



Existence of Entropy Solutions of the Anisotropic Elliptic Nonlinear Problem with Measure Data in Weighted Sobolev Space

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ABSTRACT: This paper is devoted to study the following nonlinear anisotropic elliptic unilateral problem

$$\begin{cases} Au - \operatorname{div} \phi(u) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the right hand side μ belongs to $L^1(\Omega) + W^{-1, \vec{p}'}(\Omega, \vec{\omega}^*)$. The operator $Au = - \sum_{i=1}^N \partial_i a_i(x, u, \nabla u)$ is a Leray-Lions anisotropic operator acting from $W_0^{1, \vec{p}}(\Omega, \vec{\omega})$ into its dual $W^{-1, \vec{p}'}(\Omega, \vec{\omega}^*)$ and $\phi_i \in C^0(\mathbb{R}, \mathbb{R})$.

Key Words: Entropy solutions, Anisotropic unilateral, Weighted anisotropic Sobolev space.

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1. Introduction

Suppose that Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2)$ with smooth boundary. In this paper, let us consider the following nonlinear elliptic problem

$$\begin{cases} - \sum_{i=1}^N \partial_i a_i(x, u, \nabla u) - \sum_{i=1}^N \partial_i \phi_i(u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\phi = (\phi_1, \dots, \phi_N)$ belongs to $C^0(\mathbb{R}, \mathbb{R})^N$. As regards the second member, we assume that the datum μ belongs to $L^1(\Omega) + W^{-1, \vec{p}'}(\Omega, \vec{\omega}^*)$.

The space $W^{-1, \vec{p}'}(\Omega, \vec{\omega}^*)$ is the dual space of the weighted anisotropic Sobolev space $W^{1, \vec{p}}(\Omega, \vec{\omega})$, where $1 < p_1, \dots, p_N < +\infty$ be a N real numbers and $\vec{p} = \{p_0, \dots, p_N\}$, the vector $\vec{\omega}$ denoting a vector of measurable positive functions, i.e., $\vec{\omega} = \{\omega_1, \dots, \omega_N\}$, with ω_i are weight measurable functions for all $i = 1, \dots, N$ (we refer to [1,2,13] for more details).

In this study we are using the entropy solutions who was introduced for the first time by P. Benilan et al [7], because the function ϕ_i does not belongs to $L^1_{loc}(\Omega)$ in general, then the problem (1.1) does not admit weak solution. In the case of a datum in $\mu \in L^1(\Omega) + W^{-1, \vec{p}'}(\Omega)$ the existence of entropy solutions is treated by A. Salmani, Y. Akdim and H. Redwane in [16]. Moreover, L. Boccardo, T. Gallouet and L. Orsina (see [11]) have considered the case $\phi \equiv 0$.

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The objective of our paper is to study the anisotropic unilateral nonlinear elliptic problem associated with the nonlinear problem (1.1). More precisely, we establish only the existence of entropy solutions for the following unilateral anisotropic problem,

$$\left\{ \begin{array}{l} u \geq \psi \text{ a.e. in } \Omega, \\ T_k(u) \in W_0^{1, \vec{p}}(\Omega, \vec{\omega}) \quad \forall k > 0, \\ \sum_{i=1}^N \int_{\Omega} a_i(x, u, \nabla u) \partial_i T_k(u-v) dx + \sum_{i=1}^N \int_{\Omega} \phi_i(u) \partial_i T_k(u-v) dx \\ \leq \int_{\Omega} f T_k(u-v) dx + \sum_{i=1}^N \int_{\Omega} F_i \partial_i T_k(u-v) dx, \\ \forall v \in K_{\psi} \cap L^{\infty}(\Omega), \end{array} \right. \quad (1.2)$$

in the convex class $K_{\psi} := \left\{ u \in W_0^{1, \vec{p}}(\Omega, \vec{\omega}(x)), u \geq \psi \text{ a.e. in } \Omega \right\}$, where ψ is a measurable function on Ω such that

$$\psi^+ \in W_0^{1, \vec{p}}(\Omega, \vec{\omega}) \cap L^{\infty}(\Omega). \quad (1.3)$$

This type of problem has been studied by many authors in recent years, in particular by Y. Akdim, C. Allalou and A. Salmani (see [4]) have demonstrated the existence of entropy solutions problem like (1.1). In the non weighted case $\omega_i \equiv 1$ for any $i \in \{1, \dots, N\}$, Boccardo et al. in [10] studied the existence of weak solutions for nonlinear elliptic problem (1.1) with $Au = - \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right)$, $\phi_i(u) = 0$ for $i = 1, \dots, N$ and the right-hand side is a bounded Radon measure on Ω . In addition this, we mention some works in this direction such as [5,8,12,16,6,18]. The uniqueness problem being a rather delicate one, due to a counterexample by J. Serrin (see [17], and [15] for further remarks).

One of the motivations for studying (1.1) comes from applications the mathematical modeling of physical and mechanical processes in anisotropic continuous medium.

Let us briefly summarize the contents of the paper : after a section devoted to developing the necessary functional setting as Lebesgue space with weighted and the weighted anisotropic Sobolev space, we introduce some useful technical lemmas to prove existence results and basic assumptions of our problem in section 3. In the final section we state the main result and proofs.

2. Mathematical preliminaries

Throughout this paper Ω is a bounded open subset of \mathbb{R}^N ($N \geq 2$) with smooth boundary. Almost everywhere positive and locally integrable function $\omega : \Omega \rightarrow \mathbb{R}$ will be called a weight. We shall denote by $L^p(\Omega, \omega)$ the set of all measurable functions u on Ω such that the norm

$$\|u\|_{L^p(\Omega, \omega)} \equiv \|u\|_{p, \omega} = \left(\int_{\Omega} |u|^p \omega(x) dx \right)^{\frac{1}{p}} \quad 1 \leq p < \infty \quad (2.1)$$

Let p_1, \dots, p_N be N real numbers, we define the following vectors $\vec{p} = \{p_1, \dots, p_N\}$ be a vector of exponent and $\vec{\omega} = \{\omega_1, \dots, \omega_N\}$ be a vector of weight functions, i.e., every component ω_i is a measurable function which is positive a.e. in Ω . Moreover, we assume in all our considerations that

$$\begin{aligned} (\mathbf{A}_1) \quad & \omega_i \in L_{loc}^1(\Omega) \\ (\mathbf{A}_2) \quad & \omega_i^{\frac{-1}{p_i-1}} \in L_{loc}^1(\Omega). \end{aligned}$$

for any $i = 1, \dots, N$, we denote

$$\partial_i u = \frac{\partial u}{\partial x_i} \quad \text{for } i = 1, \dots, N,$$

$$p^- = \min\{p_1, \dots, p_N\}, \quad p^+ = \max\{p_1, \dots, p_N\} \quad (2.2)$$

At present, let us consider the weighted anisotropic Sobolev space $W^{1, \vec{p}}(\Omega, \vec{w})$ is defined as follow

$$W^{1, \vec{p}}(\Omega, \vec{w}) = \left\{ u \in L^1_{loc}(\Omega) \quad \text{and} \quad D^i u \in L^{p_i}(\Omega, \omega_i), \quad i = 1, \dots, N \right\},$$

is a Banach space with respect to norm (see [13])

$$\|u\|_{1, \vec{p}, \vec{w}} = \|u\|_{L^1(\Omega)} + \sum_{i=1}^N \|\partial_i u\|_{p_i, \omega_i}. \quad (2.3)$$

We define the functional space $W_0^{1, \vec{p}}(\Omega, \vec{w})$ as the closure of $C_0^\infty(\Omega)$ in $W^{1, \vec{p}}(\Omega, \vec{w})$ with respect to the norm (2.3). Note that $C_0^\infty(\Omega)$ is dense in $W_0^{1, \vec{p}}(\Omega, \vec{w})$. By an adapted method of that of Adams [2], and by constructing an isometric isomorphism from $W^{1, \vec{p}}(\Omega, \vec{w})$ into $\prod_{i=1}^N L^{p_i}(\Omega, \omega_i)$, we can show that $(W_0^{1, \vec{p}}(\Omega, \vec{w}), \|\cdot\|_{1, \vec{p}, \vec{w}})$ is separable and reflexive if $1 \leq p_i < \infty$ and $1 < p_i < \infty$, respectively, for all $i = 1, \dots, N$. For $p_i > 1$, $W^{-1, \vec{p}'}(\Omega, \vec{\omega}^*)$ designs its dual where \vec{p}' is the conjugate of \vec{p} , i.e. $p'_i = \frac{p_i}{p_i - 1}$ and $\vec{\omega}^* = \left\{ \omega_i^* = \omega_i^{1-p'_i}, i = 1, \dots, N \right\}$.

Lemma 2.1. *Let Ω be a smooth bounded open subset of \mathbb{R}^N , and suppose that $\inf w_i(\cdot) > 0$ a.e. in Ω for all $i = 1, \dots, N$. Let (A_1) and (A_2) be satisfied, we have*

- If $p^- < N$, then $W_0^{1, \vec{p}}(\Omega, \vec{w}) \subset L^q(\Omega)$ for all $q \in [p^-, (p^-)^*]$, with $\frac{1}{(p^-)^*} = \frac{1}{p^-} - \frac{1}{N}$.
- If $p^- = N$, then $W_0^{1, \vec{p}}(\Omega, \vec{w}) \subset L^q(\Omega)$ for all $q \in [p^-, +\infty[$,
- If $p^- > N$, then $W_0^{1, \vec{p}}(\Omega, \vec{w}) \subset L^\infty(\Omega) \cap C^0(\bar{\Omega})$.

