



## Generalized Multiplicative $\alpha$ -skew Derivations on Rings

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ABSTRACT: Let  $\mathcal{R}$  be a semiprime (or prime) ring and  $\mathcal{U}$  be a nonzero ideal of  $\mathcal{R}$ . In the present paper, we study the notions of multiplicative generalized  $\alpha$ -skew derivations on ideals of  $\mathcal{R}$  and prove that if  $\mathcal{R}$  admits a multiplicative generalized  $\alpha$ -skew derivation  $G$  associated with a nonzero additive map  $d$  and an automorphism  $\alpha$ , then  $d$  is necessarily an  $\alpha$ -skew derivation of  $\mathcal{R}$ . Also, we study the structure of a semiprime ring admitting a multiplicative generalized  $\alpha$ -skew derivation satisfying more specific algebraic identities. Moreover, we also provide examples to show that the assumed restrictions cannot be relaxed.

Key Words: Semiprime ring, Multiplicative generalized  $\alpha$ -skew derivations, Commutativity.

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### 1. Introduction

Let  $\mathcal{R}$  be an associative ring with center  $Z(\mathcal{R})$ . For any  $x, y \in \mathcal{R}$ , the symbol  $[x, y]$  stands for the commutator  $xy - yx$  and the symbol  $x \circ y$  denotes the anti-commutator  $xy + yx$ . Recall that a ring  $\mathcal{R}$  is prime if  $x\mathcal{R}y = \{0\}$  implies  $x = 0$  or  $y = 0$  and  $\mathcal{R}$  is semiprime if  $x\mathcal{R}x = \{0\}$  implies  $x = 0$ . Let  $S$  be a nonempty subset of  $\mathcal{R}$ . A mapping  $F$  from  $\mathcal{R}$  to  $\mathcal{R}$  is called commutativity preserving on a subset  $S$  of  $\mathcal{R}$  if  $[x, y] = 0$  implies  $[F(x), F(y)] = 0$ , for all  $x, y \in S$ . The mapping  $F$  is called strong commutativity preserving (simply, SCP) on  $S$  if  $[x, y] = [F(x), F(y)]$ , holds for all  $x, y \in S$ . Suppose that  $\alpha$  is an automorphism of  $\mathcal{R}$ . An additive mapping  $d : \mathcal{R} \rightarrow \mathcal{R}$  is called an  $\alpha$ -skew derivation of  $\mathcal{R}$  if  $d(xy) = d(x)y + \alpha(x)d(y)$  holds for all  $x, y \in \mathcal{R}$ . Basic examples of  $\alpha$ -skew derivations are usual derivations and the mapping  $\alpha = id_{\mathcal{R}}$ , where  $id_{\mathcal{R}}$  denotes the identical mapping of  $\mathcal{R}$ . For  $\alpha = id_{\mathcal{R}}$ ,  $d$  is known as a derivation of  $\mathcal{R}$ . The notion of  $\alpha$ -skew derivation has been extended to generalized  $\alpha$ -skew derivation. An additive mapping  $F : \mathcal{R} \rightarrow \mathcal{R}$  is said to be a (right) generalized  $\alpha$ -skew derivation of  $\mathcal{R}$  if there exists an  $\alpha$ -skew derivation  $d$  of  $\mathcal{R}$  with associated automorphism  $\alpha$  such that  $F(xy) = F(x)y + \alpha(x)d(y)$  holds for all  $x, y \in \mathcal{R}$ ,  $d$  is called an associated  $\alpha$ -skew derivation of  $F$  and  $\alpha$  is called an associated automorphism of  $F$ . In particular, when  $\alpha = id_{\mathcal{R}}$ ,  $F$  is called a generalized derivation of  $\mathcal{R}$  according to the definition of [4]. A mapping  $F : \mathcal{R} \rightarrow \mathcal{R}$  satisfying  $F(xy) = F(x)\alpha(y)$  for all  $x, y \in \mathcal{R}$  is called a multiplicative  $\alpha$ -left centralizer of  $\mathcal{R}$ . For  $\alpha = id_{\mathcal{R}}$ ,  $F$  is called a left multiplier (or centralizer) of  $\mathcal{R}$ . A mapping  $d : \mathcal{R} \rightarrow \mathcal{R}$  (not necessarily additive) is called a multiplicative  $\alpha$ -skew derivation of  $\mathcal{R}$ , if  $d(xy) = d(x)y + \alpha(x)d(y)$  holds for all  $x, y \in \mathcal{R}$ , where  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  is an automorphism. A mapping  $G : \mathcal{R} \rightarrow \mathcal{R}$  (not necessarily additive) is called a multiplicative right generalized  $\alpha$ -skew derivation of  $\mathcal{R}$ , if there exists a multiplicative  $\alpha$ -skew derivation  $d : \mathcal{R} \rightarrow \mathcal{R}$  such that  $G(xy) = G(x)y + \alpha(x)d(y)$  holds for all  $x, y \in \mathcal{R}$ , where  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  is an automorphism. A mapping  $G : \mathcal{R} \rightarrow \mathcal{R}$  (not necessarily additive) is called a multiplicative left generalized  $\alpha$ -skew derivation of  $\mathcal{R}$ , if there exists a multiplicative  $\alpha$ -skew derivation  $d : \mathcal{R} \rightarrow \mathcal{R}$  of  $\mathcal{R}$  such that  $G(xy) = \alpha(x)G(y) + d(x)y$  holds for all  $x, y \in \mathcal{R}$ , where  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  is an automorphism.  $G$  is said to be a multiplicative generalized  $\alpha$ -skew derivation with associated map  $d$  if it is both a multiplicative left as well as right generalized  $\alpha$ -skew derivation.

