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Some Modular Relation on Analogous of Ramanujan's Remarkable Product of Theta-Function

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ABSTRACT: In this article, we derive new modular relations on Ramanujan's product of theta-functions $\phi(q)$ and $f(-q^2)$, which is analogous to Ramanujan's remarkable product of theta-functions and their explicit evaluations.

Key Words: Class invariant, Modular equation, Theta-function, Cubic continued fraction.

Contents

1. Introduction

Ramanujan's general theta-function [18] $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1, \\ = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{1.1}$$

Three special cases of $f(a,b)$ are as follows:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.3}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} = (q;q)_\infty, \tag{1.4}$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1.$$

On page 338 in his first notebook [18], [3] Ramanujan defines

$$a_{m,n} = \frac{ne^{\frac{-(n-1)\pi}{4}\sqrt{\frac{m}{n}}}\psi^2(e^{-\pi\sqrt{mn}})\varphi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(e^{-\pi\sqrt{\frac{m}{n}}})\varphi^2(-e^{-2\pi\sqrt{\frac{m}{n}}})}. \tag{1.5}$$

He then, on pages 338 and 339, offers a list of eighteen particular values. All these eighteen values have been established by Berndt, Chan and Zhang [4]. M. S. Mahadeva Naika and B. N. Dharmendra [10],

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also established some general theorems for explicit evaluations of the product of $a_{m,n}$ and found some new explicit values from it. Further results on $a_{m,n}$ can be found by Mahadeva Naika, Dharmendra and K. Shivashankar [12], and Mahadeva Naika and M. C. Mahesh Kumar [13]. Recently Nipen Saikia [16] established new properties of $a_{m,n}$.

In [15], Mahadeva Naika et al. defined the product

$$b_{m,n} = \frac{ne^{\frac{-(n-1)\pi}{4}\sqrt{\frac{m}{n}}}\psi^2(-e^{-\pi\sqrt{mn}})\varphi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(-e^{-\pi\sqrt{\frac{m}{n}}})\varphi^2(-e^{-2\pi\sqrt{\frac{m}{n}}})}. \tag{1.6}$$

They established general theorems for explicit evaluation of $b_{m,n}$ and obtained some particular values. Mahadeva Naika et al. [14] established general formulas for explicit values of Ramanujan's cubic continued fraction $V(q)$ in terms of the products $a_{m,n}$ and $b_{m,n}$ defined above, where

$$V(q) := \frac{q^{1/3}}{1} \underset{+}{} \frac{q+q^2}{1} \underset{+}{} \frac{q^2+q^4}{1} \underset{+}{} \frac{q^3+q^6}{1} \underset{+}{} \cdots, \quad |q| < 1, \tag{1.7}$$

and found some particular values of $V(q)$

In [5], B. N. Dharmendra defined product of theta-fuctions $d_{m,n}$ as

$$d_{m,n} = \frac{f\left(-e^{-2\pi\sqrt{\frac{n}{m}}}\right)\varphi\left(e^{-\pi\sqrt{mn}}\right)}{e^{-\frac{(m-1)\pi}{12}\sqrt{\frac{n}{m}}}f\left(-e^{-2\pi\sqrt{mn}}\right)\varphi\left(e^{-\pi\sqrt{\frac{n}{m}}}\right)}, \tag{1.8}$$

where m and n are positive real numbers. He established several properties of the product $d_{m,n}$ and proved general formulas for explicit evaluations of $d_{m,n}$ and their explicit values.