Further, the embeddings are compacts. The proof of this lemma follows from the fact that we are the embedding

$$W_0^{1, \vec{p}}(\Omega, \vec{w}) \subset W_0^{1, \vec{p}}(\Omega) \subset W_0^{1, p^-}(\Omega)$$

Remark 2.2. *A note concerning the anisotropic spaces $W_0^{1, \vec{p}}(\Omega)$ and their embedding theorems, can be found in [9].*

The rest of this paper, note by

$$\mathcal{T}_0^{1, \vec{p}}(\Omega, \vec{w}) := \left\{ u \text{ measurable in } \Omega, T_k(u) \in W_0^{1, \vec{p}}(\Omega, \vec{w}), \text{ for any } k > 0 \right\},$$

where

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k \frac{s}{|s|} & \text{if } |s| > k. \end{cases}$$

3. Basic assumptions and technical lemmas

We introduce in this section some useful technical lemmas to prove existence results, and we impart the assumptions of our problem. The functions $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ are Carathéodory functions (measurable with respect to x in Ω for every (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$ and continuous with respect to (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω) which satisfied the following conditions, for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N, \xi' \in \mathbb{R}^N$ and a. e. in $x \in \Omega$,

$$a_i(x, s, \xi) \xi_i \geq \alpha \omega_i |\xi_i|^{p_i} \quad \text{for } i = 1, \dots, N, \quad (3.1)$$

$$|a_i(x, s, \xi)| \leq \beta \omega_i^{\frac{1}{p_i}} \left(R_i(x) + \omega_i^{\frac{1}{p_i'}} |s|^{\frac{p_i}{p_i'}} + \omega_i^{\frac{1}{p_i'}} |\xi_i|^{p_i-1} \right) \quad \text{for } i = 1, \dots, N, \quad (3.2)$$

$$(a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi_i - \xi'_i) > 0 \quad \text{for } \xi_i \neq \xi'_i, \quad (3.3)$$

where R_i is a nonnegative function lying in $L^{p_i'}(\Omega)$ and $\alpha, \beta > 0$. Moreover, we suppose that

$$\phi_i \in C^0(\mathbb{R}, \mathbb{R}) \quad \text{for } i = 1, \dots, N. \quad (3.4)$$

and

$$\mu \in L^1(\Omega) + W^{-1, \vec{p}'}(\Omega, \vec{\omega}^*). \quad (3.5)$$

Lemma 3.1. [1] Let $g \in L^r(\Omega, \gamma)$ and $g_n \subset L^r(\Omega, \gamma)$ such that $\|g_n\|_{r, \gamma} \leq C$, $1 < r < \infty$, If $g_n(x) \rightarrow g(x)$ a.e. in Ω then $g_n \rightarrow g$ weakly in $L^r(\Omega, \gamma)$.

Lemma 3.2. [3] Assume that (3.1) - (3.3) hold, let $(u_n)_n$ a sequence in $W_0^{1, \vec{p}}(\Omega, \vec{\omega})$ and $u \in W_0^{1, \vec{p}}(\Omega, \vec{\omega})$, if

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1, \vec{p}}(\Omega, \vec{\omega}),$$

and

$$\sum_{i=1}^N \int_{\Omega} (a_i(x, u_n, \nabla u_n) - a_i(x, u, \nabla u)) \partial_i (u_n - u) dx \rightarrow 0,$$

then $u_n \rightarrow u$ strongly in $W_0^{1, \vec{p}}(\Omega, \vec{\omega})$.

Lemma 3.3. [3] If $u \in W_0^{1, \vec{p}}(\Omega, \vec{\omega})$ then $\sum_{i=1}^N \int_{\Omega} \partial_i u dx = 0$.

Proof. Since $u \in W_0^{1, \vec{p}}(\Omega, \vec{\omega})$ there exists $u_k \in C_0^\infty(\Omega)$ such that $u_k \rightarrow u$ strongly in $W_0^{1, \vec{p}}(\Omega, \vec{\omega})$. Moreover, since $u_k \in C_0^\infty(\Omega)$ by Green's Formula, we have

$$\sum_{i=1}^N \int_{\Omega} \partial_i u_k dx = \int_{\partial\Omega} u_k \cdot \vec{n} ds = 0 \quad (3.6)$$

Since $\partial_i u_k \rightarrow \partial_i u$ strongly in $L^{p_i}(\Omega, \omega_i)$ we have $\partial_i u_k \rightarrow \partial_i u$ strongly in $L^1(\Omega)$. We pass to limit in (3.6), we conclude that $\sum_{i=1}^N \int_{\Omega} \partial_i u dx = 0$. \square

4. Notion of solutions and main results

In this section we formulate and prove the main result of the paper. Now, we give a definition of entropy solutions for our unilateral elliptic problem (1.1).

Definition 4.1. A measurable function u is said to be an entropy solution for the obstacle problem (1.1), if $u \in \mathcal{T}_0^{1, \vec{p}}(\Omega, \vec{\omega})$ such that $u \geq \psi$ a.e. in Ω and

$$\begin{aligned} & \sum_{i=1}^N \int_{\Omega} [a_i(x, u, \nabla u) \partial_i T_k(u - \varphi) + \phi_i(u) \partial_i T_k(u - \varphi)] dx \\ & \leq \int_{\Omega} f T_k(u - \varphi) dx + \sum_{i=1}^N \int_{\Omega} F_i \partial_i T_k(u - v) dx \end{aligned} \quad (4.1)$$

for all $\varphi \in K_\psi \cap L^\infty(\Omega)$.

Theorem 4.2. Assume that (3.1) – (3.5) hold. Then there exists at least an entropy solution of problem (1.1).

Proof. The proof of Theorem 4.2 will be divided into several steps. \square

Step 1: Approximate problems. We consider the following approximate problems

$$\begin{cases} u_n \in K_\psi \\ \sum_{i=1}^N \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i(u_n - v) dx + \sum_{i=1}^N \int_{\Omega} \phi_i^n(u_n) \partial_i(u_n - v) dx \\ \leq \int_{\Omega} f_n(u_n - v) dx + \sum_{i=1}^N \int_{\Omega} F_i \partial_i T_k(u_n - v) dx \\ \forall v \in K_\psi \quad \text{and } \forall k > 0, \end{cases} \quad (4.2)$$

where $f_n = T_n(f)$ and $\phi_i^n(s) = \phi_i(T_n(s))$.

We define the operators Φ_n of K_ψ to $W_0^{-1, \vec{p}'}(\Omega, \vec{\omega}^*)$ by :

$$\langle \Phi_n u, v \rangle = \sum_{i=1}^N \int_{\Omega} \phi_i(T_n(u)) \partial_i v dx \quad \text{for all } u \in K_\psi \text{ and } v \in W_0^{1, \vec{p}}(\Omega, \vec{\omega})$$

Lemma 4.3. *The operator $B_n = A + \Phi_n$ is pseudo-monotone and coercive in the following sense; there exists $v_0 \in K_\psi$ such that*

$$\frac{\langle B_n v, v - v_0 \rangle}{\|v\|_{1, \vec{p}, \vec{\omega}}} \rightarrow +\infty \quad \text{if } \|v\|_{1, \vec{p}, \vec{\omega}} \rightarrow +\infty \quad \text{for } v \in K_\psi.$$

For the proof of Lemma 4.3, (see "Appendix").

Proposition 4.4. *Assume that (3.1) – (3.5) are fulfilled, then there exists at least one solution of the problem (4.2).*

Proof. Thanks to Lemma 4.3 and Theorem 8.2 chapter 2 in [14], there exists at least one solution to the problem (4.2). \square

Step2: A priori estimate.

Proposition 4.5. *Under the conditions (3.1) – (3.5) and if u_n is a solution of the approximate problem (4.2). Then there exists a constant C such that*

$$\sum_{i=1}^N \int_{\Omega} |\partial_i T_k(u_n)|^{p_i} \omega_i(x) dx \leq C(k+1) \quad \forall k > 0.$$

Proof. Let $v = u_n - \eta T_k(u_n^+ - \psi^+)$ where $\eta \geq 0$. Since $v \in W_0^{1, \vec{p}}(\Omega, \vec{\omega})$ and for all η small enough, we get $v \in K_\psi$. We choose v as test function in problem (4.2), we have

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i T_k(u_n^+ - \psi^+) dx + \sum_{i=1}^N \int_{\Omega} \phi_i^n(u_n) \partial_i T_k(u_n^+ - \psi^+) dx \\ \leq \int_{\Omega} f_n T_k(u_n^+ - \psi^+) dx + \sum_{i=1}^N \int_{\Omega} F_i \partial_i T_k(u_n^+ - \psi^+) dx. \end{aligned} \quad (4.3)$$

Which implies that

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i T_k(u_n^+ - \psi^+) dx \leq \int_{\Omega} f_n T_k(u_n^+ - \psi^+) dx \\ + \sum_{i=1}^N \int_{\Omega} F_i \partial_i T_k(u_n^+ - \psi^+) dx + \sum_{i=1}^N \int_{\Omega} |\phi_i^n(u_n)| |\partial_i T_k(u_n^+ - \psi^+)| dx. \end{aligned}$$

Since $\partial_i T_k(u_n^+ - \psi^+) = 0$ on the set $\{u_n^+ - \psi^+ > k\}$, we get

$$\begin{aligned} \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} a_i(x, u_n, \nabla u_n) \partial_i(u_n^+ - \psi^+) dx &\leq \int_{\Omega} f_n T_k(u_n^+ - \psi^+) dx \\ &+ \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} F_i \partial_i T_k(u_n^+ - \psi^+) dx + \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} |\phi_i^n(u_n)| |\partial_i(u_n^+ - \psi^+)| dx, \end{aligned}$$

thus, we can write

$$\begin{aligned} \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} a_i(x, u_n^+, \nabla u_n^+) \partial_i u_n^+ dx &\leq \int_{\Omega} |f_n T_k(u_n^+ - \psi^+)| dx + \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} |F_i| |\partial_i u_n^+| \omega_i^{-\frac{1}{p_i}}(x) \omega_i^{\frac{1}{p_i}}(x) dx \\ &+ \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} |F_i \partial_i \psi^+| dx + \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} |\phi_i^n(u_n)| |\partial_i u_n^+| \omega_i^{-\frac{1}{p_i}}(x) \omega_i^{\frac{1}{p_i}}(x) dx \\ &+ \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} |\phi_i^n(u_n)| |\partial_i \psi^+| dx + \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} |a_i(x, u_n^+, \nabla u_n^+) \partial_i \psi^+| dx. \end{aligned}$$