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There is also a growing literature on strong commutativity preserving (SCP) maps and derivations (for reference see [2], [3], [5], [7], [8], etc.) In [1], Ali et al. showed that if  $\mathcal{R}$  is a semiprime ring and  $f$  is an endomorphism which is a strong commutativity preserving (simply, SCP) map on a nonzero ideal  $\mathcal{U}$  of  $\mathcal{R}$ , then  $f$  is commuting on  $\mathcal{U}$ . In [12], Samman proved that an epimorphism of a semiprime ring is strong commutativity preserving if and only if it is centralizing. Derivations as well as SCP mappings have been extensively studied by researchers in the context of operator algebras, prime rings and semiprime rings too.

In [6], Daif and Bell proved that if  $\mathcal{R}$  is a semiprime ring,  $\mathcal{U}$  is a nonzero ideal of  $\mathcal{R}$  and  $d$  is a derivation of  $\mathcal{R}$  such that  $d([x, y]) = \pm[x, y]$ , for all  $x, y \in \mathcal{U}$ , then  $\mathcal{U} \subseteq Z(\mathcal{R})$ . This theorem was obtained for generalized derivations by Quadri et al. in [11]. It was further extended by Shang in [14] and by E. Koç in [9].

In the present paper, we study the concept of multiplicative (generalized) $\alpha$ -skew derivation and obtained several results. In fact, our theorems generalize, extend and complement several results viz. (Lemma 2.3, Theorem 2.4, Corollary 2.7, Theorem 2.8 and Corollary 2.10) in [9] concerning semiprime rings to a multiplicative generalized  $\alpha$ -skew derivations.

## 2. Some preliminaries

In this section, we state some well known basic identities and results of semiprime rings which will be used extensively in the forthcoming sections.

- (i)  $[x, yz] = y[x, z] + [x, y]z$ ;
- (ii)  $[xy, z] = [x, z]y + x[y, z]$ ;
- (iii)  $xy \circ z = (x \circ z)y + x[y, z] = x(y \circ z) - [x, z]y$ ;
- (vi)  $x \circ yz = y(x \circ z) + [x, y]z = (x \circ y)z + y[z, x]$ .

**Lemma 2.1.** [13, Lemma 2.1] *Let  $\mathcal{R}$  be a semiprime ring,  $\mathcal{U}$  a nonzero two-sided ideal of  $\mathcal{R}$  and  $b \in \mathcal{R}$  such that  $byb = 0$  for all  $y \in \mathcal{U}$ , then  $b = 0$ .*

**Lemma 2.2.** [10, Corollary 2] *Let  $\mathcal{R}$  be a semiprime ring with  $d$  an  $\alpha$ -skew derivation of  $\mathcal{R}$ , where  $\alpha$  is an epimorphism of  $\mathcal{R}$ . Suppose that  $[\alpha(x), d(x)] = 0$  for all  $x \in \mathcal{R}$ , then  $d(\mathcal{R})$  is contained in a central ideal of  $\mathcal{R}$ .*

Depending on the previous result we can conclude the following result:

**Lemma 2.3.** *Let  $\mathcal{R}$  be a semiprime ring which admits a nonzero  $\alpha$ -skew derivation  $d$  of  $\mathcal{R}$ , where  $\alpha$  is an epimorphism of  $\mathcal{R}$ . Suppose that  $[\alpha(x), d(x)] = 0$  for all  $x \in \mathcal{R}$ , then  $\mathcal{R}$  contains a nonzero central ideal.*

**Lemma 2.4.** [15, Proposition 2.3] *Let  $d$  be a commuting  $\alpha$ -derivation of a semiprime ring  $\mathcal{R}$ . Then  $[x, y]d(u) = d(u)[x, y] = 0$  for all  $x, y, u \in \mathcal{R}$ . In particular,  $d$  maps  $\mathcal{R}$  into its center.*

**Lemma 2.5.** *Let  $\mathcal{R}$  be a semiprime ring,  $\mathcal{U}$  a nonzero ideal of  $\mathcal{R}$  and  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  be an automorphism. If  $\mathcal{R}$  admits a multiplicative generalized  $\alpha$ -skew derivation  $G$  associated with a nonzero map  $d$  and an automorphism  $\alpha$ , then  $G(\mathcal{U}) \neq \{0\}$ .*

