set-valued Cayley type inclusions and we discuss the existence of unique solution.

Let E be a uniformly smooth Banach space with its dual E^*, for each $i = \{1,2\}$; let $N_i : E\times E \to E^*, \psi_i : E^* \to E^*, g : E \to E$ be the single-valued mappings and $P_i, Q_i, T_i, G_i : E \to CB(E)$ be the set-valued mappings. Let $H : E \to E^*$ be a strongly monotone mapping and $M : E \rightrightarrows 2^{E^*}$ be an (H,ψ)-monotone mapping. We consider the following system of set-valued inclusions.

Find $(x_i, u_i, v_i, w_i, z_i), x_i \in E, u_i \in P_i(x_1), v_i \in Q_i(x_1), w_i \in T_i(x_1)$ and $z_i \in G_i(x_1)$ such that

$$\begin{cases} 0 \in H(g(x_1)) - H(g(x_2)) + \lambda_1[N_1(u_1,v_1) + M(g(x_1), w_1) + C^{H,\lambda_1}_{M,\psi_1}(z_1)], \\ 0 \in H(g(x_2)) - H(g(x_1)) + \lambda_2[N_2(u_2,v_2) + M(g(x_2), w_2) + C^{H,\lambda_2}_{M,\psi_2}(z_2)]. \end{cases} \tag{3.1}$$

We call the system (3.1), the system of set-valued Cayley type inclusions.

Remark 3.1. *For each* $i = \{1,2\}$*, if we consider* $x_i = x, u_i = u, v_i = v, w_i = w$ *and* $z_i = z$*, then the system of set-valued Cayley type inclusions (3.1) reduces to the following Cayley type inclusion problem:*

Find $x \in E, u \in P(x), v \in Q(x), w \in T(x)$ *and* $z \in G(x)$ *such that*

$$0 \in H(g(x)) - H(g(y)) + \lambda[N(u,v) + M(g(x), w) + C^{H,\lambda}_{M,\psi}(z)],\ \forall y \in E, \lambda > 0. \tag{3.2}$$

If $g = I$, the identity mapping, $H, N \equiv 0, M(\cdot,\cdot) = M(\cdot)$, and G is a single-valued mapping, then the Cayley type inclusion problem (3.2) reduces to an equivalent problem of finding $x \in \mathcal{H}$ such that

$$0 \in M(x) + C^M_{I,\lambda}(x). \tag{3.3}$$

Problem (3.3) was studied by Ali et al. [2] using XOR-operation.

Theorem 3.2. *Let* E *be a uniformly smooth Banach space with its dual* E^**, for each* $i = \{1,2\}$*; let* $N_i : E\times E \to E^*$ *and* $g : E \to E$ *be the single-valued mappings; let* $\psi_i : E^* \to E^*$ *be a single-valued mapping such that* $\psi_i(x+y) = \psi_i(x) + \psi_i(y)$ *and* $Ker(\psi_i) = \{0\}$*, where* $Ker(\psi_i) = \{x \in E^* : \psi_i(x) = 0\}$*. Let* $H : E \to E^*$ *be a strongly monotone mapping and* $M : E \rightrightarrows 2^{E^*}$ *be an* (H,ψ)*-monotone mapping.*

Then the system of set-valued Cayley type inclusions (3.1) has a solution $(x_i, u_i, v_i, w_i, z_i)$, where $x_i \in E, u_i \in P_i(x_1), v_i \in Q_i(x_1), w_i \in T_i(x_1), z_i \in G_i(x_1)$ if and only if it satisfies following fixed point problem:

$$g(x_1) = R^{H,\lambda_1}_{M(\cdot,w_1),\psi_1}[H(g(x_2)) - \lambda_1\psi_1 \circ N_1(u_1, v_1) - \lambda_1 C^{H,\lambda_1}_{M,\psi_1}(z_1)]; \tag{3.4}$$

where,

$$g(x_2) = R^{H,\lambda_2}_{M(\cdot,w_2),\psi_2}[H(g(x_1)) - \lambda_2\psi_2 \circ N_2(u_2, v_2) - \lambda_2 C^{H,\lambda_2}_{M,\psi_2}(z_2)], \tag{3.5}$$

$\lambda_i > 0$ is a constant, $R^{H,\lambda_i}_{M(.,w_i),\psi_i} = [H + \lambda_i\psi_i \circ M(\cdot, w_i)]^{-1}$ is proximal point operator and $C^{H,\lambda_i}_{M,\psi_i} = [2R^{H,\lambda_i}_{M,\psi_i} - I]$ is a Cayley operator.

Proof. It follows from the definition of proximal point operator $R^{H,\lambda_1}_{M(.,w_i),\psi_i}$ that

$$\begin{aligned}
& H(g(x_2)) - \lambda_1\psi_1 \circ N_1(u_1, v_1) - \lambda_1 C^{H,\lambda_1}_{M,\psi_1}(z_1) \in [H + \lambda_1\psi_1 \circ M(\cdot, w_1)]g(x_1) \\
\Leftrightarrow\ & H(g(x_2)) - \lambda_1\psi_1 \circ N_1(u_1, v_1) - \lambda_1 C^{H,\lambda_1}_{M,\psi_1}(z_1) \in [H(g(x_1)) + \lambda_1\psi_1 \circ M(g(x_1), w_1)] \\
\Leftrightarrow\ & 0 \in H(g(x_1)) - H(g(x_2)) + \lambda_1\psi_1[N_1(u_1, v_1) + M(g(x_1), w_1)] + \lambda_1 C^{H,\lambda_1}_{M,\psi_1}(z_1).
\end{aligned}$$