Let K, K', L and L' denote the complete elliptic integrals of the first kind associated with the moduli k, $k' := \sqrt{1-k^2}$, l and $l' := \sqrt{1-l^2}$ respectively, where $0 < k, l < 1$. For a fixed positive integer n, suppose that

$$n\frac{K'}{K} = \frac{L'}{L}. \tag{1.9}$$

Then a modular equation of degree n is a relation between k and l induced by (1.5). Following Ramanujan, set $\alpha = k^2$ and $\beta = l^2$. Then we say β is of degree n over α.
Define

$$\chi(q) := (-q; q^2)_\infty,$$

and

$$G_n := 2^{-\frac{1}{4}}q^{-\frac{1}{24}}\chi(q),$$

where

$$q = e^{-\pi\sqrt{r}}.$$

Moreover, if $q = e^{-\pi\sqrt{\frac{n}{m}}}$ and β has degree n over α, then

$$G_{\frac{n}{m}} = (4\alpha(1-\alpha))^{\frac{-1}{24}} \tag{1.10}$$

and

$$G_{nm} = (4\beta(1-\beta))^{\frac{-1}{24}}. \tag{1.11}$$

The main purpose of this paper is to obtain several general theorems for the explicit evaluations of analogous of Ramanujan's product of theta-function of $d_{m,n}$ and also some new explicit evaluations from it.

2. Preliminary Results

In this section, we collect several identities which are useful in proving our main results.

Lemma 2.1. *[5, Theorem 3.1] We have,*

$$d_{m,1} = 1. \tag{2.1}$$

Lemma 2.2. *[5, Theorem 4.4] If n is any rational,*

$$d_{3,n} = \frac{f(-q^2)\varphi(q^3)}{q^{1/6}f(-q^6)\varphi(q)} \text{ and } P := \frac{\varphi(q)}{\varphi(q^3)}; \quad q := e^{-\pi\sqrt{\frac{n}{3}}},$$

then

$$d_{3,n}^6 = \frac{P^4 - 9}{P^4(1 - P^4)}, \quad P^4 \neq 1. \tag{2.2}$$

Lemma 2.3. *[5, Theorem 4.5] If n is any rational,*

$$d_{5,n} = \frac{f(-q^2)\varphi(q^5)}{q^{1/3}f(-q^{10})\varphi(q)} \text{ and } P := \frac{\varphi(q)}{\varphi(q^5)}; \quad q = e^{-\pi\sqrt{\frac{n}{5}}},$$

then

$$d_{5,n}^3 = \frac{5 - P^2}{P^2(P^2 - 1)}, \quad P^2 \neq 1. \tag{2.3}$$

Lemma 2.4. *[5, Theorem 4.6] If n is any rational,*

$$d_{9,n} = \frac{f(-q^2)\varphi(q^9)}{q^{2/3}f(-q^{18})\varphi(q)} \text{ and } P := \frac{\varphi(q)}{\varphi(q^9)}; \quad q = e^{-\pi\sqrt{\frac{n}{9}}},$$

then

$$d_{9,n}^3 = \left\{\frac{P - 3}{P(P - 1)}\right\}^2, \quad P \neq 1. \tag{2.4}$$

Lemma 2.5. *[8, Theorem 3.1] If* $P := \dfrac{\varphi(q)\varphi(q^2)}{\varphi(q^3)\varphi(q^6)}$ *and* $Q := \dfrac{\varphi(q)\varphi(q^6)}{\varphi(q^3)\varphi(q^2)}$, *then*

$$P + \frac{3}{P} + Q - \frac{1}{Q} = 4. \tag{2.5}$$

Lemma 2.6. *[6, Theorem 2.1] If* $P := \dfrac{\varphi(q)}{\varphi(q^3)}$ *and* $Q := \dfrac{\varphi(q^5)}{\varphi(q^{15})}$, *then*

$$(PQ)^2 + \frac{9}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 - \left(\frac{P}{Q}\right)^3 + 5\left\{\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2\right\} + 5\left\{\left(\frac{Q}{P}\right) - \left(\frac{P}{Q}\right)\right\}. \tag{2.6}$$

