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C_0-semigroups and Local Spectral Theory

H. Boua, M. Karmouni and A. Tajmouati

ABSTRACT: Let $(T(t))_{t\geq 0}$ be a C_0-semigroup of operators on a Banach space X. In this paper, we show that if there exists $t_0 > 0$ such that $T(t_0)$ has the SVEP then A has the SVEP and if $\sigma_p(A)$ has empty interior, then $T(t)$ has the SVEP for all $t \geq 0$. Also, some local spectral properties for C_0 semigroups and theirs generators and some stabilities results are also established.

Key Words: C_0-semigroup, Local spectrum, SVEP, stability.

Contents

1. Introduction

The semigroups can be used to solve a large class of problems commonly known as the Cauchy problem,

$$\begin{cases} u'(t) = Au(t) & \text{for all } t \geq 0, \\ u(0) = u_0. & . \end{cases}$$

on a Banach space X. Here A is a given linear operator with domain $D(A)$ and the initial value u_0. The solution of the previous Cauchy problem will be given by $u(t) = T(t)u_0$ for an operator semigroup $(T(t))_{t\geq 0}$ on X. In this paper, we will focus on a special class of linear semigroups called C_0 semigroups which are semigroups of strongly continuous bounded operators. Precisely, a one-parameter family $(T(t))_{t\geq 0}$ of operators on a Banach space X is called a C_0-semigroup of operators or a strongly continuous semigroup of operators if,

1. $T(0) = I$,
2. $T(t+s) = T(t)T(s)$, $\forall t, s \geq 0$,
3. $\lim\limits_{t\to 0} T(t)x = x$, $\forall x \in X$.

$(T(t))_{t\geq 0}$ has a unique infinitesimal generator A defined in domain $D(A)$ by,

$$Ax = \lim_{t\to 0} \frac{T(t)x - x}{t}, \forall x \in D(A),$$

$$D(A) = \{x \in X : \lim_{t\to 0} \frac{T(t)x - x}{t} \text{ exists}\}.$$

Also, $T(t)$ are linear and continuous operators on X for all $t \geq 0$, and A is a closed operator, see [4,8]. In order to understand the behavior of the solutions in terms of the data concerning A, one seeks information about the spectrum of $T(t)$ in terms of the spectrum of A. Unfortunately the spectral mapping theorem $e^{t\sigma(A)} = \sigma(T(t)) \setminus \{0\}$ often fails, sometimes in dramatic ways. However, the inclusion

$$e^{t\sigma(A)} \subseteq \sigma(T(t)) \setminus \{0\} \tag{1.1}$$

is always true. The aim of this paper is to develop a local spectral theory for C_0 semigroups.

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2. Preliminaries

Throughout, X denotes a complex Banach space, let A be a closed operator on X with domain $D(A)$. We denote by A^*, $R(A)$, $N(A)$, $R^\infty(A)=\bigcap_{n\geq 0}R(A^n)$, $\sigma_K(A)$, $\sigma_{su}(A)$, $\sigma(A)$, respectively the adjoint, the range, the null space, the hyper-range, the semi-regular spectrum, the surjectivity spectrum and the spectrum of A. Recall that for a closed operator A and $x\in X$, the local resolvent of A at x, $\rho_A(x)$ defined as the union of all open subset U of $\mathbb{C}$ for which there is an analytic function $f:U\to D(A)$ such that the equation $(A-\mu I)f(\mu)=x$ holds for all $\mu\in U$. The local spectrum $\sigma_A(x)$ of A at x is defined as $\sigma_A(x)=\mathbb{C}\setminus\rho_A(x)$. Evidently $\sigma_A(x)\subseteq\sigma_{su}(A)\subseteq\sigma(A)$, $\rho_A(x)$ is open and $\sigma_A(x)$ is closed.