**Proof:** Assume on the contrary that  $G(x) = 0$  for all  $x \in \mathcal{U}$ . Then  $0 = G(xr) = G(x)\alpha(r) + xd(r)$  for all  $x \in \mathcal{U}$ ,  $r \in \mathcal{R}$ . Hence by using hypothesis  $0 = xd(r)$  for all  $x \in \mathcal{U}$ ,  $r \in \mathcal{R}$ . left-Multiplying the last relation by  $d(r)$  and using Lemma 2.1, we obtain  $d = 0$ . But this contradicts our assumption.  $\square$

## 3. (SCP) multiplicative generalized $\alpha$ -skew derivations

Note that the following result is a generalization of Lemma 2.3 of [9]

**Theorem 3.1.** *Let  $\mathcal{R}$  be a semiprime ring and  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  be an automorphism. Suppose that  $\mathcal{R}$  admits a multiplicative generalized  $\alpha$ -skew derivation  $G$  associated with a nonzero additive map  $d$  and an automorphism  $\alpha$ , then  $d$  is an  $\alpha$ -skew derivation.*

**Proof:** Assume that  $G$  is a multiplicative generalized  $\alpha$ -skew derivation. Then

$$\begin{aligned} G((xy)z) &= G(xy)z + \alpha(xy)d(z) \\ &= G(x)yz + \alpha(x)d(y)z + \alpha(xy)d(z) \quad \text{for all } x, y, z \in \mathcal{R}. \end{aligned}$$

But

$$G(x(yz)) = G(x)yz + \alpha(x)d(yz) \quad \text{for all } x, y, z \in \mathcal{R}.$$

By combining the last two expressions, we obtain

$$\alpha(x)(d(yz) - d(y)z - \alpha(y)d(z)) = 0 \quad \text{for all } x, y, z \in \mathcal{R}. \quad (3.1)$$

By using semiprimeness of  $\mathcal{R}$ , we have  $d(yz) = d(y)z + \alpha(y)d(z)$  for all  $y, z \in \mathcal{R}$ . This completes the proof of our theorem.  $\square$

Letting  $\alpha = id_{\mathcal{R}}$  in theorem 3.1, we obtain [9, Lemma 2.3].

**Theorem 3.2.** *Let  $\mathcal{R}$  be a prime ring,  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  be an automorphism and  $\mathcal{U}$  a nonzero ideal of  $\mathcal{R}$ . Suppose that  $\mathcal{R}$  admits a multiplicative generalized  $\alpha$ -skew derivation  $G$  associated with a nonzero additive map  $d$  and an automorphism  $\alpha$ . If  $G$  is SCP on  $\mathcal{U}$ , then  $\mathcal{R}$  is commutative.*

**Proof:** Suppose that

$$[G(x), G(y)] = [x, y] \quad \text{for all } x, y \in \mathcal{U}. \quad (3.2)$$

Replacing  $x$  by  $xy$  in (3.2) and using it with the definition of  $G$ , it follows immediately that

$$G(x)[y, G(y)] + [\alpha(x), G(y)]d(y) + \alpha(x)[d(y), G(y)] = 0 \quad \text{for all } x, y \in \mathcal{U}. \quad (3.3)$$

Letting  $rx$  in place of  $x$  in (3.3), where  $r \in \mathcal{R}$  and using it again, we arrive at

$$d(r)x[y, G(y)] + [\alpha(r), G(y)]\alpha(x)d(y) = 0 \quad \text{for all } x, y \in \mathcal{U}, r \in \mathcal{R}. \quad (3.4)$$

Since  $\alpha$  is an automorphism, we can rewrite the above relation as

$$d(\alpha^{-1}(r))x[y, G(y)] + [r, G(y)]\alpha(x)d(y) = 0 \quad \text{for all } x, y \in \mathcal{U}, r \in \mathcal{R}. \quad (3.5)$$

Taking  $G(y)$  instead of  $r$  in (3.5), we find  $d(\alpha^{-1}(G(y)))\mathcal{R}[y, G(y)] = \{0\}$  for all  $y \in \mathcal{U}$  and primeness of  $\mathcal{R}$  gives

$$d(\alpha^{-1}(G(y))) = 0 \quad \text{or} \quad [y, G(y)] = 0 \quad \text{for all } y \in \mathcal{U}. \quad (3.6)$$

Suppose there exists  $y_0 \in \mathcal{U}$  such that  $d(\alpha^{-1}(G(y_0))) = 0$ . Replacing  $x$  by  $y_0$  and  $y$  by  $yG(y_0)$  in (3.2), we can easily get  $[G(y_0), G(y)]G(y_0) = [y_0, yG(y_0)]$  for all  $y \in \mathcal{U}$  which implies that  $[y_0, y]G(y_0) = [y_0, yG(y_0)]$  for all  $y \in \mathcal{U}$ . Developing this expression, we easily arrive at  $y[y_0, G(y_0)] = 0$  for all  $y \in \mathcal{U}$ . Since  $\mathcal{U} \neq \{0\}$ , primeness of  $\mathcal{R}$  gives  $[y_0, G(y_0)] = 0$ . Application of equation (3.6) implies that  $d$  is commuting. Using the same techniques as used in the proof of [15, Proposition 2.3], we can easily arrive at  $d(u)[x, y] = 0$  for all  $x, y, u \in \mathcal{U}$ . Replacing  $x$  by  $xt$ , where  $t \in \mathcal{R}$  in the last equation and using it, we get  $d(u)\mathcal{U}[y, t] = \{0\}$  for all  $y, u \in \mathcal{U}, t \in \mathcal{R}$ . By the primeness of  $\mathcal{R}$ , we obtain either  $d(\mathcal{U}) = \{0\}$  or  $\mathcal{U} \subseteq Z(\mathcal{R})$ . Since  $d \neq 0$ , it is clear that the second case gives the commutativity of  $\mathcal{R}$  and the first case cannot occur.  $\square$

