Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2022 (40)** : 1–8.
ISSN-0037-8712 in press
doi:10.5269/bspm.50604

New Estimates for the Fourier Transform in the Space $\mathrm{L}^2(\mathbb{R}^n)$

M. El Hamma, R. Daher, N. Djellab and Ch. Khalil

ABSTRACT: In this paper, we prove new estimates are presented for the integral $\int_{|t|\geq N} |\widehat{f}(t)|^2 dt$, where $\widehat{f}$ stands for the Fourier transform of f and $N \geq 1$, in the space $\mathrm{L}^2(\mathbb{R}^n)$ characterized by the generalized modulus of continuity of the kth order constructed with the help of the generalized spherical mean operator.

Key Words: Fourier transform, Generalized derivatives, Spherical mean operator, Continuity modulus.

Contents

1. Introduction and preliminaries

In [2], Abilov et al. proved new estimates for the Fourier transform in the space $\mathrm{L}^2(\mathbb{R})$ on certain classes of functions characterized by the generalized continuity modulus for these estimates, using a Steklov function.
In this paper, we prove the generalization of Abilov's results [2] in the Fourier transform for multivariable functions on $\mathbb{R}^n$. For this purpose, we use spherical mean operator in the place of the Steklov function.

Assume that $\mathrm{L}^2(\mathbb{R}^n)$ the space of integrable functions f with the norm

$$\|f\|_2 = \left(\int_{\mathbb{R}^n} |f(x)|^2 dx\right)^{1/2}$$

The Fourier transform for the function $f \in \mathrm{L}^1(\mathbb{R}^n)$ is defined by

$$\widehat{f}(t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix.t} dx$$

The inverse Fourier transform is defined by the formula

$$f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{f}(t) e^{ix.t} dt$$

The Plancherel theorem provides an extension of the Fourier transform to $\mathrm{L}^2(\mathbb{R}^n)$,
i.e,

$$\int_{\mathbb{R}^n} |f(x)|^2 dx = \int_{\mathbb{R}^n} |\widehat{f}(t)|^2 dt$$

Let $j_p(z)$ be a normalized Bessel function of the first kind, i.e.,

$$j_p(z) = \frac{2^p \Gamma(p+1)}{z^p} J_p(z), \;\; \forall z \in \mathbb{R}^+ \tag{1.1}$$

where $J_p(z)$ is a Bessel function of the first kind.

2010 *Mathematics Subject Classification:* 42B12.
Submitted October 30, 2019. Published June 27, 2020

Consider in $L^2(\mathbb{R}^n)$ the spherical mean operator (see [3])

$$M_h f(x) = \frac{1}{w_{n-1}} \int_{S^{n-1}} f(x + hw) dw$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$, w_{n-1} its total surface measure with respect to the usual induced measure dw.

The finite differences of the first and higher orders are defined by

$$\Delta_h f(x) = M_h f(x) - f(x) = (M_h - I) f(x)$$

$$\Delta_h^k f(x) = \Delta_h(\Delta_h^{k-1} f(x)) = (M_h - I)^k f(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} M_h^i f(x) \tag{1.2}$$

where $M_h^0 f(x) = f(x)$, $M_h^i f(x) = M_h(M_h^{i-1} f(x))$ for $i = 1, 2, ..., k$ and $k = 1, 2, ...$, I is the identity operator in $L^2(\mathbb{R}^n)$.

The k^{th} order generalized modulus of continuity of function $f \in L^2(\mathbb{R}^n)$ is defined as

$$\Omega_k(f, \delta) = \sup_{0 < h \leq \delta} \|\Delta_h^k f(x)\|_2.$$

Denote by L^2_r the class of functions $f \in L^2(\mathbb{R}^n)$ such that $D^r f \in L^2(\mathbb{R}^n)$ $r = 1, 2, ...$ (In the sense of Levi (see [5])).

where the operator $D = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator and $x = (x_1, x_2, ..., x_n)$ $D^0 f = f$, $D^i f = D(D^{i-1} f)$, $i = 1, 2, ..., r$

According to [3], we have

$$M_h f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{f}(t) j_{\frac{n-2}{2}}(|t|h) e^{ix.t} dt.$$

and

$$f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{f}(t) e^{ix.t} dt.$$

i.e

$$M_h f(x) - f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{f}(t)(j_{\frac{n-2}{2}}(|t|h) - 1) e^{ix.t} dt.$$