By Young's inequalities, we get for a positive constant λ

$$\begin{aligned} \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} a_i(x, u_n^+, \nabla u_n^+) \partial_i u_n^+ dx &\leq \int_{\Omega} f_n T_k(u_n^+ - \psi^+) dx \\ + C_1(\alpha) \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} |\phi_i^n(T_{k+\|\psi\|_{\infty}}(u_n))|^{p_i'} \omega_i^{-\frac{1}{p_i-1}}(x) dx &+ \frac{\alpha}{6} \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} |\partial_i u_n^+|^{p_i} \omega_i dx \\ + C_2(\alpha) \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} |F_i|^{p_i'} \omega_i^{-\frac{1}{p_i-1}} dx &+ \frac{\alpha}{6} \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} |\partial_i u_n^+|^{p_i} \omega_i(x) dx \\ + \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} |F_i \partial_i \psi^+| dx &+ \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} |\phi_i^n(T_{k+\|\psi\|_{\infty}}(u_n))| |\partial_i \psi^+| dx \\ + \sum_{i=1}^N \frac{\lambda^{p_i'}}{p_i} \int_{\{u_n^+ - \psi^+ \leq k\}} |a_i(x, u_n, \nabla u_n)|^{p_i'} \omega_i^{1-p_i'} dx &+ \sum_{i=1}^N \frac{1}{p_i \lambda^{p_i}} \int_{\{u_n^+ - \psi^+ \leq k\}} |\partial_i \psi^+|^{p_i} \omega_i dx. \end{aligned}$$

Using to (3.2) and taking $\lambda = \left(\frac{p'_i \alpha}{6\beta}\right)^{\frac{1}{p'_i}}$, we obtain

$$\begin{aligned}
 & \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} a_i(x, u_n, \nabla u_n) \partial_i u_n^+ dx \leq \int_{\Omega} f_n T_k(u_n^+ - \psi^+) dx \\
 & + C_1(\alpha) \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} |\phi_i^n(T_{k+\|\psi\|_{\infty}}(u_n))|^{p'_i} \omega_i^{\frac{-1}{p'_i}}(x) dx + \frac{\alpha}{6} \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} |\partial_i u_n^+|^{p_i} \omega_i(x) dx \\
 & + C_2(\alpha) \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} |F_i|^{p'_i} \omega_i^{\frac{-1}{p'_i}}(x) dx + \frac{\alpha}{6} \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} |\partial_i u_n^+|^{p_i} \omega_i(x) dx \\
 & + \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} |F_i \partial_i \psi^+| dx + \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} |\phi_i^n(T_{k+\|\psi\|_{\infty}}(u_n))| |\partial_i \psi^+| dx \\
 & + \sum_{i=1}^N \frac{\alpha}{6} \int_{\{u_n^+ - \psi^+ \leq k\}} |R_i(x)|^{p'_i} dx + \sum_{i=1}^N \frac{\alpha}{6} \int_{\{u_n^+ - \psi^+ \leq k\}} |u_n^+|^{p_i} \omega_i(x) dx \\
 & + \sum_{i=1}^N \frac{\alpha}{6} \int_{\{u_n^+ - \psi^+ \leq k\}} |\partial_i u_n^+|^{p_i} \omega_i(x) dx + \sum_{i=1}^N \frac{(6\beta)^{p_i-1}}{p_i(p'_i \alpha)^{p_i-1}} \int_{\{u_n^+ - \psi^+ \leq k\}} |\partial_i \psi^+|^{p_i} \omega_i(x) dx.
 \end{aligned}$$

According to (1.3), (3.1), (3.2), (3.3) and (\mathbf{A}_1) , (\mathbf{A}_1) , we have

$$\sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} |\partial_i u_n^+|^{p_i} \omega_i(x) dx \leq Ck + C'. \quad (4.4)$$

Since $\{x \in \Omega, u^+ \leq k\} \subset \{x \in \Omega, u^+ - \psi^+ \leq k\}$, then

$$\sum_{i=1}^N \int_{\Omega} |\partial_i T_k(u_n^+)|^{p_i} \omega_i(x) dx = \sum_{i=1}^N \int_{\{u^+ \leq k\}} |\partial_i u_n^+|^{p_i} \omega_i(x) dx \leq \sum_{i=1}^N \int_{\{u^+ - \psi^+ \leq k\}} |\partial_i u_n^+|^{p_i} \omega_i(x) dx.$$

Hence, thanks to (4.4), we get

$$\sum_{i=1}^N \int_{\Omega} |\partial_i T_k(u_n^+)|^{p_i} \omega_i(x) dx \leq kC + C' \quad \forall k > 0. \quad (4.5)$$

Similarly taking $v = u_n + T_k(u_n^-)$ as test function in approximate problem (4.2), we get

$$\sum_{i=1}^N \int_{\Omega} |\partial_i T_k(u_n^-)|^{p_i} \omega_i(x) dx \leq (k+1)C''. \quad (4.6)$$

By (4.5) and (4.6), we have

$$\sum_{i=1}^N \int_{\Omega} |\partial_i T_k(u_n^-)|^{p_i} \omega_i(x) dx \leq (k+1)C' \quad \forall k > 0.$$

□

Step 3 : Strong convergence of truncations.

Proposition 4.6. *Let u_n be a solution of approximate problem (4.2). Then there exists a measurable function u and a subsequence of u_n such that*

$$T_k(u_n) \rightarrow T(u) \quad \text{strongly in } W_0^{1, \vec{p}}(\Omega, \vec{\omega})$$

Proof. Using to Proposition 4.5, we obtain

$$\|T_k(u_n)\|_{W_0^{1,\vec{p}}(\Omega,\vec{\omega})} \leq C(k+1)^{\frac{1}{p^-}}. \quad (4.7)$$

Firstly, we will prove that $(u_n)_n$ is a Cauchy sequence in measure in Ω . For all $\lambda > 0$, we obtain $\{|u_n - u_m| > \lambda\} \subset \{|u_n| > k\} \cup \{|u_m| > k\} \cup \{|T_k(u_n) - T_k(u_m)| > \lambda\}$ which implies that

$$\begin{aligned} \text{meas}\{|u_n - u_m| > \lambda\} &\leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} \\ &\quad + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \lambda\}. \end{aligned} \quad (4.8)$$

By Hölder's inequality, Lemma 2.1 and (4.7), we have

$$\begin{aligned} k \text{meas}\{|u_n| > k\} &= \int_{\{|u_n| > k\}} |T_k(u_n)| dx \leq \int_{\Omega} |T(u_n)| dx \\ &\leq (\text{meas}(\Omega))^{\frac{1}{(p^-)^-}} \|T_k(u_n)\|_{L^{p^-}(\Omega)} \\ &\leq C(\text{meas}(\Omega))^{\frac{1}{(p^-)^-}} \|T_k(u_n)\|_{W_0^{1,\vec{p}}(\Omega,\vec{\omega})} \\ &\leq C(k+1)^{\frac{1}{p^-}}. \end{aligned}$$

Then $\text{meas}\{|u_n| > k\} \leq C\left(\frac{1}{k^{-1+p^-}} + \frac{1}{k^{p^-}}\right)^{\frac{1}{p^-}} \rightarrow 0$ as $k \rightarrow +\infty$. As results, for all $\epsilon > 0$, there exists k_0 such that $\forall k > k_0$, we get

$$\text{meas}\{|u_n| > k\} \leq \frac{\epsilon}{3} \quad \text{and} \quad \text{meas}\{|u_m| > k\} \leq \frac{\epsilon}{3}. \quad (4.9)$$

Since the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,\vec{p}}(\Omega,\vec{\omega})$ there exists a subsequence $(T_k(u_n))_n$ such that $T(u_n)$ converges to v_k a.e. in Ω , weakly in $W_0^{1,\vec{p}}(\Omega,\vec{\omega})$ and strongly in $L^{p^-}(\Omega)$ as n tends to $+\infty$. Then the sequence $(T_k(u_n))_n$ is a Cauchy sequence in measure in Ω , thus for all $\lambda > 0$, there exists n_0 such that

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \lambda\} \leq \frac{\epsilon}{3}, \quad \forall n, m \geq n_0. \quad (4.10)$$

Using (4.8), (4.9) and (4.10), then $\forall \lambda, \epsilon > 0$ we have

$$\text{meas}\{|u_n - u_m| > \lambda\} \leq \epsilon \quad \forall n, m \geq n_0.$$

Which implies that $(u_n)_n$ is a Cauchy sequence in measure in Ω , then there exists a subsequence denoted by $(u_n)_n$ such that u_n converges to a measurable function u a.e. in Ω and

$$T_k(u_n) \rightharpoonup T(u) \quad \text{weakly in } W_0^{1,\vec{p}}(\Omega,\vec{\omega}) \quad \text{and a.e. in } \Omega \quad \forall k > 0. \quad (4.11)$$

Secondly, we will show that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^N \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) (\partial_i T_k(u_n) - \partial_i T_k(u)) dx = 0. \quad (4.12)$$

Let choose $v = u_n + T_1(u_n - T_m(u_n))^-$ as test function in approximate problem (4.2), we obtain

$$\begin{aligned} & - \sum_{i=1}^N \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i T_1(u_n - T_m(u_n))^- dx - \sum_{i=1}^N \int_{\Omega} \phi_i^n(u_n) \partial_i T_1(u_n - T_m(u_n))^- dx \\ & \leq - \int_{\Omega} f_n T_1(u_n - T_m(u_n))^- dx + \sum_{i=1}^N \int_{\Omega} F_i \partial_i T_1(u_n - T_m(u_n))^- dx. \end{aligned}$$

Then,

$$\begin{aligned} \sum_{i=1}^N \int_{\{-(m+1) \leq u_n \leq -m\}} a_i(x, u_n, \nabla u_n) \partial_i u_n dx + \sum_{i=1}^N \int_{\{-(m+1) \leq u_n \leq -m\}} \phi_i(u_n) \partial_i u_n dx \\ \leq - \int_{\Omega} f_n T_1(u_n - T_m(u_n))^- dx + \sum_{i=1}^N \int_{\{-(m+1) \leq u_n \leq -m\}} F_i \partial_i u_n dx. \end{aligned} \quad (4.13)$$

We pose $\Phi_i^n(s) = \int_0^s \phi_i^n(t) \chi_{\{-(m+1) \leq t \leq -m\}} dt$. Then using the Green's formula, we obtain

$$\sum_{i=1}^N \int_{\{-(m+1) \leq u_n \leq -m\}} \phi_i(u_n) \partial_i u_n dx = \sum_{i=1}^N \int_{\Omega} \partial_i \Phi_i^n(u_n) dx = 0.$$

Then, we have

$$\begin{aligned} \sum_{i=1}^N \int_{\{-(m+1) \leq u_n \leq -m\}} a_i(x, u_n, \nabla u_n) \partial_i u_n dx \leq - \int_{\Omega} f_n T_1(u_n - T_m(u_n))^- dx \\ + \sum_{i=1}^N \int_{\{-(m+1) \leq u_n \leq -m\}} F_i \partial_i u_n dx. \end{aligned}$$