Since $\psi_1(x + y) = \psi_1(x) + \psi_1(y)$ and $Ker(\psi_1) = \{0\}$, we have

$$0 \in H(g(x_1)) - H(g(x_2)) + \lambda_1[N_1(u_1, v_1) + M(g(x_1), w_1)] + \lambda_1 C^{H,\lambda_1}_{M,\psi_1}(z_1),\ \lambda_1 > 0.$$

Similarly, we can prove that

$$0 \in H(g(x_2)) - H(g(x_1)) + \lambda_2[N_2(u_2, v_2) + M(g(x_2), w_2)] + \lambda_2 C^{H,\lambda_2}_{M,\psi_2}(z_2),\ \lambda_2 > 0.$$

This completes the proof. □

Algorithm 1. *For any arbitrary $x_1^0 \in E, u_1^0 \in P_1(x_1^0), v_1^0 \in Q_1(x_1^0), w_1^0 \in T_1(x_1^0), z_1^0 \in G_1(x_1^0)$, compute the sequences $\{x_1^n\}, \{u_1^n\}, \{v_1^n\}, \{w_1^n\}, \{z_1^n\}$ by the following iterative scheme:*

$$x_1^{n+1} = x_1^n - g(x_1^n) + R^{H,\lambda_1}_{M(\cdot,w_1^n),\psi_1}[H(g(x_2^n)) - \lambda_1\psi_1 \circ N_1(u_1^n, v_1^n) - \lambda_1 C^{H,\lambda_1}_{M,\psi_1}(z_1^n)], \lambda_1 > 0;$$

where,

$$g(x_2^n) = R^{H,\lambda_2}_{M(\cdot,w_2^n),\psi_2}[H(g(x_1^n)) - \lambda_2\psi_2 \circ N_2(u_2^n, v_2^n) - \lambda_2 C^{H,\lambda_2}_{M,\psi_2}(z_2^n)], \lambda_2 > 0;$$

and

$$\begin{aligned}
u_i^n \in P_i(x_1^n) &: \|u_i^{n+1} - u_i^n\| \le (1 + (1+n)^{-1})\mathcal{D}(P_i(x_1^{n+1}), P_i(x_1^n)); \\
v_i^n \in Q_i(x_1^n) &: \|v_i^{n+1} - v_i^n\| \le (1 + (1+n)^{-1})\mathcal{D}(Q_i(x_1^{n+1}), Q_i(x_1^n)); \\
w_i^n \in T_i(x_1^n) &: \|w_i^{n+1} - w_i^n\| \le (1 + (1+n)^{-1})\mathcal{D}(T_i(x_1^{n+1}), T_i(x_1^n)); \\
z_i^n \in G_i(x_1^n) &: \|z_i^{n+1} - z_i^n\| \le (1 + (1+n)^{-1})\mathcal{D}(G_i(x_1^{n+1}), G_i(x_1^n)).
\end{aligned}$$

Algorithm 2. *For any arbitrary $x_0 \in E, u_0 \in P(x_0), v_0 \in Q(x_0), w_0 \in T(x_0)$ and $z_0 \in G(x_0)$, compute the sequences $\{x_n\}, \{u_n\}, \{v_n\}, \{w_n\}, \{z_n\}$ by the following iterative scheme:*

$$x_{n+1} = x_n - g(x_n) + R^{H,\lambda}_{M(\cdot,w_n),\psi}[H(g(x_n)) - \lambda\psi \circ N(u_n, v_n) - \lambda C^{H,\lambda}_{M,\psi}(z_n), \lambda > 0;$$

and

$$\begin{aligned}
u_n \in P(x_n) &: \|u_{n+1} - u_n\| \le (1 + (1+n)^{-1})\mathcal{D}(P(x_{n+1}), P(x_n)); \\
v_n \in Q(x_n) &: \|v_{n+1} - v_n\| \le (1 + (1+n)^{-1})\mathcal{D}(Q(x_{n+1}), Q(x_n)); \\
w_n \in T(x_n) &: \|w_{n+1} - w_n\| \le (1 + (1+n)^{-1})\mathcal{D}(T(x_{n+1}), T(x_n)); \\
z_n \in G(x_n) &: \|z_{n+1} - z_n\| \le (1 + (1+n)^{-1})\mathcal{D}(G(x_{n+1}), G(x_n)).
\end{aligned}$$

Theorem 3.3. *Let E be a uniformly smooth Banach space with its dual E^* and modulus of smoothness $\rho_E(t) \leq ct^2$, for some $c > 0$. Let $g : E \to E$ be k-strongly accretive and Lipschitz continuous with constant μ and $H : E \to E^*$ be Lipschitz continuous with constant s with respect to g. For each $i = \{1, 2\}$; let $N_i : E \times E \to E^*$ be (α_i, β_i)-Lipschitz continuous in the first and second argument, respectively. Let $P_i, Q_i, T_i, G_i : E \to CB(E^*)$ be $\mathcal{D}$-Lipschitz continuous with constants $\delta_{P_i}, \delta_{Q_i}, \delta_{T_i}$ and δ_{G_i}, respectively. Let $\psi_i : E^* \to E^*$ be a Lipschitz continuous mapping with respect to $N_i(\cdot,\cdot)$ with constant ζ_i satisfying $\psi_i(x + y) = \psi_i(x) + \psi_i(y)$ with $Ker(\psi_i) = \{0\}$. Let $M : E \times E \rightrightarrows 2^{E^*}$ be an (H, ψ_i)-monotone mapping. Assume that there exist constants $\lambda_1, \lambda_2 > 0$ satisfying*