If we replace commutator by the anti-commutator in Theorem 3.2, then we obtain the following result:

**Theorem 3.3.** *Let  $\mathcal{R}$  be a prime ring,  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  be an automorphism and  $\mathcal{U}$  a nonzero ideal of  $\mathcal{R}$ . Suppose that  $\mathcal{R}$  admits a multiplicative generalized  $\alpha$ -skew derivation  $G$  associated with a nonzero additive map  $d$  and an automorphism  $\alpha$ . If  $G(x) \circ G(y) = x \circ y$  for all  $x, y \in \mathcal{U}$ , then  $\mathcal{R}$  is a commutative ring of characteristic 2.*

**Proof:** Suppose that

$$G(x) \circ G(y) = x \circ y \quad \text{for all } x, y \in \mathcal{U}. \quad (3.7)$$

Replacing  $x$  by  $xy$  in (3.7) and using it with definition of  $G$ , we obtain

$$G(x)[y, G(y)] + (\alpha(x) \circ G(y))d(y) + \alpha(x)[d(y), G(y)] = 0 \quad \text{for all } x, y \in \mathcal{U}. \quad (3.8)$$

Letting  $rx$  in place of  $x$  in (3.8), where  $r \in \mathcal{R}$  and using it, we arrive at

$$d(r)x[y, G(y)] = [\alpha(r), G(y)]\alpha(x)d(y) \quad \text{for all } x, y \in \mathcal{U}, r \in \mathcal{R}. \quad (3.9)$$

Since  $\alpha$  is an automorphism, The above relation reduces to

$$d(\alpha^{-1}(r))x[y, G(y)] = [r, G(y)]\alpha(x)d(y) \quad \text{for all } x, y \in \mathcal{U}, r \in \mathcal{R}. \quad (3.10)$$

Taking  $G(y)$  instead of  $r$  in (3.10), we find  $d(\alpha^{-1}(G(y)))\mathcal{R}[y, G(y)] = \{0\}$  for all  $y \in \mathcal{U}$  and primeness of  $\mathcal{R}$  gives

$$d(\alpha^{-1}(G(y))) = 0 \quad \text{or} \quad [y, G(y)] = 0 \quad \text{for all } y \in \mathcal{U}. \quad (3.11)$$

Suppose there exists  $y_0 \in \mathcal{U}$  such that  $d(\alpha^{-1}(G(y_0))) = 0$ . Replacing  $x$  by  $y_0$  and  $y$  by  $yG(y_0)$  in (3.7), we arrive at  $(G(y_0) \circ G(y))G(y_0) = y_0 \circ yG(y_0)$  for all  $y \in \mathcal{U}$  which implies that  $(y_0 \circ y)G(y_0) = y_0 \circ yG(y_0)$  for all  $y \in \mathcal{U}$ . Which can be reformulate to  $y[y_0, G(y_0)] = 0$  for all  $y \in \mathcal{U}$ . Since  $\mathcal{U} \neq \{0\}$ , primeness of  $\mathcal{R}$  implies that  $[y_0, G(y_0)] = 0$ . From which (3.11) forces that  $[y, G(y)] = 0$  for all  $y \in \mathcal{U}$ . We use the rest of the proof of Theorem 3.2, we conclude that  $\mathcal{R}$  is commutative. In this case, returning to (3.7), we deduce that  $2G(x)G(y) = 2xy$  for all  $x, y \in \mathcal{U}$ . Replacing  $y$  by  $yz$  in the last equation and using it again, we obtain that  $2G(x)\alpha(y)d(z) = 0$  for all  $x, y, z \in \mathcal{U}$ . Putting  $\alpha^{-1}(r)y\alpha^{-1}(s)$  in place of  $y$ , where  $r, s \in \mathcal{R}$  in the latter expression and using it again, we obtain  $2G(x)\mathcal{R}\alpha(y)\mathcal{R}d(z) = \{0\}$  for all  $x, y, z \in \mathcal{U}$ . By primeness of  $\mathcal{R}$  and the fact that  $\mathcal{U} \neq \{0\}$ ,  $d \neq 0$ , we find that  $2G(x) = 0$  for all  $x \in \mathcal{U}$ , according to above, it follows easily that  $2xy = 0$  for all  $x, y \in \mathcal{U}$ . By primeness of  $\mathcal{R}$ , we get  $2x = 0$  for all  $x \in \mathcal{U}$ . Taking  $rsx$  in place of  $x$ , where  $r, s \in \mathcal{R}$  and using primeness of  $\mathcal{R}$  again with  $\mathcal{U} \neq \{0\}$ , we get  $2r = 0$  for all  $r \in \mathcal{R}$ . This completes the proof of our theorem.  $\square$

The following example shows that the condition "primeness of  $\mathcal{R}$ " in Theorems 3.2, 3.3 cannot be omitted.