Lemma 2.7. *[6, Theorem 2.3] If* $P := \dfrac{\varphi(q)\varphi(q^7)}{\varphi(q^3)\varphi(q^{21})}$ *and* $Q := \dfrac{\varphi(q)\varphi(q^{21})}{\varphi(q^3)\varphi(q^7)}$, *then*

$$Q^4 - \frac{1}{Q^4} + 14\left(Q^2 - \frac{1}{Q^2}\right) + P^3 + \frac{27}{P^3} = 7\left(P + \frac{3}{P}\right)\left\{\left(Q^2 + \frac{1}{Q^2}\right) - 1\right\}. \tag{2.7}$$

Lemma 2.8. *[11, Theorem 3.2] If* $P := \dfrac{\varphi(q)}{\varphi(q^5)}$ *and* $Q := \dfrac{\varphi(q^2)}{\varphi(q^{10})}$, *then*

$$\begin{aligned}&\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 + (PQ)^2 + \left(\frac{5}{PQ}\right)^2 + 16\left(\frac{P}{Q} - \frac{Q}{P}\right)\\ &= 2\left(P^2 + \frac{5}{P^2}\right) + 2\left(Q^2 + \frac{5}{Q^2}\right) + 4.\end{aligned} \tag{2.8}$$

Lemma 2.9. *[2, Ch. 25, Entry 66, p.233] If* $P = \dfrac{\varphi(q)}{\varphi(q^5)}$ *and* $Q = \dfrac{\varphi(q^3)}{\varphi(q^{15})}$*, then*

$$PQ + \frac{5}{PQ} = -\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 + 3\left(\frac{P}{Q} + \frac{Q}{P}\right). \tag{2.9}$$

Lemma 2.10. *[11, Theorem 3.6] If* $P := \dfrac{\phi(q)\phi(q^7)}{\phi(q^5)\phi(q^{35})}$ *and* $Q := \dfrac{\phi(q)\phi(q^{35})}{\phi(q^5)\phi(q^7)}$*, then*

$$\begin{aligned} &Q^4 - \frac{1}{Q^4} - 14\left[\left(Q^3 + \frac{1}{Q^3}\right) - \left(Q^2 - \frac{1}{Q^2}\right) + 10\left(Q + \frac{1}{Q}\right)\right] + P^3 + \frac{5^3}{P^3} \\ &= 7\left\{\left(P^2 + \frac{5^2}{P^2}\right)\left(Q + \frac{1}{Q}\right) - \left(P + \frac{5}{P}\right)\left[2\left(Q^2 + \frac{1}{Q^2}\right) + 9\right]\right\}. \end{aligned} \tag{2.10}$$

3. Modular Relation Between $d_{3,n}$ and d_{3,k^2n}

In this section, we obtain some modular relation between $d_{3,n}$ and d_{3,k^2n}.

Theorem 3.1. *If* $x = d_{3,n}d_{3,4n}$ *and* $y = \dfrac{d_{3,n}}{d_{3,4n}}$*, then*

$$\begin{aligned} &\left(x^6 + \frac{1}{x^6}\right) + 25\left(x^4 + \frac{1}{x^4}\right) + 200\left(x^2 + \frac{1}{x^2}\right) + 550 = \left(y^6 + \frac{1}{y^6}\right) \\ &+ \left(y^3 + \frac{1}{y^3}\right)\left\{\left(x^5 + \frac{1}{x^5}\right) + 16\left(x^3 + \frac{1}{x^3}\right) + 71\left(x + \frac{1}{x}\right)\right\}. \end{aligned} \tag{3.1}$$

Proof. Using Lemma (2.2), $P = \frac{\varphi(q)}{\varphi(q^3)}$ can be written in terms of $r := d_{3,n}$ as