$$0 < \Theta(P) = \Delta + l_1 + l_2 + \frac{\xi_2}{k}\delta_{T_2} < 1, \tag{3.6}$$

where,
$l_1 = \Big[\dfrac{\lambda_1}{\gamma}[\zeta_1(\alpha_1\delta_{P_1} + \beta_1\delta_{Q_1}) + \Big(\dfrac{2+\gamma}{\gamma}\Big)\delta_{G_1}\Big]; l_2 = \dfrac{s\delta}{\gamma}\Big[\dfrac{\lambda_2}{k\gamma}(\zeta_2(\alpha_2\delta_{P_2} + \beta_2\delta_{Q_2}) + \Big(\dfrac{2+\gamma}{\gamma}\Big)\delta_{G_2})\Big]$ *and*
$\Delta = \sqrt{1 - 2k + 64c\delta^2} + \dfrac{s^2\mu^2}{k\gamma^2} + \xi_1\delta_{T_1}$. *In addition the following condition holds:*

$$\|R^{H,\lambda_i}_{M(\cdot,w_i^n),\psi_i}(u) - R^{H,\lambda_i}_{M(\cdot,w_i^{n-1}),\psi_i}(u)\| \leq \xi_i\|w_i^n - w_i^{n-1}\|,\ \xi_i > 0. \tag{3.7}$$

Then $(x_i, u_i, v_i, w_i, z_i), x_i \in E, u_i \in P_i(x_1), v_i \in Q_i(x_1), w_i \in T_i(x_1)$ and $z_i \in G_i(x_1)$ is unique solution of the system of set-valued Cayley type inclusions (3.1). Moreover, the iterative sequences $\{x_i^n\}, \{u_i^n\}, \{v_i^n\}, \{w_i^n\}, \{z_i^n\}$ generated by Algorithm 1 converge strogly to $(x_i, u_i, v_i, w_i, z_i)$.

Proof. It follows from Algorithm 1, Theorem 2.8 and Condition (3.7) that

$$\begin{aligned}
\|x_1^{n+1} - x_1^n\| &= \|x_1^n - g(x_1^n) + R^{H,\lambda_1}_{M(\cdot,w_1^n),\psi_1}[H(g(x_2^n)) - \lambda_1\psi_1 \circ N_1(u_1^n, v_1^n) - \lambda_1 C^{H,\lambda_1}_{M,\psi_1}(z_1^n)] \\
&\quad -x_1^{n-1} + g(x_1^{n-1}) - R^{H,\lambda_1}_{M(\cdot,w_1^{n-1}),\psi_1}[H(g(x_2^{n-1})) - \lambda_1\psi_1 \circ N_1(u_{1,}^{n-1} v_1^{n-1}) \\
&\quad -\lambda_1 C^{H,\lambda_1}_{M,\psi_1}(z_1^{n-1})]\| \\
&\leq \|x_1^n - x_1^{n-1} - (g(x_1^n) - g(x_1^{n-1}))\| \\
&\quad +\|R^{H,\lambda_1}_{M(\cdot,w_1^n),\psi_1}[H(g(x_2^n)) - \lambda_1\psi_1 \circ N_1(u_1^n, v_1^n) - \lambda_1 C^{H,\lambda_1}_{M,\psi_1}(z_1^n)] \\
&\quad -R^{H,\lambda_1}_{M(\cdot,w_1^n),\psi_1}[H(g(x_2^{n-1})) - \lambda_1\psi_1 \circ N_1(u_{1,}^{n-1} v_1^{n-1}) - \lambda_1 C^{H,\lambda_1}_{M,\psi_1}(z_1^{n-1})]\| \\
&\quad +R^{H,\lambda_1}_{M(\cdot,w_1^n),\psi_1}[H(g(x_2^{n-1})) - \lambda_1\psi_1 \circ N_1(u_{1,}^{n-1} v_1^{n-1}) - \lambda_1 C^{H,\lambda_1}_{M,\psi_1}(z_1^{n-1})]\| \\
&\quad -R^{H,\lambda_1}_{M(\cdot,w_1^{n-1}),\psi_1}[H(g(x_2^{n-1})) - \lambda_1\psi_1 \circ N_1(u_{1,}^{n-1} v_1^{n-1}) - \lambda_1 C^{H,\lambda_1}_{M,\psi_1}(z_1^{n-1})]\| \\
&\leq \|x_1^n - x_1^{n-1} - (g(x_1^n) - g(x_1^{n-1}))\| \\
&\quad +\frac{1}{\gamma}\|H(g(x_2^n)) - \lambda_1\psi_1 \circ N_1(u_1^n, v_1^n) - \lambda_1 C^{H,\lambda_1}_{M,\psi_1}(z_1^n) \\
&\quad -[H(g(x_2^{n-1})) - \lambda_1\psi_1 \circ N_1(u_{1,}^{n-1} v_1^{n-1}) - \lambda_1 C^{H,\lambda_1}_{M,\psi_1}(z_1^{n-1})]\| + \xi_1\|w_1^n - w_1^{n-1}\| \\
&\leq \|x_1^n - x_1^{n-1} - (g(x_1^n) - g(x_1^{n-1}))\| + \frac{1}{\gamma}\|H(g(x_2^n)) - H(g(x_2^{n-1}))\| \\
&\quad +\frac{\lambda_1}{\gamma}\|\psi_1 \circ N_1(u_1^n, v_1^n) - \psi_1 \circ N_1(u_1^{n-1}, v_1^{n-1})\| \\
&\quad +\frac{\lambda_1}{\gamma}\|C^{H,\lambda_1}_{M,\psi_1}(z_1^n) - C^{H,\lambda_1}_{M,\psi_1}(z_1^{n-1})\| + \xi_1\|w_1^n - w_1^{n-1}\|.
\end{aligned} \tag{3.8}$$