**Example 1:** Let  $\mathbb{Z}$  be the set of integers and let

$$\mathcal{R} = \left\{ \left( \begin{array}{ccc} 0 & 0 & 0 \\ x & y & z \\ 0 & 0 & 0 \end{array} \right) \mid x, y, z \in \mathbb{Z} \right\}, \quad \mathcal{U} = \left\{ \left( \begin{array}{ccc} 0 & 0 & 0 \\ x & 0 & z \\ 0 & 0 & 0 \end{array} \right) \mid x, z \in \mathbb{Z} \right\},$$

Let us defined  $G, \alpha : \mathcal{R} \rightarrow \mathcal{R}$  as follow:

$$G \left( \begin{array}{ccc} 0 & 0 & 0 \\ x & y & z \\ 0 & 0 & 0 \end{array} \right) = d \left( \begin{array}{ccc} 0 & 0 & 0 \\ x & y & z \\ 0 & 0 & 0 \end{array} \right) = \left( \begin{array}{ccc} 0 & 0 & 0 \\ x & 0 & z \\ 0 & 0 & 0 \end{array} \right), \quad \text{and} \quad \alpha = id_{\mathcal{R}}.$$

It is easy to see that  $\mathcal{R}$  is not prime,  $\mathcal{U}$  is a nonzero ideal of  $\mathcal{R}$  and  $G$  is a multiplicative generalized  $\alpha$ -skew derivation of  $\mathcal{R}$  associated with a nonzero additive map  $d$  and an automorphism  $\alpha$ , such that for all  $u, v \in \mathcal{U}$ :

(i)  $[G(u), G(v)] = [u, v];$

(ii)  $G(u) \circ G(v) = u \circ v;$

but  $\mathcal{R}$  is not commutative.

#### 4. Identities with multiplicative generalized $\alpha$ -skew derivations

We remark that the following theorem is a generalization of Theorem 2.4 of [9].

**Theorem 4.1.** *Let  $\mathcal{R}$  be a semiprime ring,  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  be an automorphism and  $\mathcal{U}$  a nonzero ideal of  $\mathcal{R}$ . Suppose that  $\mathcal{R}$  admits a multiplicative generalized  $\alpha$ -skew derivation  $G$  associated with a nonzero additive map  $d$  and automorphism  $\alpha$ . If  $G([\alpha(x), \alpha(y)]) = \pm\alpha(x^m)[\alpha(x^t), \alpha(y)]\alpha(x^n)$  for all  $x, y \in \mathcal{U}$ , where  $m \geq 0, n \geq 0, t > 1$  are fixed integer and  $d$  commutes with  $\alpha$ , then  $\mathcal{R}$  contains a nonzero central ideal.*

**Proof:** Suppose that

$$G([\alpha(x), \alpha(y)]) = \pm\alpha(x^m)[\alpha(x^t), \alpha(y)]\alpha(x^n) \quad \text{for all } x, y \in \mathcal{U}. \quad (4.1)$$

Substituting  $y$  with  $yx$  in (4.1) and noting that  $[\alpha(x^t), \alpha(yx)] = [\alpha(x^t), \alpha(y)]\alpha(x)$ , we get

$$G([\alpha(x), \alpha(y)]\alpha(x)) = \pm\alpha(x^m)[\alpha(x^t), \alpha(y)]\alpha(x^{n+1}) \quad \text{for all } x, y \in \mathcal{U}. \quad (4.2)$$

This can be rewritten in the form

$$G([\alpha(x), \alpha(y)]\alpha(x)) + \alpha([\alpha(x), \alpha(y)]d(\alpha(x))) = \pm\alpha(x^m)[\alpha(x^t), \alpha(y)]\alpha(x^{n+1}). \quad (4.3)$$

By using (4.1), we obtain

$$\alpha([\alpha(x), \alpha(y)]d(\alpha(x))) = 0 \quad \text{for all } x, y \in \mathcal{U}. \quad (4.4)$$

Since  $\alpha$  is an automorphism and commutes with  $d$ , (4.4) yields that

$$[\alpha(x), \alpha(y)]d(x) = 0 \quad \text{for all } x, y \in \mathcal{U}. \quad (4.5)$$

Hence the above relation leads to  $\alpha(x)\alpha(y)d(x) = \alpha(y)\alpha(x)d(x)$  for all  $x, y \in \mathcal{U}$ . Again replacing  $y$  by  $ry$ , where  $r \in \mathcal{R}$  in the last expression and use it again, to have

$$[\alpha(x), \alpha(r)]\alpha(y)d(x) = 0 \quad \text{for all } x, y \in \mathcal{U}, r \in \mathcal{R}. \quad (4.6)$$