$$P = \frac{r - 1 + \sqrt{r^2 + 34r + 1}}{2r} \tag{3.2}$$

Employing the above equation (3.2) and Lemma (2.5), we obtain

$$\begin{aligned} &(71s^4r^{10} + s^{10}r^{16} + 71s^{10}r^4 - 200s^{10}r^{10} + s^{14}r^2 + 16s^{14}r^8 - s^{14}r^{14} + s^6 + r^6 \\ &+ s^2r^{14} - 200r^6s^6 - 25s^4r^4 + 16s^8r^2 - 550s^8r^8 + 16s^8r^{14} + 71r^{12}s^6 - 25r^{12}s^{12} \\ &+ 71r^6s^{12} + 16s^2r^8 - s^2r^2 + s^{16}r^{10})(s^{32}r^{20} + 4089s^8r^{20} - 71s^8r^{26} - 7s^4r^{10} \\ &- 1425s^{10}r^{16} + s^{20}r^{32} - 5630s^{10}r^{22} + 32s^{10}r^{28} - 7s^{10}r^4 - 3530s^{10}r^{10} - 16s^{14}r^2 \\ &+ 270s^{14}r^8 - 92247s^{14}r^{14} + s^{12} + r^{12} - 969s^{14}r^{26} + 14130s^{14}r^{20} - 16s^2r^{14} \\ &- 969s^{26}r^{14} + 71s^{26}r^{20} - 23s^{26}r^{26} - 71s^{26}r^8 + s^{30}r^{24} - 16s^{30}r^{18} - s^{30}r^{12} + s^2r^8 \\ &- 23r^6s^6 - s^{20}r^2 + 4089s^{20}r^8 + s^4r^4 + 14130s^{20}r^{14} + 24062s^{20}r^{20} + 71s^{20}r^{26} \\ &+ 32s^4r^{22} + s^4r^{28} + 183s^4r^{16} - s^2r^{20} - 1425s^{16}r^{22} + 252819s^{16}r^{16} - 7s^{28}r^{22} \\ &+ 393s^8r^8 - 3530s^{22}r^{22} - 7s^{22}r^{28} + 32s^{22}r^4 - 1425s^{22}r^{16} - 5630s^{22}r^{10} + s^8r^2 \\ &+ 183s^{28}r^{16} + 32s^{28}r^{10} + s^{28}r^{28} + 183s^{16}r^{28} + r^{30}s^{24} - 16r^{30}s^{18} + 270s^8r^{14} \\ &- r^{30}s^{12} - 71r^{24}s^6 + 4089r^{24}s^{12} - 969r^{18}s^6 + 14130r^{18}s^{12} + s^{28}r^4 + 71r^{12}s^6 \\ &+ 24062r^{12}s^{12} + 71r^6s^{12} - 92247r^{18}s^{18} + 14130r^{12}s^{18} - 969r^6s^{18} - 71s^{24}r^6 \\ &+ 270r^{24}s^{18} + 183s^{16}r^4 + 4089s^{24}r^{12} + 270r^{18}s^{24} + 393r^{24}s^{24} - 1425s^{16}r^{10}) = 0. \end{aligned} \tag{3.3}$$

Here $s = d_{3,4n}$ and by examining the behavior of the above factors near $q = 0$, we can find a neighborhood about the origin, where the first factor is zero; whereas other factor is not zero in this neighborhood. By the Identity Theorem second factor vanishes identically. This completes the proof. □

Corollary 3.2.

$$\begin{aligned} &(i)\ d_{3,4}=\frac{\sqrt{3\sqrt{6}-2}+\sqrt{3\sqrt{6}-6}}{2},\\ &(ii)\ d_{3,1/4}=\frac{\sqrt{3\sqrt{6}-2}-\sqrt{3\sqrt{6}-6}}{2}. \end{aligned} \tag{3.4}$$

Proof. Employing the above Theorem (3.1) with $n=1$ and Lemma (2.1), we obtain

$$(t^8+8t^6-36t^4+8t^2+1)(t^4+4t^2+1)^2=0, \tag{3.5}$$

when $q=0$ by identical theorem first factor vanishes, whereas the second factor does not vanish.

$$t^8+8t^6-36t^4+8t^2+1=0, \tag{3.6}$$

where $t:=d_{3,4}$. The above equation can be written as,

$$T^4+4T^2-50=0. \tag{3.7}$$

where $T:=t+\frac{1}{t}$. Solving in the above equation, we get (3.4). □

Theorem 3.3. *If* $x=d_{3,n}d_{3,25n}$ *and* $y=\frac{d_{3,n}}{d_{3,25n}}$, *then*

$$\left(x^2+\frac{1}{x^2}\right)+10\left(x+\frac{1}{x}\right)+25=\left(y^3+\frac{1}{y^3}\right). \tag{3.8}$$

Proof. Employing the equation (3.2) and Lemma (2.6), we get (3.12).