By the Young's inequality, we get

$$\begin{aligned} \sum_{i=1}^N \int_{\{-(m+1) \leq u_n \leq -m\}} a_i(x, u_n, \nabla u_n) \partial_i u_n \leq - \int_{\Omega} f_n T_1(u_n - T_m(u_n))^- \\ + C(\alpha) \sum_{i=1}^N \int_{\{-(m+1) \leq u_n \leq -m\}} |F_i|^{p'_i} \omega_i^{-\frac{1}{p_i-1}}(x) dx + \frac{\alpha}{2} \sum_{i=1}^N \int_{\{-(m+1) \leq u_n \leq -m\}} |\partial_i u_n|^{p_i} \omega_i(x) dx. \end{aligned}$$

Using (3.1), we have

$$\begin{aligned} \sum_{i=1}^N \int_{\{-(m+1) \leq u_n \leq -m\}} a_i(x, u_n, \nabla u_n) \partial_i u_n \leq - \int_{\Omega} f_n T_1(u_n - T_m(u_n))^- \\ + C(\alpha) \sum_{i=1}^N \int_{\{-(m+1) \leq u_n \leq -m\}} |F_i|^{p'_i} \omega_i^{-\frac{1}{p_i-1}}(x) dx + \frac{1}{2} \sum_{i=1}^N \int_{\{-(m+1) \leq u_n \leq -m\}} a_i(x, u_n, \nabla u_n) \partial_i u_n. \end{aligned}$$

Which implies that

$$\begin{aligned} \frac{1}{2} \sum_{i=1}^N \int_{\{-(m+1) \leq u_n \leq -m\}} a_i(x, u_n, \nabla u_n) \partial_i u_n \leq - \int_{\Omega} f_n T_1(u_n - T_m(u_n))^- \\ + C(\alpha) \sum_{i=1}^N \int_{\{-(m+1) \leq u_n \leq -m\}} |F_i|^{p'_i} \omega_i^{-\frac{1}{p_i-1}}(x). \end{aligned}$$

According to Lebesgue's theorem, we get

$$\lim_{m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \int_{\Omega} f_n T_1(u_n - T_m(u_n))^- dx = 0,$$

and

$$\lim_{m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\{-(m+1) \leq u_n \leq -m\}} |F_i|^{p'_i} \omega_i^{-\frac{1}{p_i-1}}(x) dx = 0.$$

Then, we obtain

$$\lim_{m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\{-(m+1) \leq u_n \leq -m\}} a_i(x, u_n, \nabla u_n) \partial_i u_n dx = 0. \quad (4.14)$$

Similarly, we take $v = u_n - \eta T_1(u_n - T_m(u_n))^+$ as test function in approximate problem (4.2), we have

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=1}^N \int_{\{m \leq u_n \leq m+1\}} a_i(x, u_n, \nabla u_n) \partial_i u_n dx = 0. \quad (4.15)$$

We define the following function of one real variable:

$$h_m(s) = \begin{cases} 1 & \text{if } |s| \leq m \\ 0 & \text{if } |s| \geq m+1 \\ m+1-|s| & \text{if } m \leq |s| \leq m+1, \end{cases}$$

where $m > k$. Now, let consider $\varphi = u_n - \eta(T_k(u_n) - T(u))^+ h_m(u_n)$ as test function in approximate problem (4.2), we have

$$\begin{aligned} & \sum_{i=1}^N \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i (T_k(u_n) - T(u))^+ h_m(u_n) dx \\ & + \sum_{i=1}^N \int_{\Omega} a_i(x, u_n, \nabla u_n) (T_k(u_n) - T_k(u))^+ \partial_i u_n h'_m(u_n) dx \\ & + \sum_{i=1}^N \int_{\Omega} \phi_i(u_n) \partial_i (T_k(u_n) - T_k(u))^+ h_m(u_n) dx \\ & + \sum_{i=1}^N \int_{\Omega} \phi_i(u_n) \partial_i u_n (T_k(u_n) - T_k(u))^+ h'_m(u_n) dx \\ & \leq \int_{\Omega} f_n (T_k(u_n) - T_k(u))^+ h_m(u_n) dx + \sum_{i=1}^N \int_{\Omega} F_i \partial_i (T_k(u_n) - T(u))^+ h_m(u_n) dx \\ & + \sum_{i=1}^N \int_{\Omega} F_i (T_k(u_n) - T_k(u))^+ \partial_i u_n h'_m(u_n) dx. \end{aligned} \quad (4.16)$$

Combining (4.14) and (4.15), we have the second integral in (4.16) converges to zero as n and m tend to $+\infty$. Since $h_m(u_n) = 0$ if $|u_n| > m+1$, we get

$$\begin{aligned} & \sum_{i=1}^N \int_{\Omega} \phi_i(u_n) \partial_i (T_k(u_n) - T_k(u))^+ h_m(u_n) dx \\ & = \sum_{i=1}^N \int_{\Omega} \phi_i(T_{m+1}(u_n)) h_m(u_n) \partial_i (T_k(u_n) - T_k(u))^+ dx. \end{aligned}$$

Using Lebesgue's theorem, we have $\phi_i(T_{m+1}(u_n)) h_m(u_n) \rightarrow \phi_i(T(u)) h_m(u)$ in $L^{p'_i}(\Omega, \omega_i^*)$ and $\partial_i T_k(u_n) \rightharpoonup \partial_i T(u)$ weakly in $L^{p_i}(\Omega, \omega_i(x))$ as n goes to $+\infty$, then the third integral in (4.16) converges to zero as n and m tend to $+\infty$. We set $\Phi_i(u_n) = \int_0^{u_n} \phi_i(t) (T_k(t) - T_k(u))^+ \chi_{\{m \leq |t| \leq m+1\}} dt$.

Then by Lemma 3.3, we obtain the fourth integral in (4.16) converges to zero as n and m tend to $+\infty$. Using Lebesgue's theorem, we have the first integral on the right hand in (4.16) converges to zero as n and m tend to $+\infty$. Moreover, since $F_i h_m(u_n) \rightarrow F_i h_m(u)$ in $L^{p'_i}(\Omega, \omega_i^*)$ and $\partial_i (T_k(u_n) - T_k(u)) \rightharpoonup 0$

weakly in $L^{p_i}(\Omega, \omega_i)$ we get the second integral on the right hand in (4.16) converges to zero as n and m tend to $+\infty$.

By Young's inequality, we have

$$\begin{aligned} \left| \sum_{i=1}^N \int_{\Omega} F_i(T_k(u_n) - T_k(u))^+ \partial_i u_n h'_m(u_n) \right| &\leq \left| - \sum_{i=1}^N \int_{\{m \leq u_n \leq m+1\}} F_i(T_k(u_n) - T_k(u))^+ \partial_i u_n \omega_i^{-\frac{1}{p_i}} \omega_i^{\frac{1}{p_i}} \right| \\ &+ \left| \sum_{i=1}^N \int_{\{-(m+1) \leq u_n \leq -m\}} F_i(T_k(u_n) - T_k(u))^+ \partial_i u_n \omega_i^{-\frac{1}{p_i}} \omega_i^{\frac{1}{p_i}} \right| \\ &\leq \sum_{i=1}^N C(\alpha) \int_{\{m \leq u_n \leq m+1\}} |F_i|^{p'_i} (T_k(u_n) - T_k(u))^+ \omega_i^{-\frac{1}{p_i-1}} \\ &\quad + \frac{\alpha}{2} \sum_{i=1}^N \int_{\{m \leq u_n \leq m+1\}} |\partial_i u_n|^{p_i} \omega_i (T_k(u_n) - T_k(u))^+ \\ &\quad + \sum_{i=1}^N C(\alpha) \int_{\{-(m+1) \leq u_n \leq -m\}} |F_i|^{p'_i} (T_k(u_n) - T_k(u))^+ \omega_i^{-\frac{1}{p_i-1}} \\ &\quad + \frac{\alpha}{2} \sum_{i=1}^N \int_{\{-(m+1) \leq u_n \leq -m\}} |\partial_i u_n|^{p_i} \omega_i (T_k(u_n) - T_k(u))^+. \end{aligned}$$

Since $0 \leq \int_{\{m \leq u_n \leq m+1\}} |F_i|^{p'_i} (T_k(u_n) - T_k(u))^+ \omega_i^{-\frac{1}{p_i-1}} \leq \int_{\Omega} |F_i|^{p'_i} (T_k(u_n) - T_k(u))^+ \omega_i^{-\frac{1}{p_i-1}}$, $T_k(u_n) \rightarrow T_k(u)$ a.e. in Ω as $n \rightarrow \infty$ and $|F_i|^{p'_i} (T_k(u_n) - T_k(u))^+ \omega_i^{-\frac{1}{p_i-1}} \leq 2k |F_i|^{p'_i} \omega_i^{-\frac{1}{p_i-1}} \in L^1(\Omega)$ we have

$$\lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\{m \leq u_n \leq m+1\}} |F_i|^{p'_i} (T_k(u_n) - T_k(u))^+ \omega_i^{-\frac{1}{p_i-1}} = 0 \quad (4.17)$$

Similarly, we have

$$\lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\{-(m+1) \leq u_n \leq -m\}} |F_i|^{p'_i} (T_k(u_n) - T_k(u))^+ \omega_i^{-\frac{1}{p_i-1}} = 0. \quad (4.18)$$

Using (3.1), (4.14), (4.15) and Lebesgue's theorem, we obtain

$$\lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\{-(m+1) \leq u_n \leq -m\}} |\partial_i u_n|^{p_i} (T_k(u_n) - T_k(u))^+ \omega_i(x) dx = 0, \quad (4.19)$$

and

$$\lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\{m \leq u_n \leq m+1\}} |\partial_i u_n|^{p_i} (T_k(u_n) - T_k(u))^+ \omega_i dx = 0. \quad (4.20)$$