Let $f(z)=\displaystyle\sum_{i=0}^{\infty}x_i(z-\mu)^i$ (in a neighborhood of μ) be the Taylor expansion of f. It is easy to see that $\mu\in\rho_A(x)$ if and only if there exists a sequence such that $(x_i)_{i\geq 0}\subseteq D(A)$, $(A-\mu)x_0=x$, $(A-\mu)x_{i+1}=x_i$, and $\sup_i||x_i||^{\frac{1}{i}}<\infty$, see [5,7].

For any arbitrary closed set Ω in the complex field, the spectral subspace associated to Ω is :

$$X_A(\Omega)=\{x\in X:\sigma_A(x)\subseteq\Omega\}$$

$X_A(\Omega)$ is a hyperinvariant subspace of A not always closed, see [6].

Next, let A be a closed operator, A is said to have the single valued extension property at $\lambda_0\in\mathbb{C}$ (SVEP) if for every open disc $D_{\lambda_0}\subseteq\mathbb{C}$ centered at λ_0, the only analytic function $f:D_{\lambda_0}\longrightarrow D(A)$ which satisfies the equation $(A-zI)f(z)=0$ for all $z\in D_{\lambda_0}$ is the function $f\equiv 0$. A is said to have the SVEP if A has the SVEP for every $\lambda\in\mathbb{C}$. Denote by

$$S(A)=\{\lambda\in\mathbb{C}: A \text{ has not the SVEP at } \lambda\}.$$

$X_A(\emptyset)=\{0\}$ implies $S(A)=\emptyset$ [1]. If A is bounded, then $X_A(\emptyset)$ is closed if and only if $X_A(\emptyset)=\{0\}$ if and only if $S(A)=\emptyset$ [6].

Note that $\mu\in S(A)$ if and only if there exists a sequence $(x_i)_{i\geq 0}\subseteq D(A)$ not all of them equal to zero such that $(A-\mu)x_{i+1}=x_i$, with $x_0=0$ and $\sup_i||x_i||^{\frac{1}{i}}<\infty$, see [5].

Let $(T(t))_{t\geq 0}$ be a C_0 semigroup with generator A, we introduce the following operator acting on X and depending on the parameters $\lambda\in\mathbb{C}$ and $t\geq 0$,

$$B_\lambda(t)x=\int_0^t e^{\lambda(t-s)}T(s)xds, \text{ for all } x\in X.$$

It is well known that $B_\lambda(t)$ is a bounded operator on X and we have ([4,8]):

$$\begin{gathered}(e^{\lambda t}-T(t))^n x=(\lambda-A)^n B_\lambda^n(t)x, \quad \text{for all } x\in X \text{ and all } n\in\mathbb{N}\\ (e^{\lambda t}-T(t))^n x=B_\lambda^n(t)(\lambda-A)^n x, \quad \text{for all } x\in D(A^n) \text{ and all } n\in\mathbb{N};\\ R^\infty(e^{\lambda t}-T(t))\subseteq R^\infty(\lambda-A);\\ N(\lambda-A)^n\subseteq N(e^{\lambda t}-T(t))^n.\end{gathered}$$

Recall that some spectral inclusions for various reduced spectra are studied in [3], [4] and [8]. The authors proved that

$$e^{t\nu(A)}\subseteq\nu(T(t))$$

where $\nu\in\{\sigma_{ap},\sigma_K\}$, approximate point spectrum and semi-regular spectrum, also we have equality where $\nu\in\{\sigma_p,\sigma_r\}$ point spectrum and residual spectrum. In the next two sections, we will prove a spectral inclusion for local spectrum and a framing of $S(;)$ which characterizes it. Some related stability results are also presented.

3. Local Spectral Theory

Theorem 3.1. *For the generator A of a strongly continuous semigroup $(T(t))_{t\geq 0}$, then for all $t\geq 0$, we have*

$$e^{tS(A)}\subseteq S(T(t))\backslash\{0\}\subseteq e^{t\ int(\sigma_p(A))}$$

Proof. Let $e^{\lambda_0 t} \notin S(T(t))$ $t \geq 0$, then $T(t)$ has SVEP at $e^{\lambda_0 t}$. Let us show that $\lambda_0 \notin S(A)$. Let D_{λ_0} the open disc centered at λ_0, $f : D_{\lambda_0} \longrightarrow D(A)$ an analytic function such that for all $\mu \in D_{\lambda_0}$, $(\mu - A)f(\mu) = 0$. Show that $f \equiv 0$.