□

Corollary 3.4.

$$\begin{aligned} &(i)\ d_{3,25}=\frac{1+4k+k^2+3\sqrt{5+2k+2k^2}}{6},\\ &(ii)\ d_{3,1/25}=\frac{1+4k+k^2-3\sqrt{5+2k+2k^2}}{6}, \end{aligned} \tag{3.9}$$

where $k:=10^{\frac{1}{3}}$.

Proof. Employing the above Theorem (3.3) with $n=1$ and Lemma (2.1), we obtain

$$-10t^2-t+1-10t^4-t^5-25t^3+t^6=0, \tag{3.10}$$

where $t:=d_{3,25}$. The above equation can be written as,

$$T^3-T^2-13T-23=0, \tag{3.11}$$

where $T:=t+\frac{1}{t}$. Solving in the above equation, we get (3.9). □

Theorem 3.5. *If* $x=d_{3,n}d_{3,49n}$ *and* $y=\frac{d_{3,n}}{d_{3,49n}}$, *then*

$$\begin{aligned} \left(x^3+\frac{1}{x^3}\right)&=\left(y^4+\frac{1}{y^4}\right)-14\left(y^3+\frac{1}{y^3}\right)\\ &+77\left(y^2+\frac{1}{y^2}\right)-210\left(y+\frac{1}{y}\right)+294. \end{aligned} \tag{3.12}$$

Proof. Using the equation (3.2) and Lemma (2.7), we obtain (3.12). □

Corollary 3.6.

$$d_{3,49} = \frac{91 + 14k + k^2 + 3\sqrt{1029 + 294k + 42k^2}}{42}, \tag{3.13}$$

where $k := 98^{\frac{1}{3}}$.

Proof. Employing the above Theorem (3.5) with $n = 1$ and Lemma (2.1), we obtain

$$1 + 77t^2 - 210t^5 + t^8 - 15t + 294t^4 - 15t^7 + 77t^6 - 210t^3 = 0, \tag{3.14}$$

where $t := d_{3,49}$. The above equation can be written as,

$$T^4 - 15T^3 + 73T^2 - 165T + 142 = 0, \tag{3.15}$$

where $T := t + \dfrac{1}{t}$. Solving in the above equation, we get (3.13). □

4. Modular Relation Between $d_{5,n}$ and d_{5,k^2n}

In this section, we obtain some modular relation between $d_{5,n}$ and d_{5,k^2n}.

Theorem 4.1. *If* $x = d_{5,n}d_{5,4n}$ *and* $y = \dfrac{d_{5,n}}{d_{5,4n}}$, *then*

$$\begin{aligned}\left(x^3 + \frac{1}{x^3}\right) + 13\left(x^2 + \frac{1}{x^2}\right) + 52\left(x + \frac{1}{x}\right) + 82 &= \left(\{xy\}^{\frac{3}{2}} + \frac{1}{\{xy\}^{\frac{3}{2}}}\right)\\ + \left(y^3 + \frac{1}{y^3}\right) + \left(y^{\frac{3}{2}} + \frac{1}{y^{\frac{3}{2}}}\right)&\left\{\left(x^{\frac{5}{2}} + \frac{1}{x^{\frac{5}{2}}}\right) + 8\left(x^{\frac{3}{2}} + \frac{1}{x^{\frac{3}{2}}}\right) + 19\left(x^{\frac{1}{2}} + \frac{1}{x^{\frac{1}{2}}}\right)\right\}.\end{aligned} \tag{4.1}$$