Since g is k-strongly accretive and Lipschitz continuous with constant μ, then from Lemma 2.1, we have

$$\begin{aligned}
\|x_1^n - x_1^{n-1} &- (g(x_1^n) - g(x_1^{n-1}))\|^2 \\
&\leq \|x_1^n - x_1^{n-1}\|^2 + 2\langle j((x_1^n - x_1^{n-1}) - g(x_1^n) + g(x_1^{n-1})), -(g(x_1^n) - g(x_1^{n-1}))\rangle \\
&\leq \|x_1^n - x_1^{n-1}\|^2 - 2\langle j(x_1^n - x_1^{n-1}), g(x_1^n) - g(x_1^{n-1})\rangle \\
&\quad +2\langle j(x_1^n - x_1^{n-1} - g(x_1^n) + g(x_1^{n-1})) - j(x_1^n - x_1^{n-1}), -(g(x_1^n) - g(x_1^{n-1}))\rangle \\
&\leq \|x_1^n - x_1^{n-1}\|^2 - 2k\|x_1^n - x_1^{n-1}\|^2 + 4d^2\rho_E\left(\frac{4\|g(x_1^n) - g(x_1^{n-1})\|}{d}\right) \\
&\leq (1 - 2k + 64c\mu^2)\|x_1^n - x_1^{n-1}\|^2,
\end{aligned}$$

which implies that

$$\|x_1^n - x_1^{n-1} - (g(x_1^n) - g(x_1^{n-1}))\| \leq \sqrt{1 - 2k + 64c\mu^2}\|x_1^n - x_1^{n-1}\|. \tag{3.9}$$

Using the Lipschitz continuities of H and g, we get

$$\|H(g(x_2^n)) - H(g(x_2^{n-1}))\| \leq s\mu\|x_2^n - x_2^{n-1}\|. \tag{3.10}$$

Since ψ_1 is ζ_1-Lipschitz continuous with respect to $N_1(\cdot,\cdot)$ and $N_1(\cdot,\cdot)$ is (α_1,β_1)-Lipschitz continuous with respect to first and second argument, respectively, P_1 and Q_1 are $\mathcal{D}$-Lipschitz continuous with constants δ_{P_1} and δ_{Q_1}, respectively, then we have

$$\begin{aligned}
\|\psi_1 \circ N_1(u_1^n, v_1^n) - \psi_1 \circ N_1(u_1^{n-1}, v_1^{n-1})\| &\leq \zeta_1\|N_1(u_1^n, v_1^n) - N_1(u_1^{n-1}, v_1^{n-1})\| \\
&\leq \zeta_1\|N_1(u_1^n, v_1^n) - N_1(u_1^{n-1}, v_1^n)\| \\
&\quad +\zeta_1\|N_1(u_1^{n-1}, v_1^n) - N_1(u_1^{n-1}, v_1^{n-1})\| \\
&\leq \zeta_1\alpha_1\|u_1^n - u_1^{n-1}\| + \zeta_1\beta_1\|v_1^n - v_1^{n-1}\| \\
&\leq \zeta_1\alpha_1[1 + (1+n)^{-1}]\mathcal{D}(P_1(x_1^n), P_1(x_1^{n-1})) \\
&\quad +\zeta_1\beta_1[1 + (1+n)^{-1}]\mathcal{D}(Q_1(x_1^n), Q_1(x_1^{n-1})) \\
&\leq \zeta_1[1 + (1+n)^{-1}][\alpha_1\delta_{P_1} + \beta_1\delta_{Q_1}]\|x_1^n - x_1^{n-1}\|.
\end{aligned} \tag{3.11}$$

Since $C_{M,\psi_1}^{H,\lambda_1}$ is $\left(\dfrac{2+\gamma}{\gamma}\right)$-Lipschitz continuous and G_1 is $\mathcal{D}$-Lipschitz continuous with constant δ_{G_1}, then we have

$$\begin{aligned}
\|C_{M,\psi_1}^{H,\lambda_1}(z_1^n) - C_{M,\psi_1}^{H,\lambda_1}(z_1^{n-1})\| &\leq \left(\frac{2+\gamma}{\gamma}\right)\|z_1^n - z_1^{n-1}\| \\
&\leq \left(\frac{2+\gamma}{\gamma}\right)[1 + (1+n)^{-1}]\mathcal{D}(G_1(x_1^n), G_1(x_1^{n-1})) \\
&\leq \left(\frac{2+\gamma}{\gamma}\right)[1 + (1+n)^{-1}]\delta_{G_1}\|x_1^n - x_1^{n-1}\|.
\end{aligned} \tag{3.12}$$