Consider the analytic function $\varphi_t : \mu \in D_{\lambda_0} \longrightarrow e^{t\mu}$. For all $\mu \in D_{\lambda_0}$, $\varphi_t'(\mu) = te^{\mu t} \neq 0$. By the inverse function theorem, there exists a neighborhood V of λ_0 such that $V \subseteq D_{\lambda_0}$, $\varphi_t(V)$ is open and the function $\varphi_t : V \longrightarrow \varphi_t(V)$ is bijective. The function $\varphi_t^{-1} : \varphi_t(V) \longrightarrow V$ is analytic and therefore the function $g : z \in \varphi_t(V) \longrightarrow f(\varphi_t^{-1}(z))$ is analytic. Moreover, for all $z \in \varphi_t(V)$, there exists a $\mu \in V$ such that $z = e^{t\mu}$. Furthermore,

$$\begin{aligned}(z - T(t))g(z) &= (\mu - A)B_\mu(t)f(\varphi_t^{-1}(z)) \\ &= (\mu - A)B_\mu(t)f(\mu) \\ &= B_\mu(t)(\mu - A)f(\mu) = 0.\end{aligned}$$

Thus $g \equiv 0$, then $f \equiv 0$ on V, hence $f \equiv 0$ on D_{λ_0}. Hence $\lambda_0 \notin S(A)$.

On the other hand $S(T(t))\backslash\{0\} \subseteq int(\sigma_p(T(t))\backslash\{0\}) = int(e^{t\sigma_p(A)}) \subseteq e^{t\ int(\sigma_p(A))}$. So the proof is complete.

□

In the following, we give a sufficient condition to show that the spectral subspace $X_A(\emptyset)$ is closed for all $t > 0$.

Corollary 3.2. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup, with generator A, then:*
$X_{T(t)}(\emptyset) = \{0\}$ for some $t \geq 0$ implies that A has the SVEP.

Proof. Let $t \geq 0$ such that $X_{T(t)}(\emptyset) = \{0\}$, that implies that $S(T(t)) = \emptyset$, by theorem 3.1 we have $S(A) = \emptyset$. □

Corollary 3.3. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup, with infinitesimal generator A.*

1. *If $T(t_0)$ has the SVEP for some $t_0 \geq 0$, then A has the SVEP.*
2. *If $\sigma_p(A)$ has empty interior, then $T(t)$ has the SVEP for all $t \geq 0$,.*

Example 3.4. *We consider the left translation group $T(t)_{t\in\mathbb{R}}$ on $X = C_0(\mathbb{R})$. Then $\sigma(A) = i\mathbb{R}$ and $\sigma(T(t)) = \{z \in \mathbb{C} : |z| = 1\}$, so A has the SVEP. According to corollary 3.3, $T(t)$ has the SVEP for all $t > 0$. Then $\sigma_{su}(T(t)) = \{z \in \mathbb{C} : |z| = 1\}$.*

Example 3.5. *A C_0-semigroup $(T(t))_{t\geq 0}$ is called periodic if there exists $t_0 > 0$ such that $T(t_0) = I$, so $T(t_0)$ has the SVEP. From corollary 3.3, the infinitesimal generator A of $(T(t))_{t\geq 0}$ has the SVEP.*

To continue the development of a spectral theory for semigroups and theirs generators, we prove that the formula (1.1) holds for local spectrum.