Combining (4.17)-(4.20), we have the third integral on the right hand in (4.16) converges to zero as n and m tend to $+\infty$. We conclude

$$\lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i (T_k(u_n) - T_k(u))^+ h_m(u_n) dx \leq 0,$$

as results

$$\begin{aligned} &\lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\{T_k(u_n) - T_k(u) \geq 0, |u_n| \leq k\}} a_i(x, u_n, \nabla u_n) \partial_i (T_k(u_n) - T_k(u)) h_m(u_n) dx \\ &- \lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\{T_k(u_n) - T_k(u) \geq 0, |u_n| > k\}} a_i(x, u_n, \nabla u_n) \partial_i T_k(u) h_m(u_n) dx \leq 0. \end{aligned}$$

Since $h_m(u_n) = 0$ in $\{|u_n| > m + 1\}$, we have

$$\begin{aligned} & \sum_{i=1}^N \int_{\{T_k(u_n) - T_k(u) \geq 0, |u_n| > k\}} a_i(x, u_n, \nabla u_n) \partial_i T_k(u) h_m(u_n) dx \\ &= \sum_{i=1}^N \int_{\{T_k(u_n) - T_k(u) \geq 0, |u_n| > k\}} a_i(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \partial_i T_k(u) h_m(u_n) dx. \end{aligned}$$

Since $(a_i(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)))_{n \geq 0}$ is bounded in $L^{p'_i}(\Omega, \omega_i^*)$ we have $a_i(x, T_{m+1}(u_n), \nabla T(u))$ converges to Y_m^i weakly in $L^{p'_i}(\Omega, \omega_i^*)$. Hence

$$\begin{aligned} \lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\{T_k(u_n) - T_k(u) \geq 0, |u_n| > k\}} a_i(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \partial_i T_k(u) h_m(u_n) dx \\ = \lim_{m \rightarrow +\infty} \sum_{i=1}^N \int_{\{|u| > k\}} Y_m^i \partial_i T_k(u) h_m(u) dx = 0, \end{aligned}$$

which implies that

$$\lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\{T_k(u_n) - T_k(u) \geq 0\}} a_i(x, T_k(u_n), \nabla T_k(u_n)) \partial_i (T_k(u_n) - T_k(u)) h_m(u_n) dx \leq 0. \quad (4.21)$$

Moreover, we have $a_i(x, T_k(u_n), \nabla T_k(u)) h_m(u_n) \rightarrow a_i(x, T_k(u), \nabla T_k(u)) h_m(u)$ in $L^{p'_i}(\Omega, \omega_i^*)$ and $\partial_i (T_k(u_n) - T_k(u))$ converges to 0 weakly in $L^{p_i}(\Omega, \omega_i)$ then

$$\lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\{T_k(u_n) - T_k(u) \geq 0\}} a_i(x, T_k(u_n), \nabla T_k(u)) \partial_i (T_k(u_n) - T_k(u)) h_m(u_n) dx = 0. \quad (4.22)$$

Using (3.3), (4.21) and (4.22), we deduce

$$\begin{aligned} \lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\{T_k(u_n) - T_k(u) \geq 0\}} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) \\ \partial_i (T_k(u_n) - T_k(u)) h_m(u_n) dx = 0. \end{aligned} \quad (4.23)$$

Similarly, we consider $\varphi = u_n + (T_k(u_n) - T_k(u))^- h_m(u_n)$ as test function in approximate problem (4.2), we have

$$\begin{aligned} \lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\{T_k(u_n) - T_k(u) \leq 0\}} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) \\ \partial_i (T_k(u_n) - T_k(u)) h_m(u_n) dx = 0. \end{aligned} \quad (4.24)$$

Using (4.23) and (4.24), we get

$$\begin{aligned} \lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) \\ \partial_i (T_k(u_n) - T_k(u)) h_m(u_n) dx = 0. \end{aligned} \quad (4.25)$$

Now, we prove

$$\begin{aligned} \lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) \\ \partial_i (T_k(u_n) - T_k(u)) (1 - h_m(u_n)) dx = 0. \end{aligned} \quad (4.26)$$

Let $\varphi = u_n + T_k(u_n)^-(1 - h_m(u_n))$ as test function in approximate problem (1.1), we have

$$\begin{aligned}
& - \sum_{i=1}^N \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i T_k(u_n)^-(1 - h_m(u_n)) dx + \sum_{i=1}^N \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i u_n T_k(u_n)^- h'_m(u_n) dx \\
& - \sum_{i=1}^N \int_{\Omega} \phi_i(u_n) \partial_i T_k(u_n)^-(1 - h_m(u_n)) dx + \sum_{i=1}^N \int_{\Omega} \phi_i(u_n) T_k(u_n)^- \partial_i u_n h'_m(u_n) dx \\
& \leq - \int_{\Omega} f_n T_k(u_n)^-(1 - h_m(u_n)) dx - \sum_{i=1}^N \int_{\Omega} F_i \partial_i T_k(u_n)^-(1 - h_m(u_n)) dx \\
& + \sum_{i=1}^N \int_{\Omega} F_i T_k(u_n)^- \partial_i u_n h'_m(u_n) dx.
\end{aligned} \tag{4.27}$$

According to (4.14) and (4.15), we have

$$\lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i u_n T_k(u_n)^- h'_m(u_n) dx = 0.$$

Then the second integral in (4.27) converges to zero as n and m goes to $+\infty$. Since $\partial_i T_k(u_n)^- \rightarrow \partial_i T_k(u)^-$ in $L^{p_i}(\Omega, \omega_i)$ and $\phi_i(T_k(u_n))(1 - h_m(u_n)) \rightarrow \phi_i(T_k(u))(1 - h_m(u))$ strongly in $L^{p_i}(\Omega, \omega_i^*)$, we get

$$\begin{aligned}
& \lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\Omega} \phi_i(u_n) \partial_i T_k(u_n)^-(1 - h_m(u_n)) dx \\
& = \lim_{m \rightarrow +\infty} \sum_{i=1}^N \int_{\Omega} \phi_i(T_k(u)) \partial_i T_k(u)^-(1 - h_m(u)) dx.
\end{aligned}$$

Thanks to Lebesgue's theorem, we obtain

$$\lim_{m \rightarrow +\infty} \sum_{i=1}^N \int_{\Omega} \phi_i(T_k(u)) \partial_i T_k(u)^-(1 - h_m(u)) dx = 0.$$

Hence the third integral in (4.27) converges to zero as n and m tends to $+\infty$. We set $\Phi_i^n(t) = \int_0^t \phi_i(s) T_k(s)^- h'_m(s) ds$, in view to Green's Formula, we have

$$\sum_{i=1}^N \int_{\Omega} \phi_i^n(u_n) \partial_i u_n T_k(u_n)^- h'_m(u_n) dx = \sum_{i=1}^N \int_{\Omega} \partial_i \Phi_i^n(u_n) dx = 0.$$

Then the fourth integral in (4.27) converges to zero as n and m tend to $+\infty$. Using to Lebesgue's theorem, we get the integral on the right hand in (4.27) converges to zero as n and m goes to $+\infty$. Since $F_i(1 - h_m(u_n)) \rightarrow F_i(1 - h_m(u))$ in $L^{p_i}(\Omega, \omega_i^*)$ and $\partial_i T_k(u_n) \rightarrow \partial_i T_k(u)$ in $L^{p_i}(\Omega, \omega_i)$ as n tends to $+\infty$, we have

$$\lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\Omega} F_i \partial_i T_k(u_n)^-(1 - h_m(u_n)) = \sum_{i=1}^N \int_{\Omega} F_i \partial_i T_k(u)^-(1 - h_m(u)).$$

Also, we have $F_i(1 - h_m(u)) \rightarrow 0$ in $L^{p_i}(\Omega, \omega_i^*)$ as m tends to $+\infty$ and $\partial_i T_k(u)^- \in L^{p_i}(\Omega, \omega_i)$ thus the second integral on the right hand in (4.27) converges to zero as n and m tend to $+\infty$. Using

Young's Inequality and (3.1), we get

$$\begin{aligned}
& \left| \sum_{i=1}^N \int_{\Omega} F_i \partial_i u_n T_k(u_n)^- h'_m(u_n) \omega_i^{-\frac{1}{p_i}} \omega_i^{\frac{1}{p_i}} \right| \\
& \leq C(\alpha) \sum_{i=1}^N \int_{\Omega} |F_i|^{p'_i} T_k(u_n)^- h'_m(u_n) \omega_i^{-\frac{1}{p_i-1}} + \alpha \sum_{i=1}^N \int_{\Omega} \omega_i |\partial_i u_n|^{p_i} T_k(u_n)^- h'_m(u_n) \\
& \leq C(\alpha) \sum_{i=1}^N \int_{\Omega} |F_i|^{p'_i} T_k(u_n)^- \chi_{\{-(m+1) \leq u_n \leq -m\}} \omega_i^{-\frac{1}{p_i-1}} + k \sum_{i=1}^N \int_{\{-(m+1) \leq u_n \leq -m\}} a_i(x, u_n, \nabla u_n) \partial_i u_n.
\end{aligned}$$

In sight to Lebesgue's theorem and (4.14), we obtain the third integral on the right hand in (4.27) converges to zero as n and m tend to $+\infty$, we conclude

$$\lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\{u_n \leq 0\}} a_i(x, u_n, \nabla u_n) \partial_i T_k(u_n) (1 - h_m(u_n)) dx = 0. \quad (4.28)$$

Next, for η small enough, we choose $\varphi = u_n - \eta T_k(u_n^+ - \psi^+) (1 - h_m(u_n))$ as test function in approximate problem (4.2), we have

$$\begin{aligned}
& \sum_{i=1}^N \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i T_k(u_n^+ - \psi^+) (1 - h_m(u_n)) dx - \sum_{i=1}^N \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i u_n T_k(u_n^+ - \psi^+) h'_m(u_n) dx \\
& + \sum_{i=1}^N \int_{\Omega} \phi_i(u_n) \partial_i T_k(u_n^+ - \psi^+) (1 - h_m(u_n)) dx - \sum_{i=1}^N \int_{\Omega} \phi_i(u_n) \partial_i u_n T_k(u_n^+ - \psi^+) h'_m(u_n) dx \\
& \leq \int_{\Omega} f_n T_k(u_n^+ - \psi^+) (1 - h_m(u_n)) dx + \sum_{i=1}^N \int_{\Omega} F_i \partial_i T_k(u_n^+ - \psi^+) (1 - h_m(u_n)) dx \\
& - \sum_{i=1}^N \int_{\Omega} F_i \partial_i u_n T_k(u_n^+ - \psi^+) h'_m(u_n) dx
\end{aligned} \quad (4.29)$$

thanks to (4.14) and (4.15), we have the second integral in the left hand in (4.29) converges to zero as n and m tend to $+\infty$. For the third integral in the left hand in (4.29), we obtain

$$\begin{aligned}
\sum_{i=1}^N \int_{\Omega} \phi_i(u_n) \partial_i T_k(u_n^+ - \psi^+) (1 - h_m(u)) & = \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} \phi_i(u_n) \partial_i u_n^+ (1 - h_m(u_n)) \\
& - \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} \phi_i(u_n) \partial_i \psi^+ (1 - h_m(u_n)).
\end{aligned}$$