Also, T_1 is $\mathcal{D}$-Lipschitz continuous with constant δ_{T_1}, then we have

$$\begin{aligned}
\|w_1^n - w_1^{n-1}\| &\leq [1 + (1+n)^{-1}]\mathcal{D}(T_1(x_1^n), T_1(x_1^{n-1})) \\
&\leq [1 + (1+n)^{-1}]\delta_{T_1}\|x_1^n - x_1^{n-1}\|.
\end{aligned} \tag{3.13}$$

Thus, from (3.8)–(3.13), we get

$$\begin{aligned}
\|x_1^{n+1} - x_1^n\| \leq \Big\{&\sqrt{1 - 2k + 64c\mu^2} + [1 + (1+n)^{-1}][\frac{\lambda_1}{\gamma}[\zeta_1(\alpha_1\delta_{P_1} + \beta_1\delta_{Q_1}) \\
&+ \left(\frac{2+\gamma}{\gamma}\right)\delta_{G_1}] + \xi_1\delta_{T_1}]\Big\}\|x_1^n - x_1^{n-1}\| + \frac{s\mu}{\gamma}\|x_2^n - x_2^{n-1}\|.
\end{aligned} \tag{3.14}$$

Again, using the fact that g is k-strongly accretive and Lipschitz continuous with constant μ, we have

$$\begin{aligned}
\|g(x_2^n) - g(x_2^{n-1})\|\|x_2^n - x_2^{n-1}\| &\geq \langle g(x_2^n) - g(x_2^{n-1}), j(x_2^n - x_2^{n-1})\rangle \\
&\geq k\|x_2^n - x_2^{n-1}\|^2,
\end{aligned}$$

which implies that

$$
\begin{aligned}
\|x_2^n - x_2^{n-1}\| &\leq \frac{1}{k}\|g(x_2^n) - g(x_2^{n-1})\| \\
&= \frac{1}{k}\|R^{H,\lambda_2}_{M(\cdot,w_2^n),\psi_2}[H(g(x_1^n)) - \lambda_2\psi_2 \circ N_2(u_2^n, v_2^n) - \lambda_2 C^{H,\lambda_2}_{M,\psi_2}(z_2^n)] \\
&\quad - R^{H,\lambda_2}_{M(\cdot,w_2^{n-1}),\psi_2}[H(g(x_1^{n-1})) - \lambda_2\psi_2 \circ N_2(u_2^{n-1}, v_2^{n-1}) - \lambda_2 C^{H,\lambda_2}_{M,\psi_2}(z_2^{n-1})]\| \\
&\leq \frac{1}{k\gamma}\|H(g(x_1^n)) - H(g(x_1^{n-1}))\| + \frac{\lambda_2}{k\gamma}\|\psi_2 \circ N_2(u_2^n, v_2^n) - \psi_2 \circ N_2(u_2^{n-1}, v_2^{n-1})\| \\
&\quad + \frac{\lambda_2}{k\gamma}\|C^{H,\lambda_2}_{M,\psi_2}(z_2^n) - C^{H,\lambda_2}_{M,\psi_2}(z_2^{n-1})\| + \frac{\xi_2}{k}\|w_2^n - w_2^{n-1}\|.
\end{aligned} \tag{3.15}
$$

Since ψ_2 is Lipschitz continuous with constant ζ_2 with respect to $N_2(\cdot,\cdot)$ and $N_2(\cdot,\cdot)$ is (α_2, β_2)-Lipschitz continuous with respect to first and second argument, respectively, P_2 and Q_2 are $\mathcal{D}$-Lipschitz continuous with constant δ_{P_2} and δ_{Q_2}, respectively, then we have

$$
\begin{aligned}
\|\psi_2 \circ N_2(u_2^n, v_2^n) - \psi_2 \circ N_2(u_2^{n-1}, v_2^{n-1})\| &\leq \zeta_2\|N_2(u_2^n, v_2^n) - N_2(u_2^{n-1}, v_2^{n-1})\| \\
&\leq \zeta_2[1 + (1+n)^{-1}][\alpha_2\delta_{P_2} + \beta_2\delta_{Q_2}]\|x_1^n - x_1^{n-1}\|.
\end{aligned} \tag{3.16}
$$

Since H and g are Lipschitz continuities with constant s and μ, respectively, then we have

$$
\|H(g(x_1^n)) - H(g(x_1^{n-1}))\| \leq s\mu\|x_1^n - x_1^{n-1}\|. \tag{3.17}
$$

Again, it follows from the fact that $C^{H,\lambda_2}_{M,\psi_2}$ is $\left(\frac{2+\gamma}{\gamma}\right)$-Lipschitz continuous and G_2 is $\mathcal{D}$-Lipschitz continuous with constant δ_{G_2}, then we have

$$
\begin{aligned}
\|C^{H,\lambda_1}_{M,\psi_2}(z_2^n) - C^{H,\lambda_1}_{M,\psi_2}(z_2^{n-1})\| &\leq \left(\frac{2+\gamma}{\gamma}\right)\|z_2^n - z_2^{n-1}\| \\
&\leq \left(\frac{2+\gamma}{\gamma}\right)[1 + (1+n)^{-1}]\delta_{G_2}\|x_1^n - x_1^{n-1}\|.
\end{aligned} \tag{3.18}
$$