By Parseval's identity, we obtain

$$\|M_h f(x) - f(x)\|_2^2 = \int_{\mathbb{R}^n} |\widehat{f}(t)|^2 (j_{\frac{n-2}{2}}(|t|h) - 1)^2 dt$$

Lemma 1.1. *Let $f \in L^2_r$. Then*

$$\|\Delta_h^k D^r f(x)\|_2^2 = \int_{\mathbb{R}^n} |t|^{2r} (1 - j_{\frac{n-2}{2}}(|t|h))^{2k} |\widehat{f}(t)|^2 dt \tag{1.3}$$

Proof. We have

$$\widehat{Df}(t) = (-1)^r |t|^r \widehat{f}(t)$$

Then

$$\widehat{M_h^i D^r f}(t) = (-1)^r |t|^r \left(j_{\frac{n-2}{2}}(h|t|) \right)^i \widehat{f}(t)$$

From formula (1.2), we conclude that the Fourier transform of $\Delta_h^k D^r f(x)$ is $(-1)^r|t|^r\left(j_{\frac{n-2}{2}}(h|t|)-1\right)^k \widehat{f}(t)$. By Plancherel identity, we have the result. □

2. Main Result

Befor presenting the theorems and their proofs, for convenience, we intoduce the notation

$$m_l(f)=\left(\int_{|t|\geq l}|\widehat{f}(t)|^2dt\right)^{1/2}$$

Theorem 2.1. *Let $r\in\mathbb{N}\cup\{0\}$ and $k\in\mathbb{N}$. If $f\in L^2_r$, then*

$$m_N(f)\leq N^{-r}\Omega_k(D^r f,\frac{c}{N}),$$

where $c>0$ is a fixed constant and $N\to\infty$.

Proof. In the terms of $j_p(z)$, we have (see [1])

$$|j_p(z)|\leq 1. \quad (2.1)$$

$$1-j_p(z)=O(1),\ \ z\geq 1. \quad (2.2)$$

$$1-j_p(z)=O(z^2),\ \ 0\leq z\leq 1. \quad (2.3)$$

$$\sqrt{hz}J_p(hz)=O(1),\ \ hz\geq 0 \quad (2.4)$$

Let $f\in L^2_r$. By Hölder inequality, we have

$$\begin{aligned}
&\int_{|t|\geq N}|\widehat{f}(t)|^2dt-\int_{|t|\geq N}|\widehat{f}(t)|^2 j_{\frac{n-2}{2}}(h|t|))dt=\int_{|t|\geq N}(1-j_{\frac{n-2}{2}}(h|t|)|\widehat{f}(t)|^2dt\\
=&\int_{|t|\geq N}(1-j_{\frac{n-2}{2}}(h|t|))|\widehat{f}(t)|^{2-\frac{1}{k}}|\widehat{f}(t)|^{\frac{1}{k}}dt\\
\leq&\left(\int_{|t|\geq N}|\widehat{f}(t)|^2dt\right)^{\frac{2k-1}{2k}}\left(\int_{|t|\geq N}(1-j_{\frac{n-2}{2}}(h|t|))^{2k}|\widehat{f}(t)|^2dt\right)^{\frac{1}{2k}}\\
\leq&\left(\int_{|t|\geq N}|\widehat{f}(t)|^2dt\right)^{\frac{2k-1}{2k}}\left(\int_{|t|\geq N}\frac{1}{|t|^{2r}}(1-j_{\frac{n-2}{2}}(h|t|))^{2k}|t|^{2r}|\widehat{f}(t)|^2dt\right)^{\frac{1}{2k}}\\
\leq&\ N^{\frac{-r}{k}}\left(\int_{|t|\geq N}|\widehat{f}(t)|^2dt\right)^{\frac{2k-1}{2k}}\left(\int_{|t|\geq N}(1-j_{\frac{n-2}{2}}(h|t|))^{2k}|t|^{2r}|\widehat{f}(t)|^2dt\right)^{\frac{1}{2k}}.
\end{aligned}$$

From formula (1.3), we have the inequality

$$\int_{|t|\geq N}(1-j_{\frac{n-2}{2}}(h|t|))^{2k}|t|^{2r}|\widehat{f}(t)|^2dt\leq\|\Delta_h^k D^r f(x)\|_2^2$$