Substituting  $y$  by  $\alpha^{-1}(y)x$  in (4.6), it follows that

$$[\alpha(x), \alpha(r)]y\alpha(x)d(x) = 0 \quad \text{for all } x, y \in \mathcal{U}, r \in \mathcal{R}. \quad (4.7)$$

Right-Multiplying (4.6) by  $\alpha(x)$ , the above yields that

$$[\alpha(x), \alpha(r)]\alpha(y)d(x)\alpha(x) = 0 \quad \text{for all } x, y \in \mathcal{U}, r \in \mathcal{R}. \quad (4.8)$$

By subtracting (4.6) from (4.8), we get

$$[\alpha(x), \alpha(r)]\alpha(y)[\alpha(x), d(x)] = 0 \quad \text{for all } x, y \in \mathcal{U}, r \in \mathcal{R}. \quad (4.9)$$

Replacing  $r$  by  $\alpha^{-1}(d(x))$  in the latter relation, we obtain

$$[\alpha(x), d(x)]\alpha(y)[\alpha(x), d(x)] = 0 \quad \text{for all } x, y \in \mathcal{U}. \quad (4.10)$$

Therefore,  $[\alpha(x), d(x)]\alpha(\mathcal{U})[\alpha(x), d(x)] = \{0\}$  for all  $x \in \mathcal{U}$ . Using Lemma 2.1 gives  $[\alpha(x), d(x)] = 0$  for all  $x \in \mathcal{U}$ . Furthermore, by using Lemma 2.1 and Lemma 2.3, we find that  $\mathcal{R}$  contains a nonzero central ideal.  $\square$

If we put  $\alpha = id_{\mathcal{R}}$  in Theorem 4.1, we immediately get the following corollary from the above theorem:

**Corollary 4.1.** [9, Theorem 2.4] *Let  $\mathcal{R}$  be a semiprime ring and  $\mathcal{U}$  a nonzero ideal of  $\mathcal{R}$ . Suppose that  $\mathcal{R}$  admits a multiplicative generalized derivation  $G$  associated with a nonzero additive map  $d$ . If  $G([x, y]) = \pm x^m[x, y]x^n$  for all  $x, y \in \mathcal{U}$ , where  $m \geq 0, n \geq 0$  are fixed integers, then  $\mathcal{R}$  contains a nonzero central ideal.*

Recall that if we put  $m = n = 0$  in Theorem 4.1, we obtain the following corollary.

**Corollary 4.2.** *Let  $\mathcal{R}$  be a semiprime ring and  $\mathcal{U}$  a nonzero ideal of  $\mathcal{R}$ . Suppose that  $\mathcal{R}$  admits a multiplicative generalized  $\alpha$ -skew derivation  $G$  associated with a nonzero additive map  $d$  and an automorphism  $\alpha$ . If  $G([\alpha(x), \alpha(y)]) = \pm[\alpha(x), \alpha(y)]$  for all  $x, y \in \mathcal{U}$ , then  $\mathcal{R}$  contains a nonzero central ideal.*

By putting  $\alpha = id_{\mathcal{R}}$  and  $m = n = 0$  in Theorem 4.1, we can easily conclude the following corollary:

**Corollary 4.3.** [9, Corollary 2.5] *Let  $\mathcal{R}$  be a semiprime ring,  $\mathcal{U}$  a nonzero ideal of  $\mathcal{R}$ . Suppose that  $\mathcal{R}$  admits a multiplicative generalized derivation  $G$  associated with a nonzero additive map  $d$ . If  $G([x, y]) = \pm[x, y]$  for all  $x, y \in \mathcal{U}$ , then  $\mathcal{R}$  contains a nonzero central ideal.*

The following example shows that the condition "semiprimeness of  $\mathcal{R}$ " in Theorem 4.1 cannot be omitted.

**Example 2:** Let  $\mathbb{Z}$  be the set of integers and let

$$\mathcal{R} = \left\{ \begin{pmatrix} x & y & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\}, \quad \mathcal{U} = \left\{ \begin{pmatrix} 0 & y & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid y, z \in \mathbb{Z} \right\},$$

Let us defined  $G, \alpha : \mathcal{R} \rightarrow \mathcal{R}$  as follow:

$$G \begin{pmatrix} x & y & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = d \begin{pmatrix} x & y & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & y & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and } \alpha = id_{\mathcal{R}}.$$

It is clair that  $\mathcal{R}$  is not semiprime,  $\mathcal{U}$  is nonzero ideal of  $\mathcal{R}$  and  $G$  is a multiplicative generalized  $\alpha$ -skew derivation associated with a nonzero additive map  $d$  and automorphism  $\alpha$ , such that for all  $u, v \in \mathcal{U}$ , and integers  $m \geq 0, n \geq 0, t > 1$ ;

- (i)  $G([\alpha(u), \alpha(v)]) = \pm\alpha(u^m)[\alpha(u^t), \alpha(v)]\alpha(u^n)$ ;
- (ii)  $G([\alpha(u), \alpha(v)]) = \pm[\alpha(u), \alpha(v)]$ ;
- (iii)  $G([u, v]) = \pm u^m[u^t, v]u^n$ ;
- (vi)  $G([u, v]) = \pm[u, v]$ ,

but  $\mathcal{R}$  is not commutative.