Theorem 3.6. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on X with infinitesimal generator A. The following spectral inclusion holds :*

$$e^{t\sigma_A(x)} \subseteq \sigma_{T(t)}(x)\backslash\{0\}, \text{ for all } t \geq 0 \text{ and } x \in X.$$

Proof. Let $e^{\lambda t} \notin \sigma_{T(t)}(x)$, then there exists $(x_i)_{i\geq 0} \subseteq X$, such that

$$(e^{\lambda t} - T(t))x_0 = x,\ \ (e^{\lambda t} - T(t))x_i = x_{i-1} \text{ and } \sup \|x_i\|^{\frac{1}{i}} < \infty.$$

Let $y_i = B_\lambda^{i+1}(t)x_i$, then $(y_i)_{i\geq 0} \subseteq D(A)$ and $y_0 = B_\lambda(t)x_0$. We have :

$$\begin{aligned}(\lambda - A)y_i &= (\lambda - A)B_\lambda(t)B_\lambda^i(t)x_i \\ &= (e^{\lambda t} - T(t))B_\lambda^i(t)x_i \\ &= B_\lambda^i(t)(e^{\lambda t} - T(t))x_i \\ &= B_\lambda^i(t)x_{i-1} \\ &= y_{i-1}\end{aligned}$$

and

$$\sup \|y_i\|^{\frac{1}{i}} < \infty$$

So that $\lambda \notin \sigma_A(x)$
□

Remark 3.1. *The spectral inclusion for local spectrum is strict. Indeed, let $(T(t))_{t\geq 0}$ be a quasi-nilpotent C_0 semigroup with infinitesimal generator A, and $0 \neq x \in X$. We have $\sigma_{T(t)}(x) = \{0\}$, but $e^{t\sigma_A(x)} = \emptyset$.*

4. Stability Results.

Let $(T(t))_{t\geq 0}$ ba a C_0-semigroup on X with infinitesimal generator A. $(T(t))_{t\geq 0}$ is said to be strongly stable if $\lim_{t\to\infty} ||T(t)x|| = 0$ for all $x \in X$. We say that $(T(t))_{t\geq 0}$ is uniformly stable if $\lim_{t\to\infty} ||T(t)|| = 0$. In [2], A. Elkoutri and M. A. Taoudi showed that $(T(t))_{t\geq 0}$ is strongly stable if $\sigma_K(A) \cap i\mathbb{R} = \emptyset$. In the following, we give a stability result for strongly continuous semigroups using the local spectrum:

Proposition 4.1. *Let A be the generator of a bounded strongly continuous semigroup $(T(t))_{t\geq 0}$. If $\sigma_A(x) \cap i\mathbb{R} = \emptyset$ for all $x \in X$, then $(T(t))_{t\geq 0}$ is strongly stable.*

Proof. If $\sigma_A(x) \cap i\mathbb{R} = \emptyset$, for all $x \in X$. Then,

$$\sigma_{su}(A) \cap i\mathbb{R} = \bigcup_{x\in X} \sigma_A(x) \cap i\mathbb{R} = \bigcup_{x\in X} (\sigma_A(x) \cap i\mathbb{R}) = \emptyset.$$

As $\sigma_K(A) \cap i\mathbb{R} \subseteq \sigma_{su}(A) \cap i\mathbb{R} = \emptyset$, then $\sigma_K(A) \cap i\mathbb{R} = \emptyset$. According to [2, corollary 2.1], $(T(t))_{t\geq 0}$ is strongly stable. □

Proposition 4.2. *Let A be the generator of a bounded strongly continuous semigroup $(T(t))_{t\geq 0}$. Then, the following assertions are equivalent:*

1. *$(T(t))_{t\geq 0}$ is uniformly stable;*
2. *for all $x \in X$, there exists $t_0 > 0$ such that $\sigma_{T(t_0)}(x) \cap \Gamma = \emptyset$*

where Γ stands for the unit circle of $\mathbb{C}$.

Proof. According to [2, corollary 2.2] and [3, Theorem 3.2] , it suffices to show that $\sigma_{(T(t_0))}(x) \cap \Gamma = \emptyset$ implies that $\sigma_K(T(t_0)) \cap \Gamma = \emptyset$. Indeed: If $\sigma_{(T(t_0))}(x) \cap \Gamma = \emptyset$ for all $x \in X$, then

$$\sigma_{su}(T(t_0)) \cap \Gamma = \bigcup_{x\in X} \sigma_{(T(t_0))}(x) \cap \Gamma = \bigcup_{x\in X} (\sigma_{T(t_0)}(x) \cap \Gamma) = \emptyset.$$

As $\sigma_K(T(t_0)) \cap \Gamma \subseteq \sigma_{su}(T(t_0)) \cap \Gamma = \emptyset$, then $\sigma_K(T(t_0)) \cap \Gamma = \emptyset$.