Therefore,

$$\begin{aligned}
\int_{|t|\geq N}|\widehat{f}(t)|^2dt\quad&\leq\int_{|t|\geq N}|\widehat{f}(t)|^2 j_{\frac{n-2}{2}}(h|t|)d\xi\\
&+\ N^{\frac{-r}{k}}\left(\int_{|t|\geq N}|\widehat{f}(t)|^2dt\right)^{\frac{2k-1}{2k}}\|\Delta_h^k D^r f(x)\|_2^{\frac{1}{k}}
\end{aligned}$$

From formulas (1.1) and (2.4), we have

$$j_{\frac{n-2}{2}}(x)=O(x^{\frac{-n+1}{2}})$$

Then

$$(1-O(Nh)^{\frac{-n+1}{2}})\int_{|t|\geq N}|\widehat{f}(t)|^2dt=O\left(N^{\frac{-r}{k}}\left(\int_{|t|\geq N}|\widehat{f}(t)|^2dt\right)^{\frac{2k-1}{2k}}\|\Delta_h^k\mathrm{D}^rf(x)\|_2^{\frac{1}{k}}\right)$$

Setting $h=\frac{c}{N}$ in the last inequality and choose $c>0$ such that $(1-O(c^{\frac{-n+1}{2}}))\geq\frac{1}{2}$.

Therefore

$$\int_{|t|\geq N}|\widehat{f}(t)|^2dt=O(N^{\frac{-r}{k}})\left(\int_{|t|\geq N}|\widehat{f}(t)|^2dt\right)^{\frac{2k-1}{2k}}\|\Delta_h^k\mathrm{D}^rf(x)\|_2^{\frac{1}{k}}$$

Then

$$\left(\int_{|t|\geq N}|\widehat{f}(t)|^2dt\right)^{\frac{1}{2k}}=O\left(N^{\frac{-r}{k}}\right)\|\Delta_{\frac{c}{N}}^k\mathrm{D}^rf(x)\|_2^{\frac{1}{k}}$$

i.e.,

$$m_N(f)=O\left(N^{-r}\Omega_k(\mathrm{D}^rf,\frac{c}{N})\right).$$

This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let $f\in\mathrm{L}^2(\mathbb{R}^n)$. Then*

$$\Omega_k(f,\delta)=O\left(N^{-2k}\left(\sum_{l=1}^{N}l^{4k-1}m_l^2(f)\right)^{1/2}\right),$$

where $k=1,2,...$ and $N\to+\infty$.

Proof. Let $f\in\mathrm{L}^2(\mathbb{R}^n)$. Then

$$\|\Delta_h^kf(x)\|_2^2=\int_{\mathbb{R}^n}(1-j_{\frac{n-2}{2}}(|t|h))^{2k}|\widehat{f}(t)|^2dt$$

Let $N=[\frac{1}{h}]$, where $0<h<1$. From formula (2.3), we have

$$
\begin{aligned}
& \int_{\mathbb{R}^n} = \int_{|t|<N} + \int_{|t|\geq N} \\
= \ & O(1)\left[h^{4k}\int_{|t|<N}|t|^{4k}|\widehat{f}(t)|^2dt + \int_{|t|\geq N}|\widehat{f}(t)|^2dt\right] \\
= \ & O(\frac{1}{N^{4k}})\left[\int_{|t|<N}|t|^{4k}|\widehat{f}(t)|^2dt + N^{4k}\int_{|t|\geq N}|\widehat{f}(t)|^2dt\right] \\
= \ & O(\frac{1}{N^{4k}})\left[\sum_{i=1}^{N}\int_{i-1\leq|t|<i}|t|^{4k}|\widehat{f}(t)|^2dt + N^{4k}\int_{|t|\geq N}|\widehat{f}(t)|^2dt\right] \\
= \ & O(\frac{1}{N^{4k}})\left[\sum_{i=1}^{N}i^{4k}\int_{i-1\leq|t|<i}|\widehat{f}(t)|^2dt + N^{4k}\int_{|t|\geq N}|\widehat{f}(t)|^2dt\right] \\
= \ & O(\frac{1}{N^{4k}})\left[\sum_{i=1}^{N}i^{4k}\left(\int_{|t|\geq i}|\widehat{f}(t)|^2dt - \int_{|t|\geq i-1}|\widehat{f}(t)|^2dt\right) + N^{4k}\int_{|t|\geq N}|\widehat{f}(t)|^2dt\right] \\
= \ & O\left(\frac{1}{N^{4k}}\right)\sum_{i=1}^{N}((i+1)^{4k}-i^{4k})\int_{|t|\geq i}|\widehat{f}(t)|^2dt.
\end{aligned}
$$