Also, T_2 is $\mathcal{D}$-Lipschitz continuous with constant δ_{T_2}, then we have

$$
\begin{aligned}
\|w_2^n - w_2^{n-1}\| &\leq [1 + (1+n)^{-1}]\mathcal{D}(T_2(x_1^n), T_2(x_1^{n-1})) \\
&\leq [1 + (1+n)^{-1}]\delta_{T_2}\|x_1^n - x_1^{n-1}\|.
\end{aligned} \tag{3.19}
$$

Thus, from (3.15)–(3.19), we have

$$
\begin{aligned}
\|x_2^n - x_2^{n-1}\| \leq\ & \Big\{\frac{s\mu}{k\gamma} + [1 + (1+n)^{-1}]\Big[\frac{\lambda_2}{k\gamma}[\zeta_2(\alpha_2\delta_{P_2} + \beta_2\delta_{Q_2}) + \left(\frac{2+\gamma}{\gamma}\right)\delta_{G_2}] \\
&+ \frac{\xi_2}{k}\delta_{T_2}\Big]\Big\}\|x_1^n - x_1^{n-1}\|.
\end{aligned} \tag{3.20}
$$

It follows from (3.14) and (3.20) that

$$
\begin{aligned}
\|x_1^{n+1} - x_1^n\| \leq\ & \Big\{\sqrt{1 - 2k + 64c\mu^2} + [1 + (1+n)^{-1}][\frac{\lambda_1}{\gamma}[\zeta_1(\alpha_1\delta_{P_1} + \beta_1\delta_{Q_1}) \\
&+ \left(\frac{2+\gamma}{\gamma}\right)\delta_{G_1}] + \xi_1\delta_{T_1}]\Big\}\|x_1^n - x_1^{n-1}\| \\
&+ \frac{s\mu}{\gamma}\Big\{\frac{s\mu}{k\gamma} + [1 + (1+n)^{-1}]\Big[\frac{\lambda_2}{k\gamma}[\zeta_2(\alpha_2\delta_{P_2} + \beta_2\delta_{Q_2}) \\
&+ \left(\frac{2+\gamma}{\gamma}\right)\delta_{G_2}] + \frac{\xi_2}{k}\delta_{T_2}\Big]\Big\}\|x_1^n - x_1^{n-1}\|.
\end{aligned} \tag{3.21}
$$

Thus, we have

$$\|x_1^{n+1} - x_1^n\| \leq \Theta(P_n)\|x_1^n - x_1^{n-1}\|, \tag{3.22}$$

where,

$$\begin{aligned}\Theta(P_n) \;=\; & \Delta_n + [1+(1+n)^{-1}]\Big\{\frac{\lambda_1}{\gamma}[\zeta_1(\alpha_1\delta_{P_1} + \beta_1\delta_{Q_1}) + \Big(\frac{2+\gamma}{\gamma}\Big)\delta_{G_1}] \\ & + \frac{s\mu}{\gamma}\Big[\frac{\lambda_2}{k\gamma}[\zeta_2(\alpha_2\delta_{P_2} + \beta_2\delta_{Q_2}) + \Big(\frac{2+\gamma}{\gamma}\Big)\delta_{G_2}] + \frac{\xi_2}{k}\delta_{T_2}\Big]\Big\},\end{aligned} \tag{3.23}$$

$\Delta_n = \sqrt{1 - 2k + 64c\mu^2} + \dfrac{s^2\mu^2}{k\gamma^2} + [1+(1+n)^{-1}]\xi_1\delta_{T_1}$.

We see that $\Theta(P_n) \to \Theta(P)$ as $n \to \infty$, where $\Theta(P) = \Delta + l_1 + l_2 + \dfrac{\xi_2}{k}\delta_{T_2}$ and $\Delta = \sqrt{1-2k+64c\delta^2} + \dfrac{s^2\mu^2}{k\gamma^2} + \xi_1\delta_{T_1}$; $l_1 = \dfrac{\lambda_1}{\gamma}[\zeta_1(\alpha_1\delta_{P_1} + \beta_1\delta_{Q_1}) + \Big(\dfrac{2+\gamma}{\gamma}\Big)\delta_{G_1}]$; $l_2 = \dfrac{s\mu}{\gamma}\Big[\dfrac{\lambda_2}{k\gamma}[\zeta_2(\alpha_2\delta_{P_2} + \beta_2\delta_{Q_2}) + \Big(\dfrac{2+\gamma}{\gamma}\Big)\delta_{G_2}]$. It follows from the condition (3.6) that $0 < \Theta(P) < 1$, and consequently by (3.22), $\{x_1^n\}$ is a Cauchy sequence in E. Similarly by (3.20) and (3.22), it follows that $\{x_2^n\}$ is also a Cauchy sequence in E. Since E is complete, then there exist $x_1, x_2 \in E$ such that $x_1^n \to x_1$ and $x_2^n \to x_2$ as $n \to \infty$. It follows from Algorithm 1 that

$$\begin{aligned}\|u_i^{n+1} - u_i^n\| &\leq (1+(1+n)^{-1})\mathcal{D}(P_i(x_1^{n+1}), P_i(x_1^n)) \\ &\leq (1+(1+n)^{-1})\delta_{P_i}\|x_1^n - x_1^{n-1}\|.\end{aligned} \tag{3.24}$$