Proof. Using Lemma (2.3), $P = \frac{\varphi(q)}{\varphi(q^5)}$ can be written in terms of $r := d_{5,n}$ as

$$P^2 = \frac{r^3 - 1 + \sqrt{r^6 + 18r^3 + 1}}{2r^3} \tag{4.2}$$

Employing the above equation (4.2) and Lemma (2.8), we obtain

$$\begin{aligned}&(8r^4s + 8r^4s^7 - 82r^4s^4 - rs + 19r^2s^5 + rs^7 + 8rs^4 - 13r^2s^2 + r^5s^8 - r^7s^7\\ &+ 8r^7s^4 + 19r^5s^2 + r^7s - 52r^5s^5 + r^8s^5 - 13r^6s^6 + 19r^6s^3 + 19s^6r^3 + r^3\\ &+ s^3 - 52r^3s^3)(r^{16}s^{10} - r^{15}s^6 + r^{15}s^{12} + r^{12}s^{15} + r^4s + 22r^4s^7 + 101r^4s^4\\ &+ 43r^8s^{14} - 101r^7s^{13} + 58r^7s^{10} - rs^{10} + r^2s^{14} - 3r^2s^5 + 43r^2s^8 + 16r^2s^{11}\\ &- 8rs^7 + rs^4 + r^2s^2 - 25r^8s^{11} + 3387r^8s^8 - 19r^4s^{13} + 253r^4s^{10} + r^{14}s^{14}\\ &+ 16r^{14}s^5 + 43r^{14}s^8 - 3r^{14}s^{11} - 101r^{13}s^7 - 19r^{13}s^4 + r^{14}s^2 + 19r^{13}s^{10}\\ &- 11r^{13}s^{13} + 16r^5s^{14} - 398r^5s^{11} + r^{10}s^{16} - r^{10}s + 19r^{10}s^{13} + 58r^{10}s^7\\ &- 25r^5s^8 + 253r^{10}s^4 + 1358r^{10}s^{10} - 2523r^7s^7 + 22r^7s^4 - 3r^5s^2 - 8r^7s\\ &+ 43r^8s^2 + 16r^{11}s^2 + 1358r^6s^6 + 19r^6s^3 + 19s^6r^3 - 101r^9s^3 - 101r^3s^9\\ &+ 58r^9s^6 - 19r^{12}s^3 + 253s^{12}r^6 - 19s^{12}r^3 + 58s^9r^6 + 101r^{12}s^{12} + 22r^9s^{12}\\ &- 398r^{11}s^5 - 3r^{11}s^{14} + r^6 + s^6 - 422r^{11}s^{11} - 11r^3s^3 + 253r^{12}s^6 + 22r^{12}s^9\\ &- 2523r^9s^9 - 25r^{11}s^8 - s^{15}r^6 - 8s^{15}r^9 - 8r^{15}s^9 - 422r^5s^5 - 25r^8s^5) = 0.\end{aligned} \tag{4.3}$$

Here $s = d_{5,4n}$ and by examining the behavior of the above factors near $q = 0$, we can find a neighborhood about the origin, where the first factor is zero; whereas other factors are not zero in this neighborhood. By the Identity Theorem second factor vanishes identically. This completes the proof. □

Corollary 4.2.

$$(i)\ \ d_{5,4}=\frac{a+\sqrt{a^2-4a}-2}{2},\qquad (ii)\ \ d_{5,1/4}=\frac{a+\sqrt{a^2-4a}-2}{2}, \tag{4.4}$$

where $a:=\sqrt{11+5\sqrt{5}}$.