Since

$$(n+1)^q - n^q \leq 2^q n^{q-1},\ q>1$$

The previous inequality implies that

$$\int_{\mathbb{R}^n}(1-j_{\frac{n-2}{2}}(|t|h))^{2k}|\widehat{f}(t)|^2dt = O\left(\frac{1}{N^{4k}}\right)\sum_{i=1}^{N}i^{4k-1}m_i^2(f)$$

i.e.,

$$\|\Delta_h^k f(x)\|_2^2 = O\left(\frac{1}{N^{4k}}\right)\sum_{i=1}^{N}i^{4k-1}m_i^2(f)$$

which implies

$$\Omega_k(f,\delta) = O\left(\frac{1}{N^{2k}}\left(\sum_{i=1}^{N}i^{4k-1}m_i^2(f)\right)^{1/2}\right)$$

This theorem is proved. □

Theorem 2.3. *Let $f \in L^2(\mathbb{R}^n)$. If the serie*

$$\sum_{i=1}^{+\infty}i^{r-1}m_i(f),\ r=1,2,...$$

converge, then $f \in L_r^2$ and

$$\Omega_k(D^r f) = O\left(N^{-4k}\sum_{i=1}^{N}i^{2r+4k-1}m_i^2(f)\right) + \sum_{i=[\frac{N}{2}]}^{N}i^{2r-1}m_i(f),$$

where $k=1,2...$ and $N\to+\infty$

Proof. Let $f \in \mathrm{L}^2(\mathbb{R}^n)$, we have

$$\begin{aligned}\|\mathrm{D}^r f\|_2^2 &= \int_{\mathbb{R}^n} |t|^{2r} |\widehat{f}(t)|^2 dt \\ &= \sum_{i=0}^{+\infty} \int_{i \le |t| \le i+1} |t|^{2r} |\widehat{f}(t)|^2 dt\end{aligned}$$

using an Abel transformation we obtain

$$\|\mathrm{D}^r f\|_2^2 \le m_0^2(f) + 2r \sum_{i=1}^{+\infty} (i+1)^{2r-1} m_i^2(f)$$

From the inequality $i + 1 \le 2i$ we conclude that

$$\|\mathrm{D}^r f\|_2^2 \le C(m_0^2(f) + \sum_{i=1}^{+\infty} i^{2r-1} m_i^2(f))$$

where $C > 0$ is a positive constant.

Hence

$$\|\mathrm{D}^r f\|_2^2 = O\left(\sum_{i=1}^{+\infty} i^{r-1} m_i(f)\right)$$

Since the serie $\sum_{i=1}^{+\infty} i^{r-1} m_i(f)$, $r = 1, 2, ..,$ converge then $f \in \mathrm{L}_r^2$.

From Lemma 1.1, we have

$$\|\Delta_h^k \mathrm{D}^r f(x)\|_2^2 = \int_{\mathbb{R}^n} |t|^{2r} (1 - j_{\frac{n-2}{2}}(|t|h))^{2k} |\widehat{f}(t)|^2 dt$$

Let $N = [\frac{1}{h}]$. Then

$$\begin{aligned}\|\Delta_h^k \mathrm{D}^r f(x)\|_2 &\le \left(\int_{|t|<N} + \int_{|t|\ge N}\right)^{1/2} \\ &\le \left(\int_{|t|<N}\right)^{1/2} + \left(\int_{|t|\ge N}\right)^{1/2} \\ &= (\mathrm{I}_1)^{1/2} + (\mathrm{I}_2)^{1/2}\end{aligned}$$

We have

$$\mathrm{I}_1 = \int_{|t|<N} |t|^{2r} (1 - j_{\frac{n-2}{2}}(|t|h))^{2k} |\widehat{f}(t)|^2 dt$$