$$\begin{aligned}\|v_i^{n+1} - v_i^n\| &\leq (1+(1+n)^{-1})\mathcal{D}(Q_i(x_1^{n+1}), Q_i(x_1^n)) \\ &\leq (1+(1+n)^{-1})\delta_{Q_i}\|x_1^n - x_1^{n-1}\|.\end{aligned} \tag{3.25}$$

$$\begin{aligned}\|w_i^{n+1} - w_i^n\| &\leq (1+(1+n)^{-1})\mathcal{D}(T_i(x_1^{n+1}), T_i(x_1^n)) \\ &\leq (1+(1+n)^{-1})\delta_{T_i}\|x_1^n - x_1^{n-1}\|.\end{aligned} \tag{3.26}$$

$$\begin{aligned}\|z_i^{n+1} - z_i^n\| &\leq (1+(1+n)^{-1})\mathcal{D}(G_i(x_1^{n+1}), G_i(x_1^n)) \\ &\leq (1+(1+n)^{-1})\delta_{G_i}\|x_1^n - x_1^{n-1}\|.\end{aligned} \tag{3.27}$$

Clearly, from (3.24)–(3.27), we know that $\{u_i^n\}, \{v_i^n\}, \{w_i^n\}$ and $\{z_i^n\}$ are also Cauchy sequences in E. Let $u_i^n \to u_i, v_i^n \to v_i, w_i^n \to w_i$ and $z_i^n \to z_i$ as $n \to \infty$. Thus, by Theorem 3.2, we conclude that $(x_i, u_i, v_i, w_i, z_i), x_i \in E, u_i \in P_i(x_1), v_i \in Q_i(x_1), w_i \in T_i(x_1)$ and $z_i \in G_i(x_1)$ is a solution of the system of set-valued Cayley type inclusions (3.1). Next, we show that $u_i \in P_i(x_1), v_i \in Q_i(x_1), w_i \in T_i(x_1)$ and $z_i \in G_i(x_1)$. Since,

$$\begin{aligned}d(u_i, P_i(x_1)) &\leq \|u_i - u_i^n\| + d(u_i^n, P_i(x_1)) \\ &\leq \|u_i - u_i^n\| + \mathcal{D}(P_i(x_1^n), P_i(x_1)) \\ &\leq \|u_i - u_i^n\| + \delta_{P_i}\|x_1^n - x_1\| \to 0 \ as \ n \to \infty,\end{aligned} \tag{3.28}$$

which shows that $d(u_i, P_i(x_1)) = 0$, and hence $u_i \in P_i(x_1)$. Similarly, one can show that $v_i \in Q_i(x_1), w_i \in T_i(x_1)$ and $z_i \in G_i(x_1)$, respectively. Now, we prove the uniqueness of the solution $(x_i, u_i, v_i, w_i, z_i)$. Let $(x_i', u_i', v_i', w_i', z_i'), x_i' \in E, u_i' \in P_i(x_1'), v_i' \in Q_i(x_1'), w_i' \in T_i(x_1')$ and $z_i' \in G_i(x_1')$ be another solution of the system of set-valued Cayley type inclusions (3.1), then it follows from Theorem 3.2 that

$$g(x_1') = R^{H,\lambda_1}_{M(\cdot,w_1'),\psi_1}[H(g(x_2')) - \lambda_1\psi_1 \circ N_1(u_1', v_1') - \lambda_1 C^{H,\lambda_1}_{M,\psi_1}(z_1')];$$

where,

$$g(x_2') = R^{H,\lambda_2}_{M(\cdot,w_2'),\psi_2}[H(g(x_1')) - \lambda_2\psi_2 \circ N_2(u_2', v_2') - \lambda_2 C^{H,\lambda_2}_{M,\psi_2}(z_2')].$$

Now following the same arguments as mentioned from (3.8)–(3.22), we have

$$\begin{aligned}\|x_1 - x_1'\| \leq\; & \|x_1 - g(x_1) + R^{H,\lambda_1}_{M(\cdot,w_1),\psi_1}[H(g(x_2)) - \lambda_1\psi_1 \circ N_1(u_1, v_1) - \lambda_1 C^{H,\lambda_1}_{M,\psi_1}(z_1)] \\ & - x_1' + g(x_1') - R^{H,\lambda_1}_{M(\cdot,w_1'),\psi_1}[H(g(x_2')) - \lambda_1\psi_1 \circ N_1(u_1', v_1') - \lambda_1 C^{H,\lambda_1}_{M,\psi_1}(z_1')]\| \\ \leq\; & \Theta(P)\|x_1 - x_1'\|.\end{aligned} \tag{3.29}$$

Since $0 < \Theta(P) < 1$, thus we have $x_1 = x_1^{'}$. Similarly, one can show that $x_2 = x_2^{'}, u_i = u_i^{'}, v_i = v_i^{'}, w_i = w_i^{'}$ and $z_i = z_i^{'}$. Therefore $(x_i, u_i, v_i, w_i, z_i), x_i \in E, u_i \in P_i(x_1), v_i \in Q_i(x_1), w_i \in T_i(x_1)$ and $z_i \in G_i(x_1)$ is unique solution of the system of set-valued Cayley type inclusions (3.1). □