It is easy to see that the following result is a generalization of [9, Corollary 2.7].

**Corollary 4.4.** *Let  $\mathcal{R}$  be a semiprime ring,  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  be an automorphism and  $\mathcal{U}$  a nonzero ideal of  $\mathcal{R}$ . Suppose that  $\mathcal{R}$  admits a generalized  $\alpha$ -skew derivation  $G$  associated with a nonzero additive map  $d$  and an automorphism  $\alpha$ . If  $G$  satisfies one of the following conditions, then  $\mathcal{R}$  contains a nonzero central ideal*

- (i)  $G(\alpha(xy)) = \alpha(xy)$  for all  $x, y \in \mathcal{U}$ ;
- (ii)  $G(\alpha(xy)) = -\alpha(xy)$  for all  $x, y \in \mathcal{U}$ ;
- (iii) For each  $x, y \in \mathcal{U}$ , either  $G(\alpha(xy)) = \alpha(xy)$  or  $G(\alpha(xy)) = -\alpha(xy)$ .

**Proof:** (i) Suppose  $G(\alpha(xy)) = \alpha(xy)$  for all  $x, y \in \mathcal{U}$ , it follows that

$$\begin{aligned} G(\alpha(xy) - \alpha(yx)) &= G(\alpha(xy)) - G(\alpha(yx)) \\ &= \alpha(xy) - \alpha(yx) \quad \text{for all } x, y \in \mathcal{U}. \end{aligned}$$

This leads to  $G([\alpha(x), \alpha(y)]) = [\alpha(x), \alpha(y)]$  for all  $x, y \in \mathcal{U}$ . Hence by using Corollary 4.2, we find that  $\mathcal{R}$  contains a nonzero central ideal.

(ii) We can find the required result if we use the same arguments as used in (i).

(iii) For each  $x \in \mathcal{U}$ , we put  $U_x^1 = \{y \in \mathcal{U} \mid G(\alpha(xy)) = \alpha(xy)\}$  and  $U_x^2 = \{y \in \mathcal{U} \mid G(\alpha(xy)) = -\alpha(xy)\}$ . So  $(\mathcal{U}, +) = \mathcal{U}_x^1 \cup \mathcal{U}_x^2$ , but a group cannot be the union of two proper subgroups. Thus we have  $\mathcal{U} = \mathcal{U}_x^1$  or  $\mathcal{U} = \mathcal{U}_x^2$ . By a similar technique in (i) or (ii), we complete the proof of our theorem.  $\square$

In the case when  $\alpha = id_{\mathcal{R}}$ , we obtain the following Corollary:

**Corollary 4.5.** [9, Corollary 2.7] *Let  $\mathcal{R}$  be a semiprime ring and  $\mathcal{U}$  a nonzero ideal of  $\mathcal{R}$ . Suppose that  $\mathcal{R}$  admits a generalized derivation  $G$  associated with a nonzero additive map  $d$ . If  $G$  satisfies any one of the following conditions, then  $\mathcal{R}$  contains a nonzero central ideal*

(i)  $G(y) = xy$  for all  $x, y \in \mathcal{U}$ ;

(ii)  $G(xy) = -xy$  for all  $x, y \in \mathcal{U}$ ;

(iii) For each  $x, y \in \mathcal{U}$ , either  $G(xy) = xy$  or  $G(xy) = -xy$ .

**Theorem 4.2.** *Let  $\mathcal{R}$  be a semiprime ring,  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  be an automorphism and  $\mathcal{U}$  a nonzero ideal of  $\mathcal{R}$ . Suppose that  $\mathcal{R}$  admits a multiplicative generalized  $\alpha$ -skew derivation  $G$  associated with a nonzero additive map  $d$  and an automorphism  $\alpha$ . If  $G(\alpha(x) \circ \alpha(y)) = \pm \alpha(x^m)(\alpha(x^t) \circ \alpha(y))\alpha(x^n)$  for all  $x, y \in \mathcal{U}$ , where  $m \geq 0, n \geq 0, t > 1$  are fixed integers and  $d$  commutes with  $\alpha$ , then  $\mathcal{R}$  contains a nonzero central ideal.*

**Proof:** Suppose that

$$G(\alpha(x) \circ \alpha(y)) = \pm \alpha(x^m)(\alpha(x^t) \circ \alpha(y))\alpha(x^n) \quad \text{for all } x, y \in \mathcal{U}. \quad (4.11)$$

Replacing  $y$  by  $yx$  in (4.11) and using the fact that  $\alpha(x^t) \circ \alpha(yx) = (\alpha(x^t) \circ \alpha(y))\alpha(x)$ , we get

$$G((\alpha(x) \circ \alpha(y))\alpha(x)) = \pm \alpha(x^m)(\alpha(x^t) \circ \alpha(y))\alpha(x^{n+1}) \quad \text{for all } x, y \in \mathcal{U}. \quad (4.12)$$