By formula (2.3), we have

$$\begin{aligned}
\mathrm{I}_1 &= O(h^{4k})\int_{|t|<N}|t|^{2r+4k}|\widehat{f}(t)|^2dt\\
&= O(h^{4k})\sum_{i=1}^{N}\int_{i-1\leq|t|<i}|t|^{2r+4k}|\widehat{f}(t)|^2dt\\
&= O(h^{4k})\sum_{i=1}^{N}i^{2r+4k}\int_{i-1\leq|t|<i}|\widehat{f}(t)|^2dt\\
&= O(h^{4k})\sum_{i=1}^{N}i^{2r+4k}\left(\int_{|t|\geq i}|\widehat{f}(t)|^2dt-\int_{|t|\geq i-1}|\widehat{f}(t)|^2dt\right)\\
&= O(h^{4k})\sum_{i=1}^{N}i^{2r+4k-1}\int_{|t|\geq i}|\widehat{f}(t)|^2dt.
\end{aligned}$$

i.e

$$\mathrm{I}_1 = O(h^{4k})\sum_{i=1}^{N}i^{2r+4k-1}m_i^2(f).$$

Now we estimate I_2, we have

$$\begin{aligned}
\mathrm{I}_2 &= \int_{|t|\geq N}|t|^{2r}(1-j_{\frac{n-2}{2}}(|t|h))^{2k}|\widehat{f}(t)|^2dt\\
&= \sum_{i=1}^{\infty}\int_{2^{i-1}N\leq|t|\leq 2^iN}|t|^{2r}(1-j_{\frac{n-2}{2}}(|t|h))^{2k}|\widehat{f}(t)|^2dt\\
&= O\left(\sum_{i=1}^{\infty}(2^iN)^{2r}\int_{2^{i-1}N\leq|t|\leq 2^iN}|\widehat{f}(t)|^2dt\right)\\
&= O\left(N^{2r}\sum_{i=1}^{\infty}2^{2ri}m^2_{2^{i-1}N}(f)\right)
\end{aligned}$$

It follows that

$$(\mathrm{I}_2)^{1/2} = O\left(N^r\sum_{i=1}^{\infty}2^{ri}m_{2^{i-1}N}(f)\right)$$

Applying the relations

$$\int_{2^{l-2}N}^{2^{l-1}N}y^{r-1}dy = \frac{1}{r}N^r2^{r(l-2)}(2^r-1)$$

Using the fact the sequence $m_N(f)$, $N = 1, 2, ...,$ is monotonically decreasing, we can show (see [6]) that

$$(\mathrm{I}_2)^{1/2} = O\left(\sum_{i=[\frac{N}{2}]}^{N}i^{2r-1}m_i(f)\right)$$

then this theorem is proved. □

Remark:

Theorems 2.1 and 2.2 imply

$$m_N(f) = O(N^\nu) \Longleftrightarrow \|\Delta_h f\|_2 = O(h^\nu),$$

where $0 < \nu < 2$, $0 < h < 1$, $N \to +\infty$.

This result was proved in [4].

Acknowledgments

The authors would like to thank the referee for his valuable comments and suggestions.

References

1. V. A. Abilov and F. V. Abilova, *Approximation of Functions by Fourier-Bessel Sums.* Izv. Vyssh. Uchebn. Zaved. Mat., No. 8, 3-9, (2001).
2. V. A. Abilov, F. V. Abilova and M. K. Kerimov, *Some New Estimates of the Fourier Transform in the Space* $\mathbb{L}_2(\mathbb{R})$, Comput. Math. Math. Physi, Vol. 53, No. 9, pp. 1231-1238, (2013).
3. W. O. Bray and M. A. Pinsky, *Growth properties of Fourier transforms via moduli of continuity*, Journal of Functional Analysis, 255, 2265-2285, (2008).
4. R. Daher and M. El Hamma, *On Estimates for the Fourier transform in the Space* $\mathrm{L}^2(\mathbb{R}^n)$, C. R. Acad. Sci. Paris, Ser. I 352, 235-240, (2014).
5. S.M. Nikol'skii, *Approximation of Functions of Several Variables and Embedding Theorems*, (Nauka, Moscow)[in Russian], (1996).
6. A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, (Macmillan, New York, 1963).

M. El Hamma, R. Daher, N. Djellab and Ch. Khalil
Université Hassan II,Faculté des Sciences Aïn Chock,
Département de mathématiques et informatique,
Laboratoire Topologie, Algèbre, Géométrie et Mathématiques Discrètes, Casablanca, Morocco.
E-mail address: m_elhamma@yahoo.fr
E-mail address: rjdaher024@gmail.com
E-mail address: nisrine.djellab@gmail.com
E-mail address: khalil.chaimaa.92@gmail.com