We set $\Phi_i(s) = \int_0^\pi \phi_i(t) (1 - h_m(t)) \chi_{\{t - \psi^+ \leq k\}} \cdot \chi_{\{t > 0\}}$ and by Lemma 3.3, we have

$$\sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} \phi_i(u_n) \partial_i u_n^+ (1 - h_m(u_n)) = 0,$$

thanks to Lebesgue's theorem and (3.4), we get

$$\lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} \phi_i(u_n) \partial_i u_n T_k(u_n^+ - \psi^+) h'_m(u_n) dx = 0.$$

Then the fourth integral in the left hand in (4.29) converges to zero as n and m tend to $+\infty$. In addition, by the Lebesgue's theorem, we obtain the first integral in the right hand in (4.29) converges to zero as n and m tend to $+\infty$. Using the Young's inequality, we have

$$\begin{aligned}
& \sum_{i=1}^N \int_{\Omega} F_i \partial_i T_k(u_n^+ - \psi^+)(1 - h_m(u)) \\
&= \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} F_i \partial_i u_n^+ \omega_i^{-\frac{1}{p_i}} \omega_i^{\frac{1}{p_i}} (1 - h_m(u_n)) - \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} F_i \partial_i \psi^+(1 - h_m(u_n)) \\
&\leq c(\alpha) \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} |F_i|^{p_i'} \omega_i^{-\frac{1}{p_i-1}} (1 - h_m(u_n)) + \alpha \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} |\partial_i u_n^+|^{p_i} \omega_i (1 - h_m(u_n)) \\
&\quad + \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} F_i \partial_i \psi^+(1 - h_m(u_n)) \\
&\leq c(\alpha) \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} |F_i|^{p_i'} \omega_i^{-\frac{1}{p_i-1}} (1 - h_m(u_n)) + \alpha \sum_{i=1}^N \int_{\Omega} |\partial_i T_{k+\|\psi\|_{\infty}}(u_n^+)|^{p_i} \omega_i (1 - h_m(u_n)) \\
&\quad + \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} F_i \partial_i \psi^+(1 - h_m(u_n)).
\end{aligned}$$

By Lebesgue's theorem, we have

$$\lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} |F_i|^{p_i'} \omega_i^{-\frac{1}{p_i-1}} (1 - h_m(u_n)) = 0$$

and

$$\lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} F_i \partial_i \psi^+(1 - h_m(u_n)) = 0.$$

Since $\partial_i T_{k+\|\psi\|_{\infty}}(u_n^+) \rightharpoonup \partial_i T_{k+\|\psi\|_{\infty}}(u^+)$ weakly in $L^{p_i}(\Omega, \omega_i)$ and $(1 - h_m(u_n)) \rightarrow (1 - h_m(u))$ strongly in $L^{p_i'}(\Omega, \omega_i^*)$ we have

$$\lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\Omega} |\partial_i T_{k+\|\psi\|_{\infty}}(u_n^+)|^{p_i} \omega_i (1 - h_m(u_n)) = \sum_{i=1}^N \int_{\Omega} |\partial_i T_{k+\|\psi\|_{\infty}}(u^+)|^{p_i} \omega_i (1 - h_m(u)).$$

Thanks to Lebesgue's theorem again, we have

$$\lim_{m \rightarrow +\infty} \sum_{i=1}^N \int_{\Omega} |\partial_i T_{k+\|\psi\|_{\infty}}(u^+)|^{p_i} \omega_i (1 - h_m(u)) = 0.$$

Thus the second integral in the right hand in (4.29) converges to zero as n and m tend to $+\infty$. Furthermore, by Young's inequality, we obtain

$$\begin{aligned}
& \left| \sum_{i=1}^N \int_{\Omega} F_i \partial_i u_n T_k(u_n^+ - \psi^+) h'_m(u_n) \right| = \left| - \sum_{i=1}^N \int_{\{m \leq u_n \leq m+1\}} F_i \omega_i^{-\frac{1}{p_i}} \omega_i^{\frac{1}{p_i}} \partial_i u_n T_k(u_n^+ - \psi^+) \right| \\
&\leq c(\alpha) \sum_{i=1}^N \int_{\{m \leq u_n \leq m+1\}} |F_i|^{p_i'} \omega_i^{-\frac{1}{p_i-1}} T_k(u_n^+ - \psi^+) + \alpha \sum_{i=1}^N \int_{\{m \leq u_n \leq m+1\}} |\partial_i u_n|^{p_i} \omega_i T_k(u_n^+ - \psi^+).
\end{aligned}$$

According to (3.1) and (4.15), Lebesgue's theorem, the third integral in the right hand in (4.29) converges to zero as n and m tend to $+\infty$.

Hence, the first integral in the right hand in (4.29) satisfies that

$$\sum_{i=1}^N \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i T_k(u_n^+ - \psi^+) (1 - h_m(u_n)) \leq \epsilon_1(n, m),$$

then, we have

$$\begin{aligned} \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} a_i(x, u_n, \nabla u_n) \partial_i u_n^+ (1 - h_m(u_n)) \\ \leq \epsilon_1(n, m) + \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} a_i(x, u_n, \nabla u_n) \partial_i \psi^+ (1 - h_m(u_n)). \end{aligned}$$

Using (3.1), Young's inequality and Lebesgue's theorem, we have

$$\sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \leq k\}} a_i(x, u_n, \nabla u_n) \partial_i u_n^+ (1 - h_m(u_n)) \leq \epsilon_1(n, m) + \epsilon_2(n, m).$$

Thus, since $\{u_n^+ \leq k\} \subset \{u_n^+ - \psi^+ \leq k\}$, we get

$$\lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\{u_n^+ \leq k\}} a_i(x, u_n, \nabla u_n) \partial_i u_n^+ (1 - h_m(u_n)) = 0,$$

which implies that

$$\lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\{0 \leq u_n\}} a_i(x, u_n, \nabla u_n) \partial_i T_k(u_n) (1 - h_m(u_n)) = 0. \quad (4.30)$$

Using (4.28) and (4.30), we have

$$\lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sum_{i=1}^N \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i T_k(u_n) (1 - h_m(u_n)) = 0. \quad (4.31)$$

Furthermore, we get

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) (\partial_i T_k(u_n) - \partial_i T_k(u)) \\ = \sum_{i=1}^N \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) (\partial_i T_k(u_n) - \partial_i T_k(u)) h_m(u_n) \\ + \sum_{i=1}^N \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n))) \partial_i T_k(u_n) (1 - h_m(u_n)) \\ - \sum_{i=1}^N \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n))) \partial_i T_k(u) (1 - h_m(u_n)) \\ - \sum_{i=1}^N \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u))) (\partial_i T_k(u_n) - \partial_i T_k(u)) (1 - h_m(u_n)). \end{aligned}$$

Combining (4.25) and (4.31), the first and the second integrals on the right hand side converge to zero as n and m tend to $+\infty$. Since $(a_i(x, T_k(u_n), \nabla T_k(u_n)))_n$ is bounded in $L^{p'_i}(\Omega, \omega_i^*)$

and $\partial_i T_k(u)(1 - h_m(u))$ converge to zero in $L^{p_i}(\Omega, \omega_i)$ as n and m tend to $+\infty$. Then, the third integral on the right hand side converge to zero as n and m tend to $+\infty$. Also, since $a_i(x, T_k(u_n), \nabla T_k(u_n))(1 - h_m(u))$ converges to $a_i(x, T_k(u), \nabla T_k(u))(1 - h_m(u))$ strongly in $L^{p'_i}(\Omega, \omega_i^*)$ and $\partial_i T_k(u_n) \rightharpoonup \partial_i T_k(u)$ weakly in $L^{p_i}(\Omega, \omega_i)$ we obtain the fourth integral on the right hand side converge to zero as n and m tend to $+\infty$. Hence, we get (4.12).

By (4.11), (4.12) and Lemma 3.2, we have

$T_k(u_n) \rightarrow T_k(u)$ strongly in $W_0^{1, \vec{p}}(\Omega, \vec{\omega})$ and a.e. in $\Omega \quad \forall k > 0$. \square

Step 4: Passing to the limit. Let $\varphi \in K_\psi \cap L^\infty(\Omega)$, we chose $v = u_n - T_k(u_n - \varphi)$ as test function in approximate problem (4.2), for n large enough ($n > k + \|\varphi\|_\infty$), we have

$$\begin{aligned} & \sum_{i=1}^N \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i T_k(u_n - \varphi) dx + \sum_{i=1}^N \int_{\Omega} \phi_i^n(u_n) \partial_i T_k(u_n - \varphi) dx \\ & \leq \int_{\Omega} f_n T_k(u_n - \varphi) dx + \sum_{i=1}^N \int_{\Omega} F_i \partial_i T_k(u_n - \varphi) dx, \end{aligned}$$

therefore,

$$\begin{aligned} & \sum_{i=1}^N \int_{\Omega} a_i(x, T_{k+\|\varphi\|_\infty}(u_n), \nabla T_{k+\|\varphi\|_\infty}(u_n)) \partial_i T_k(u_n - \varphi) dx \\ & \quad + \sum_{i=1}^N \int_{\Omega} \phi_i(T_{k+\|\varphi\|_\infty}(u_n)) \partial_i T_k(u_n - \varphi) dx \\ & \leq \int_{\Omega} f_n T_k(u_n - \varphi) dx + \sum_{i=1}^N \int_{\Omega} F_i \partial_i T_k(u_n - \varphi) dx. \end{aligned}$$

As $T_k(u_n) \rightarrow T(u)$ strongly in $W_0^{1, \vec{p}}(\Omega, \vec{\omega})$ and a.e. in $\Omega \quad \forall k > 0$, we get

$$a_i(x, T_{k+\|\varphi\|_\infty}(u_n), \nabla T_{k+\|\varphi\|_\infty}(u_n)) \rightharpoonup a_i(x, T_{k+\|\varphi\|_\infty}(u), \nabla T_{k+\|\varphi\|_\infty}(u)) \quad \text{weakly in } L^{p'_i}(\Omega, \omega_i^*),$$

$$\phi_i(T_{k+\|\varphi\|_\infty}(u_n)) \rightarrow \phi_i(T_{k+\|\varphi\|_\infty}(u)) \quad \text{strongly in } L^{p'_i}(\Omega, \omega_i^*)$$

and

$$\partial_i T_k(u_n - \varphi) \rightarrow \partial_i T_k(u - \varphi) \quad \text{strongly in } L^{p_i}(\Omega, \omega_i).$$

we can pass to limit in

$$\left\{ \begin{array}{l} u_n \in K_\psi \\ \sum_{i=1}^N \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i T_k(u_n - \varphi) dx + \sum_{i=1}^N \int_{\Omega} \phi_i^n(u_n) \partial_i T_k(u_n - \varphi) dx \\ \leq \int_{\Omega} f_n T_k(u_n - \varphi) dx + \sum_{i=1}^N \int_{\Omega} F_i \partial_i T_k(u_n - \varphi) dx. \\ \forall \varphi \in K_\psi \cap L^\infty(\Omega) \quad \text{and} \quad \forall k > 0, \end{array} \right.$$

this completes the proof of Theorem 4.2.