Proof. Employing the above Theorem (4.1) with $n=1$ and Lemma (2.1), we obtain

$$8t+8t^7-66t^4-32t^5+6t^2+t^8+6t^6-32t^3+1=0, \tag{4.5}$$

where $t:=d_{5,4}$. The above equation can be written as,

$$T^4+8T^3+2T^2-56T-76=0. \tag{4.6}$$

where $T:=t+\dfrac{1}{t}$. Solving in the above equation, we get (4.4). □

Theorem 4.3. *If* $x=d_{5,n}d_{5,9n}$ *and* $y=\dfrac{d_{5,n}}{d_{5,9n}}$, *then*

$$\left(x^3+\frac{1}{x^3}\right)+18\left\{\left(x^{\frac{3}{2}}+\frac{1}{x^{\frac{3}{2}}}\right)\left(y^{\frac{3}{2}}+\frac{1}{y^{\frac{3}{2}}}\right)\right\}=\left(y^6+\frac{1}{y^6}\right)-45\left(y^3+\frac{1}{y^3}\right)+162. \tag{4.7}$$

Proof. Using the equation (4.2) and Lemma (2.9), we obtain

$$\begin{aligned}&(s^{12}-45r^3s^9-18r^6s^9-r^9s^9-18r^3s^6+162r^6s^6-18r^9s^6-r^3s^3+r^{12}\\&-18r^6s^3-45r^9s^3)(369s^9n-881s^{12}-406s^6+203s^6n+n-1476s^9\\&-36s^3+27s^3n-1)=0\end{aligned} \tag{4.8}$$

Here $s=d_{5,9n}$, $n=\sqrt{s^6+18s^3+1}$ and by examining the behavior of the above factors near $q=0$, we can find a neighborhood about the origin, where the first factor is zero; whereas other factors are not zero in this neighborhood. By the Identity Theorem second factor vanishes identically. This completes the proof. □

Corollary 4.4.

$$(i)\ \ d_{5,9}=\left(31+8\sqrt{15}\right)^{\frac{1}{3}},\qquad (ii)\ \ d_{5,1/9}=\left(31-8\sqrt{15}\right)^{\frac{1}{3}}. \tag{4.9}$$

Proof. If $n=1$ in the above Theorem (4.3),we obtain (4.9). □

Theorem 4.5. *If* $x=d_{5,n}d_{5,49n}$ *and* $y=\dfrac{d_{5,n}}{d_{5,49n}}$, *then*

$$\begin{aligned}&\left(x^3+\frac{1}{x^3}\right)+14\left(x^{\frac{3}{2}}+\frac{1}{x^{\frac{3}{2}}}\right)\left\{\left(y^{\frac{3}{2}}+\frac{1}{y^{\frac{3}{2}}}\right)+6\left(y^{\frac{1}{2}}+\frac{1}{y^{\frac{1}{2}}}\right)\right\}\\&=\left(y^4+\frac{1}{y^4}\right)-7\left(y^3+\frac{1}{y^3}\right)-35\left(y^2+\frac{1}{y^2}\right)+42\left(y+\frac{1}{y}\right)+392.\end{aligned} \tag{4.10}$$

Proof. Using the equation (4.2) and Lemma (2.10), we obtain (4.10). □

Corollary 4.6.

$$d_{5,49} = \frac{a^2 + 20a + 889 + \sqrt{a^4 + 2142a^2 + 35560a + 1860201 + 36960\sqrt{15}}}{6a}, \tag{4.11}$$

where $a := (26747 + 924\sqrt{15})^{\frac{1}{3}}$

Proof. Employing the above Theorem (4.5) with $n = 1$ and Lemma (2.1), we obtain

$$(t^6 - 20t^5 - 160t^4 - 342t^3 - 160t^2 - 20t + 1)(t-1)^2 = 0, \tag{4.12}$$

when $q = 0$ by identical theorem first factor vanishes, whereas the second factor does not vanish.

$$t^6 - 20t^5 - 160t^4 - 342t^3 - 160t^2 - 20t + 1 = 0, \tag{4.13}$$

where $t := d_{5,49}$. The above equation can be written as,

$$T^3 - 20T^2 - 163T - 302 = 0. \tag{4.14}$$

where $T := t + \frac{1}{t}$. Solving in the above equation, we get (4.11). □

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