Corollary 3.4. *Let E be a uniformly smooth Banach space with its dual E^* and modulus of smoothness $\rho_E(t) \leq ct^2$, for some $c > 0$. Let $g : E \to E$ be k-strongly accretive and Lipschitz continuous with constant μ and $H : E \to E^*$ be Lipschitz continuous with constant s with respect to g. Let $N : E \times E \to E^*$ be (α, β)-Lipschitz continuous in the first and second argument, respectively. Let $P, Q, T, G : E \to CB(E^*)$ be $\mathcal{D}$-Lipschitz continuous with constants $\delta_P, \delta_Q, \delta_T$ and δ_G, respectively. Let $\psi : E^* \to E^*$ be a Lipschitz continuous mapping with respect to $N(\cdot,\cdot)$ with constant ζ satisfying $\psi(x+y) = \psi(x)+\psi(y)$ with $Ker(\psi) = \{0\}$ and $M : E \times E \rightrightarrows 2^{E^*}$ be an (H, ψ)-monotone mapping. Assume that there exists a constant $\lambda > 0$ satisfying*

$$0 < \Theta(P) = \Delta + l < 1, \tag{3.30}$$

where,

$$\Delta = \sqrt{1 - 2k + 64c\mu^2} + \xi\delta_T;\ l = \frac{1}{\gamma}\Big[s\mu + \frac{\lambda}{\gamma}[\zeta(\alpha\delta_P + \beta\delta_Q) + \Big(\frac{2+\gamma}{\gamma}\Big)\delta_G\Big].$$

In addition the following condition holds:

$$\|R^{H,\lambda}_{M(\cdot,w^n),\psi}(u) - R^{H,\lambda}_{M(\cdot,w^{n-1}),\psi}(u)\| \leq \xi\|w^n - w^{n-1}\|,\ \xi > 0. \tag{3.31}$$

Then $(x, u, v, w, z), x \in E, u \in P(x), v \in Q(y), w \in T(x)$ and $z \in G(x)$ is unique solution of Cayley type inclusion problem (3.2). Moreover, the iterative sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$,$\{w_n\}$, $\{z_n\}$ generated by Algorithm 2 converge strogly to (x, u, v, w, z).

Example 3.5. *Let $E = \mathbb{R}$ with the usual inner product and norm. Let $g : E \to E, H : E \to E^*, \psi : E^* \to E^*$ and $N : E \times E \to E^*$ be the mappings defined by $g(x) = \dfrac{x}{4}, H(x) = 50x - \dfrac{7}{20}, \psi(x) = 5x$ and $N(x, y) = \dfrac{x}{15} + \dfrac{y}{20}, \forall x, y \in \mathbb{R}$, respectively. Then, it is easy to verify that g is $\dfrac{1}{5}$-strongly accretive and $\dfrac{1}{3}$-Lipschitz continuous, H is $\dfrac{51}{4}$-Lipschitz continuous with respect to g, ψ is $\dfrac{36}{60}$-Lipschitz continuous with respect to N and satisfies $\psi(x + y) = \psi(x) + \psi(y)$ $M(x) = \{14x\}, H(x) = 50x - \dfrac{7}{20}$ and $\psi(x) = 5x - \dfrac{1}{2}, \forall x \in \mathbb{R}$.*

$$\langle\psi \circ M(x) - \psi \circ M(y), x - y\rangle = 70(x - y)^2 \geq 0$$

Thus, $\psi \circ M$ is a monotone mapping. It is easy to see that

$$(H + \psi \circ M)(x) = 120x - \frac{17}{20}, \forall x \in \mathbb{R},$$

i.e., $(H + \psi \circ M)$ is surjective. Hence, M is (H, ψ)-monotone.

For $\lambda = 1$, the proximal point operator associated to (H, ψ)-monotone mapping defined by (2.3) is given by

$$R^{H,\lambda}_{M,\psi}(x) = (H + \lambda\psi \circ M)^{-1}(x) = \frac{x}{120} + \frac{17}{2400}, \forall x \in \mathbb{R}.$$

Now,

$$\|R^{H,\lambda}_{M,\psi}(x) - R^{H,\lambda}_{M,\psi}(y)\| = \frac{1}{120}\|x - y\| \leq \frac{1}{n}\|x - y\|, \forall x \in \mathbb{R}, n \leq 120.$$

Thus, the proximal point operator $R^{H,\lambda}_{M,\psi}$ is $\dfrac{1}{n}$–Lipschitz continuous, for $n \leq 120$.
The Cayley operator $C^{H,\lambda}_{M,\psi}$ defined by (2.4) is given by

$$C^{H,\lambda}_{M,\psi}(x) = \frac{-x - 1}{3}, \forall x \in \mathbb{R}.$$

$$\text{Also, } \|C^{H,\lambda}_{M,\psi}(x) - C^{H,\lambda}_{M,\psi}(y)\| = \frac{1}{120}\|x-y\| \leq \left(\frac{2}{n}+1\right)\|x-y\|, \forall x \in \mathbb{R}, n = 1,2,3.$$

Thus, the Cayley operator $C^{H,\lambda}_{M,\psi}$ is $\left(\frac{2}{n}+1\right)$–Lipschitz continuous, for $n = 1,2,3$.

4. Concluding Remarks

In this paper, we considered and studied a system of set-valued Cayley type inclusions involving Cayley operator and (H,ψ)-monotone operator in real Banach spaces, which includes many inclusion problems studied in the literature as special cases. We proved that Cayley operator associated with the (H,ψ)-monotone operator is Lipschitz type continuous. Existence and uniqueness of the approximate solution is proved. Moreover, we suggested an iterative algorithm for the system of set-valued Cayley type inclusions and the strong convergence of the sequences generated by the proposed algorithm is discussed.

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