5. Appendix

Proof of lemma 4.3

Firstly, we will prove that the operator B_n is pseudo-monotone. Let $(u_k)_k$ be a sequence in $W_0^{1, \vec{p}}(\Omega, \vec{\omega})$ such that

$$\left\{ \begin{array}{l} u_k \rightarrow u \quad \text{weakly } W_0^{1, \vec{p}}(\Omega, \vec{\omega}) \\ B_n u_k \rightarrow \chi \quad \text{weakly in } W_0^{-1, \vec{p}' }(\Omega, \vec{\omega}^*) \\ \limsup_{k \rightarrow +\infty} \langle B_n u_k, u_k \rangle \leq \langle \chi, u \rangle. \end{array} \right.$$

We will show that $\chi = B_n u$ and $\langle B_n u_k, u_k \rangle \rightarrow \langle \chi, u \rangle$ as $k \rightarrow +\infty$. Since $W_0^{1, \vec{p}}(\Omega, \vec{\omega}) \hookrightarrow L^{p^-}(\Omega)$, then $u_k \rightarrow u$ strongly in $L^{p^-}(\Omega)$ and a.e. in Ω for a subsequence denoted again $(u_k)_k$. Since $(u_k)_k$ is bounded in $W_0^{1, \vec{p}}(\Omega, \vec{\omega})$ by (3.2), we have $(a_i(x, u_k, \nabla u_k))_k$ is bounded in $L^{p'_i}(\Omega, \omega_i^*)$. Then there exists a function $\varphi_i \in L^{p'_i}(\Omega, \omega_i^*)$ such that

$$a_i(x, u_k, \nabla u_k) \rightarrow \varphi_i \quad \text{as } k \rightarrow +\infty. \quad (5.1)$$

Moreover, since $(\phi_i^n(u_k))_k$ is bounded in $L^{p'_i}(\Omega, \omega_i^*)$ and $\phi_i^n(u_k) \rightarrow \phi_i^n(u)$ a.e. in Ω , we have

$$\phi_i^n(u_k) \rightarrow \phi_i^n(u) \quad \text{strongly in } L^{p'_i}(\Omega, \omega_i^*) \quad \text{as } k \rightarrow +\infty. \quad (5.2)$$

For all $v \in W_0^{1, \vec{p}}(\Omega, \vec{\omega})$ using (5.1) and (5.2), we obtain

$$\begin{aligned} \langle \chi, v \rangle &= \lim_{k \rightarrow +\infty} \langle B_n u_k, v \rangle \\ &= \lim_{k \rightarrow +\infty} \sum_{i=1}^N \int_{\Omega} a_i(x, u_k, \nabla u_k) \partial_i v \, dx + \lim_{k \rightarrow +\infty} \sum_{i=1}^N \int_{\Omega} \phi_i^n(u_k) \partial_i v \, dx \\ &= \sum_{i=1}^N \int_{\Omega} \varphi_i \partial_i v \, dx + \sum_{i=1}^N \int_{\Omega} \phi_i^n(u) \partial_i v \, dx. \end{aligned}$$

Hence, we get

$$\begin{aligned} \limsup_{k \rightarrow +\infty} \langle B_n u_k, u_k \rangle &= \limsup_{k \rightarrow +\infty} \left[\sum_{i=1}^N \int_{\Omega} a_i(x, u_k, \nabla u_k) \partial_i u_k \, dx + \sum_{i=1}^N \int_{\Omega} \phi_i^n(u_k) \partial_i u_k \, dx \right] \\ &= \limsup_{k \rightarrow +\infty} \sum_{i=1}^N \int_{\Omega} a_i(x, u_k, \nabla u_k) \partial_i u_k \, dx + \sum_{i=1}^N \int_{\Omega} \phi_i^n(u) \partial_i u \, dx \\ &\leq \langle \chi, u \rangle \\ &= \sum_{i=1}^N \int_{\Omega} \varphi_i \partial_i u \, dx + \sum_{i=1}^N \int_{\Omega} \phi_i^n(u) \partial_i u \, dx \end{aligned}$$

as a result

$$\limsup_{k \rightarrow +\infty} \sum_{i=1}^N \int_{\Omega} a_i(x, u_k, \nabla u_k) \partial_i u_k \, dx \leq \sum_{i=1}^N \int_{\Omega} \varphi_i \partial_i u \, dx. \quad (5.3)$$

Thanks to (3.3), we have $\sum_{i=1}^N \int_{\Omega} (a_i(x, u_k, \nabla u_k) - a_i(x, u_k, \nabla u)) (\partial_i u_k - \partial_i u) \, dx > 0$. Then

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega} a_i(x, u_k, \nabla u_k) \partial_i u_k \, dx &\geq - \sum_{i=1}^N \int_{\Omega} a_i(x, u_k, \nabla u) \partial_i u \, dx + \sum_{i=1}^N \int_{\Omega} a_i(x, u_k, \nabla u_k) \partial_i u \, dx \\ &\quad + \sum_{i=1}^N \int_{\Omega} a_i(x, u_k, \nabla u) \partial_i u_k \, dx. \end{aligned}$$

By (5.1), we obtain

$$\liminf_{k \rightarrow +\infty} \sum_{i=1}^N \int_{\Omega} a_i(x, u_k, \nabla u_k) \partial_i u_k \, dx \geq \sum_{i=1}^N \int_{\Omega} \varphi_i \partial_i u \, dx. \quad (5.4)$$

Using (5.3) and (5.4), we have

$$\lim_{k \rightarrow +\infty} \sum_{i=1}^N \int_{\Omega} a_i(x, u_k, \nabla u_k) \partial_i u_k \, dx = \sum_{i=1}^N \int_{\Omega} \varphi_i \partial_i u \, dx. \quad (5.5)$$

Which implies that,

$$\begin{aligned}
\lim_{k \rightarrow +\infty} \langle B_n u_k, u_k \rangle &= \lim_{k \rightarrow +\infty} \sum_{i=1}^N \int_{\Omega} a_i(x, u_k, \nabla u_k) \partial_i u_k \, dx + \lim_{k \rightarrow +\infty} \sum_{i=1}^N \int_{\Omega} \phi_i^n(u_k) \partial_i u_k \, dx \\
&= \sum_{i=1}^N \int_{\Omega} \varphi_i \partial_i u \, dx + \sum_{i=1}^N \int_{\Omega} \phi_i^n(u) \partial_i u \, dx \\
&= \langle \chi, u \rangle.
\end{aligned}$$

Moreover, since $a_i(x, u_k, \nabla u)$ converges to $a_i(x, u, \nabla u)$ strongly in $L^{p'_i}(\Omega, \omega_i)$ by (5.5), we get

$$\sum_{i=1}^N \int_{\Omega} (a_i(x, u_k, \nabla u_k) - a_i(x, u, \nabla u)) (\partial_i u_k - \partial_i u) \, dx = 0.$$

By Lemma 3.2, we have u_k converges to u strongly in $W_0^{1, \vec{p}}(\Omega, \vec{\omega})$ and a.e. in Ω , then $a_i(x, u_k, \nabla u)$ converges to $a_i(x, u, \nabla u)$ weakly in $L^{p'_i}(\Omega, \omega_i)$ and $\phi_i^n(u)$ converges to $\phi_i^n(u)$ strongly in $L^{p_i}(\Omega, \omega_i)$. Then for all $v \in W_0^{1, \vec{p}}(\Omega, \vec{\omega})$ we obtain

$$\begin{aligned}
\langle \chi, v \rangle &= \lim_{k \rightarrow +\infty} \langle B_n u_k, v \rangle \\
&= \lim_{k \rightarrow +\infty} \sum_{i=1}^N \int_{\Omega} a_i(x, u_k, \nabla u_k) \partial_i v \, dx + \lim_{k \rightarrow +\infty} \sum_{i=1}^N \int_{\Omega} \phi_i^n(u_k) \partial_i v \, dx \\
&= \sum_{i=1}^N \int_{\Omega} a_i(x, u, \nabla u) \partial_i v \, dx + \sum_{i=1}^N \int_{\Omega} \phi_i^n(u) \partial_i v \, dx \\
&= \langle B_n u, v \rangle
\end{aligned}$$

which implies that $B_n u = \chi$. Secondly, it remains to show that the operator B_n is coercive. For all $u, v \in W_0^{1, \vec{p}}(\Omega, \vec{\omega})$ and by the generalized Hölder's type inequality inequality, we have