This leads to

$$G(\alpha(x) \circ \alpha(y))\alpha(x) + \alpha(\alpha(x) \circ \alpha(y))d(\alpha(x)) = \pm \alpha(x^m)(\alpha(x^t) \circ \alpha(y))\alpha(x^{n+1}). \quad (4.13)$$

Applying (4.12) in the above relation, we obtain

$$\alpha(\alpha(x) \circ \alpha(y))d(\alpha(x)) = 0 \quad \text{for all } x, y \in \mathcal{U}. \quad (4.14)$$

But since  $\alpha$  is an automorphism and commutes with  $d$ , then (4.14) yields that

$$(\alpha(x) \circ \alpha(y))d(x) = 0 \quad \text{for all } x, y \in \mathcal{U}. \quad (4.15)$$

From relation (4.15) one can conclude that  $-\alpha(x)\alpha(y)d(x) = \alpha(y)\alpha(x)d(x)$  for all  $x, y \in \mathcal{U}$ . Again substituting  $ry$  in place of  $y$ , where  $r \in \mathcal{R}$  in the last expression and using it again, it follows that

$$[\alpha(x), \alpha(r)]\alpha(y)d(x) = 0 \quad \text{for all } x, y \in \mathcal{U}, r \in \mathcal{R}. \quad (4.16)$$

Further, the rest of the proof can be obtained from the proof of Theorem 4.1, after (4.6) and this completes the proof of our theorem.  $\square$

By depending on the pervious theorem and by placing  $\alpha = id_{\mathcal{R}}$  or  $m = n = 0$ , we obtain the following results.

**Corollary 4.6.** [9, Theorem 2.8] *Let  $\mathcal{R}$  be a semiprime ring and  $\mathcal{U}$  a nonzero ideal of  $\mathcal{R}$ . Suppose that  $\mathcal{R}$  admits a multiplicative generalized derivation  $G$  associated with a nonzero additive map  $d$ . If  $G(x \circ y) = \pm x^m(x \circ y)x^n$  for all  $x, y \in \mathcal{U}$ , where  $m \geq 0, n \geq 0, t > 1$  are fixed integers, then  $\mathcal{R}$  contains a nonzero central ideal.*

**Corollary 4.7.** *Let  $\mathcal{R}$  be a semiprime ring,  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  be an automorphism and  $\mathcal{U}$  a nonzero ideal of  $\mathcal{R}$ . Suppose that  $\mathcal{R}$  admits a multiplicative generalized  $\alpha$ -skew derivation  $G$  associated with a nonzero additive map  $d$  and automorphism  $\alpha$ . If  $G(\alpha(x) \circ \alpha(y)) = \pm (\alpha(x) \circ \alpha(y))$  for all  $x, y \in \mathcal{U}$ , and  $d$  commutes with  $\alpha$ , then  $\mathcal{R}$  contains a nonzero central ideal.*

**Corollary 4.8.** [9, Corollary 2.10] *Let  $\mathcal{R}$  be a semiprime ring and  $\mathcal{U}$  a nonzero ideal of  $\mathcal{R}$ . Suppose that  $\mathcal{R}$  admits a multiplicative generalized derivation  $G$  associated with a nonzero additive map  $d$ . If  $G(x \circ y) = \pm(x \circ y)$  for all  $x, y \in \mathcal{U}$ , then  $\mathcal{R}$  contains a nonzero central ideal.*

The following example shows that the condition "semiprimeness of  $\mathcal{R}$ " is necessary in Theorem 4.2 and can not be omitted.

**Example 3:** Let  $\mathbb{Z}$  be the set of integers, and let

$$\mathcal{R} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & y & z \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\}, \quad U = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & y & 0 \end{pmatrix} \mid x, y \in \mathbb{Z} \right\},$$

Let us defined  $G, \alpha : \mathcal{R} \rightarrow \mathcal{R}$  as follow:

$$G \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & y & z \end{pmatrix} = d \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & y & z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & y & 0 \end{pmatrix}, \quad \text{and } \alpha = id_{\mathcal{R}}.$$

It is clair that  $\mathcal{R}$  is not semiprime ring,  $\mathcal{U}$  is nonzero ideal of  $\mathcal{R}$  and  $G$  is a multiplicative generalized  $\alpha$ -skew derivation associated with a nonzero additive map  $d$  and automorphism  $\alpha$ , such that for all  $u, v \in \mathcal{U}$ , and integers  $m \geq 0, n \geq 0, t > 1$ ;

(i)  $G(\alpha(u) \circ \alpha(v)) = \pm \alpha(u^m)(\alpha(u^t) \circ \alpha(v))\alpha(x^n),$

(ii)  $G(\alpha(u) \circ \alpha(v)) = \pm(\alpha(u) \circ \alpha(v)),$

(iii)  $G(u \circ v) = \pm u^m(u^t \circ v)u^n,$

(vi)  $G(u \circ v) = \pm(u \circ v),$

but  $\mathcal{R}$  is not commutative.

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