□

Example 4.3. *Consider the Heat equation in $L^p(0, \pi)$.*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x), (t,x) \in \mathbb{R}^+ \times (0,\pi) \\ u(t,0) = 0 = u(t,\pi), t \geq 0 \\ u(0,x) = f(x) \quad x \in (0,\pi) \end{cases}.$$

Let $p > 2$. On $X = L^p(0, \pi)$ consider the operator defined by

$$Af(x) = f''(x)$$

with domain $D(A) = W^{2,p}(0, \pi) \cap W_0^{1,p}(0, \pi)$, $x \in (0, \pi)$ where

$$W_0^{1,p} = \{f \in W^{1,p}(0, \pi) : f(0) = 0 = f(\pi)\}.$$

The operator A is self-adjoint. For each $f \in W^{2,p}(0,\pi) \cap W_0^{1,p}(0,\pi)$ the unique solution of the equation is given by

$$u(t,x) = (T(t)f)(x).$$

The spectrum of A is $\sigma(A) = \{-n^2; n \geq 1\}$. Since $int(\sigma(A)) = \emptyset$, then A has the SVEP. So, $T(t)$ has the SVEP for all $t > 0$. Since $\sigma_A(x) \cap i\mathbb{R} = \emptyset$ for all $x \in X$, then $(T(t))_{t\geq 0}$ is strongly stable.

Example 4.4. *On the Banach space $X := L^1[\frac{\alpha}{2}, 1]$ define the operator :*

$$Af := -f' - (\mu + b)f \text{ with } D(A) := \{f \in W^{1,1}[\tfrac{\alpha}{2}, 1] : f(\tfrac{\alpha}{2}) = 0\},$$

μ is a positive continuous function on $[\frac{\alpha}{2}, 1]$ and b a continuous function with $b(s) > 0$ for $s \in (\alpha, 1)$, $b(s) = 0$ otherwise. The operator A generates a C_0 semigroup $(T(t))_{t\geq 0}$ on X given by :

$$T(t)f(s) = \begin{cases} e^{-\int_{s-t}^{s}(\mu(\tau)+b(\tau))d\tau}.f(s-t) & \text{for } s - t > \frac{\alpha}{2}. \\ 0, & \text{else where.} \end{cases}$$

The spectrum A is empty. Hence A has the SVEP, so $T(t)$ has the SVEP for all $t > 0$. Furthermore, $(T(t))_{t\geq 0}$ is a nilpotent semigroup, so $\sigma(T(t)) = \{0\}$. Hence $\sigma_*(T(t)) = \{0\}$, where $\sigma_* = \sigma_s, \sigma_{ap}, \sigma_k, \sigma_e$. Since, $T(t)$ has the SVEP, then $\sigma_{T(t)}(x) = \{0\}$ for all $x \in X \setminus \{0\}$, so $\sigma_{T(t)}(x) \cap \Gamma = \emptyset$, thus $(T(t))_{t\geq 0}$ is uniformly stable.

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Hamid Boua,
Mohammed First University,
Pluridisciplinary Faculty of Nador,
Laboratory of Analysis, Geometry and Applications,
Nador, Morocco.
E-mail address: h.boua@ump.ac.ma

and

Mohammed Karmouni,
Cadi Ayyad University,
Multidisciplinary Faculty,
Safi, Morocco.
E-mail address: mohammed.karmouni@uca.ma

and

Abdelaziz Tajmouati,
Sidi Mohamed Ben Abdellah University,
Faculty of Sciences Dhar Al Mahraz,
Laboratory of Mathematical Analysis and Applications,
Fez, Morocco.
E-mail address: abdelaziz.tajmouati@usmba.ac.ma