$$\begin{aligned}
|\langle \Phi_n u, v \rangle| &\leq \sum_{i=1}^N \int_{\Omega} \phi_i(T_n(u)) \partial_i v \omega_i^{-\frac{1}{p_i}}(x) \omega_i^{\frac{1}{p_i}}(x) \, dx \\
&\leq \sum_{i=1}^N \left(\int_{\Omega} \left| \phi_i(T_n(u)) \omega_i^{-\frac{1}{p_i}}(x) \right|^{p'_i} \, dx \right)^{\frac{1}{p'_i}} \left(\int_{\Omega} |\partial_i v \omega_i^{\frac{1}{p_i}}(x)|^{p_i} \, dx \right)^{\frac{1}{p_i}} \\
&\leq \sum_{i=1}^N \left(\int_{\Omega} \sup_{|s| \leq n} |\phi_i(s)|^{p'_i} \omega_i^{-\frac{p'_i}{p_i}}(x) \, dx \right)^{\frac{1}{p'_i}} \left(\int_{\Omega} |\partial_i v|^{p_i} \omega_i(x) \, dx \right)^{\frac{1}{p_i}} \\
&\leq \sum_{i=1}^N \left(\int_{\Omega} (\sup_{|s| \leq n} |\phi_i(s)| + 1)^{p'_i} \omega_i^{-\frac{1}{p_i-1}}(x) \, dx \right)^{\frac{1}{p'_i}} \left(\int_{\Omega} |\partial_i v|^{p_i} \omega_i(x) \, dx \right)^{\frac{1}{p_i}} \\
&\leq \sum_{i=1}^N \left(\sup_{|s| \leq n} |\phi_i(s)| + 1 \right) \left(\int_{\Omega} \omega_i^{-\frac{1}{p_i-1}}(x) \, dx \right)^{\frac{1}{p'_i}} \left(\int_{\Omega} |\partial_i v|^{p_i} \omega_i(x) \, dx \right)^{\frac{1}{p_i}} \\
&\leq C(n) \|v\|_{W_0^{1, \vec{p}}(\Omega, \vec{\omega})},
\end{aligned}$$

which implies that $\frac{|\langle \Phi_n u, v \rangle|}{\|v\|_{1, \vec{p}, \vec{\omega}}} \leq C(n)$. Let $v_0 \in K_\psi$, thanks to Hölder's inequality and (3.2), by the following continuous embeddings $W_0^{1, p_i}(\Omega, \omega_i) \hookrightarrow L^{p_i}(\Omega, \omega_i)$, we have

$$\begin{aligned}
|\langle Av, v_0 \rangle| &\leq \sum_{i=1}^N \int_{\Omega} a_i(x, v, \nabla v) \partial_i v_0 \omega_i^{-\frac{1}{p_i}}(x) \omega_i^{\frac{1}{p_i}}(x) dx \\
&\leq \sum_{i=1}^N \left(\int_{\Omega} \left| a_i(x, v, \nabla v) \omega_i^{-\frac{1}{p_i}}(x) \right|^{p'_i} dx \right)^{\frac{1}{p'_i}} \left(\int_{\Omega} \left| \partial_i v_0 \omega_i^{\frac{1}{p_i}}(x) \right|^{p_i} dx \right)^{\frac{1}{p_i}} \\
&\leq \beta \sum_{i=1}^N \left(\int_{\Omega} R_i^{p'_i}(x) + |v|^{p_i} \omega_i(x) + |\partial_i v|^{p_i} \omega_i(x) dx \right)^{\frac{1}{p'_i}} \left(\int_{\Omega} |\partial_i v_0|^{p_i} \omega_i(x) dx \right)^{\frac{1}{p_i}} \\
&\leq \beta \sum_{i=1}^N \left(C_1 + C_2 \int_{\Omega} |\partial_i v|^{p_i} \omega_i(x) dx + \int_{\Omega} |\partial_i v|^{p_i} \omega_i(x) dx \right)^{\frac{1}{p'_i}} \left(\int_{\Omega} |\partial_i v_0|^{p_i} \omega_i(x) dx \right)^{\frac{1}{p_i}} \\
&\leq \beta \sum_{i=1}^N C_1^{\frac{1}{p'_i}} \left(1 + \frac{C_2 + 1}{C_1} \sum_{i=1}^N \int_{\Omega} |\partial_i v|^{p_i} \omega_i dx \right)^{\frac{1}{p'_i}} \left(\int_{\Omega} |\partial_i v_0|^{p_i} \omega_i(x) dx \right)^{\frac{1}{p_i}} \\
&\leq \beta C_4 \sum_{i=1}^N \left(1 + \frac{C_2 + 1}{C_1} \sum_{i=1}^N \int_{\Omega} |\partial_i v|^{p_i} \omega_i dx \right)^{\frac{1}{p'_i}} \left(\int_{\Omega} |\partial_i v_0|^{p_i} \omega_i(x) dx \right)^{\frac{1}{p_i}} \\
&\leq \beta C_4 \sum_{i=1}^N \left(1 + C_3 \left(\sum_{i=1}^N \int_{\Omega} |\partial_i v|^{p_i} \omega_i(x) dx \right)^{\frac{1}{p'_i}} \right) \left(\int_{\Omega} |\partial_i v_0|^{p_i} \omega_i(x) dx \right)^{\frac{1}{p_i}} \\
&\leq \beta C_4 \left(1 + C_3 \left(\sum_{i=1}^N \int_{\Omega} |\partial_i v|^{p_i} \omega_i(x) dx \right)^{\frac{1}{p'_i}} \right) \sum_{i=1}^N \left(\int_{\Omega} |\partial_i v_0|^{p_i} \omega_i(x) dx \right)^{\frac{1}{p_i}} \\
&\leq \beta C_4 \left(1 + C_3 \left(\sum_{i=1}^N \int_{\Omega} |\partial_i v|^{p_i} \omega_i(x) dx \right)^{\frac{1}{p'_i}} \right) \|v_0\|_{W_0^{1, \vec{p}}(\Omega, \vec{\omega})}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{|\langle Av, v - v_0 \rangle|}{\|v\|_{W_0^{1, \vec{p}}(\Omega, \vec{\omega})}} &\geq \alpha \frac{\sum_{i=1}^N \int_{\Omega} |\partial_i v|^{p_i} \omega_i(x) dx}{\|v\|_{W_0^{1, \vec{p}}(\Omega, \vec{\omega})}} - \frac{\beta C_4 \|v_0\|_{W_0^{1, \vec{p}}(\Omega, \vec{\omega})}}{\|v\|_{W_0^{1, \vec{p}}(\Omega, \vec{\omega})}} \\
&\quad - \frac{\beta C_4 C_3}{\|v\|_{W_0^{1, \vec{p}}(\Omega, \vec{\omega})}} \left(\sum_{i=1}^N \int_{\Omega} |\partial_i v|^{p_i} \omega_i(x) dx \right)^{\frac{1}{p'_i}} \|v_0\|_{W_0^{1, \vec{p}}(\Omega, \vec{\omega})}.
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{|\langle Av, v - v_0 \rangle|}{\|v\|_{W_0^{1, \vec{p}}(\Omega, \vec{\omega})}} &\geq \alpha \frac{\sum_{i=1}^N \int_{\Omega} |\partial_i v|^{p_i} \omega_i(x) dx}{\|v\|_{W_0^{1, \vec{p}}(\Omega, \vec{\omega})}} \left[1 - \frac{\beta}{\alpha} C_4 C_3 \left(\sum_{i=1}^N \int_{\Omega} |\partial_i v|^{p_i} dx \right)^{\frac{1}{p'_i} - 1} \|v_0\|_{W_0^{1, \vec{p}}(\Omega, \vec{\omega})} \right] \\
&\quad - \frac{\beta C_4 \|v_0\|_{W_0^{1, \vec{p}}(\Omega, \vec{\omega})}}{\|v\|_{W_0^{1, \vec{p}}(\Omega, \vec{\omega})}}.
\end{aligned}$$

(5.6)

Using to Jensen's inequality, we obtain

$$\begin{aligned} \|v\|_{W_0^{1,\vec{p}}(\Omega,\vec{\omega})}^{p_-^+} &= \left(\sum_{i=1}^N \left(\int_{\Omega} |\partial_i v|^{p_i} \omega_i(x) dx \right)^{\frac{1}{p_i}} \right)^{p_-^+} \\ &\leq \left(\sum_{i=1}^N \left(\int_{\Omega} |\partial_i v|^{p_i} \omega_i dx \right)^{\frac{1}{p_i^+}} \right)^{p_-^+} \\ &\leq C \sum_{i=1}^N \int_{\Omega} |\partial_i v|^{p_i} \omega_i(x) dx, \end{aligned}$$

where

$$p_-^+ = \begin{cases} p^- & \text{if } \|\partial_i v\|_{L^{p_i}(\Omega,\omega_i)} \geq 1, \\ p^+ & \text{if } \|\partial_i v\|_{L^{p_i}(\Omega,\omega_i)} < 1. \end{cases}$$

Then,

$$\frac{\sum_{i=1}^N \int_{\Omega} |\partial_i v|^{p_i} \omega_i dx}{\|v\|_{W_0^{1,\vec{p}}(\Omega,\vec{\omega})}} \rightarrow +\infty \text{ and } \sum_{i=1}^N \int_{\Omega} |\partial_i v|^{p_i} \omega_i dx \rightarrow +\infty \text{ as } \|v\|_{W_0^{1,\vec{p}}(\Omega,\vec{\omega})} \rightarrow +\infty.$$

From (5.6), we have $\frac{|\langle Av, v - v_0 \rangle|}{\|v\|_{W_0^{1,\vec{p}}(\Omega,\vec{\omega})}} \rightarrow +\infty$ as $\|v\|_{1,\vec{p},\vec{\omega}} \rightarrow +\infty$.

Since $\frac{\langle \Phi_n v, v \rangle}{\|v\|_{W_0^{1,\vec{p}}(\Omega,\vec{\omega})}}$ and $\frac{\langle \Phi_n v, v_0 \rangle}{\|v\|_{W_0^{1,\vec{p}}(\Omega,\vec{\omega})}}$ are bounded, then we get

$$\frac{\langle B_n v, v, -v_0 \rangle}{\|v\|_{W_0^{1,\vec{p}}(\Omega,\vec{\omega})}} = \frac{\langle Av, v - v_0 \rangle}{\|v\|_{W_0^{1,\vec{p}}(\Omega,\vec{\omega})}} + \frac{\langle \Phi_n v, v, -v_0 \rangle}{\|v\|_{W_0^{1,\vec{p}}(\Omega,\vec{\omega})}} \rightarrow +\infty \text{ as } \|v\|_{W_0^{1,\vec{p}}(\Omega,\vec{\omega})} \rightarrow +\infty.$$

We conclude that the operator $B_n = A + \Phi_n$ is coercive.

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