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On the Sum of the Powers of A_α Eigenvalues of Graphs and A_α-energy Like Invariant*

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ABSTRACT: For a connected simple graph G with A_α eigenvalues $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ and a real number β, let $S^\alpha_\beta(G) = \sum_{i=1}^{n} \rho_i^\beta$ be the sum of the β^{th} powers of the A_α eigenvalues of graph G. In this paper, we obtain various bounds for the graph invariant $S^\alpha_\beta(G)$ in terms of different graph parameters. As a consequence, we obtain the bounds for the quantity $IE^{A_\alpha}(G) = S^\alpha_{\frac{1}{2}}(G)$, the A_α energy-like invariant of the graph G.

Key Words: Adjacency matrix, A_α matrix, Degree regular graph, Signless Laplacian Matrix.

Contents

1. Introduction

Let $G(V, E)$ be a simple graph with n vertices and m edges and having vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. This is referred as (n, m) graph. The set of vertices adjacent to $v \in V(G)$, denoted by $N(v)$, is the *neighborhood* of v. The *degree* of v, denoted by $d_G(v)$ (we simply write d_v if it is clear from the context) is the cardinality of $N(v)$. A graph is called *regular* if each of its vertices have the same degree. The adjacency matrix $A = (a_{ij})$ of G is a $(0,1)$-square matrix of order n whose (i,j)-entry is equal to 1, if v_i is adjacent to v_j and equal to 0, otherwise. Let $D(G) = diag(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of vertex degrees $d_i = d_G(v_i), i = 1, 2, \ldots, n$ of graph G. The matrices $L(G) = A(G) - D(G)$ and $Q(G) = A(G) + D(G)$ are called the Laplacian matrix and the signless Laplacian matrix, respectively. It is well known that both $L(G)$ and $Q(G)$ are positive semidefinite matrices having real eigenvalues so that their eigenvalues can be ordered as $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$ and $q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G)$, respectively.

Nikiforov [10] proposed to study the convex combinations $A_\alpha(G)$ of $A(G)$ and $D(G)$ defined by $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, $0 \leq \alpha \leq 1$. It is obvious that $A(G) = A_0(G)$, $D(G) = A_1(G)$ and $2A_{\frac{1}{2}}(G) = D(G) + A(G) = Q(G)$. We further note that $A_\alpha - A_\gamma = (\alpha-\gamma)(D(G) - A(G)) = (\alpha-\gamma)L(G)$. As $A_\alpha(G)$ is a symmetric matrix, for $\alpha \in \left[\frac{1}{2}, 1\right]$, clearly $A_\alpha(G)$ is positive semidefinite and so the A_α eigenvalues of G can be taken as $\rho_1(G) \geq \rho_2(G) \geq \cdots \geq \rho_n(G)$. In this setup, the matrices $A(G), Q(G)$ and $D(G)$ were seen in a new light and very interesting results were deduced in [3,10,11,14,17].

For a real number β ($\beta \neq 0, 1$), Zhou [18] considered the graph invariant $s_\beta(G)$, the sum of β^{th} powers of the Laplacian eigenvalues of G. In particular, for $\beta = \frac{1}{2}$, $s_{\frac{1}{2}}(G) = \sum_{i=1}^{n} \sqrt{\mu_i} = LEL(G)$, known as Laplacian-energy-like invariant, was investigated in [9]. Similarly for $\beta = -1$, we have $ns_{-1}(G) = n \sum_{i=1}^{n} \frac{1}{\mu_i} = Kf(G)$, called the Kirchhoff index [4] of the graph G. We note that the cases $\beta = 0, 1$ are trivial as $s_0(G) = n - 1$ and $s_1(G) = Tr(L(G)) = 2m$, where Tr is the trace of the matrix. More about $LEL(G)$ and $Kf(G)$ can be found in [13] and the references therein.

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Akbari et al. [1] introduced the sum of the β^{th} powers of the signless Laplacian eigenvalues of G as $s_\beta^+(G)=\sum\limits_{i=1}^{n} q_i^\beta$. Again for $\beta=0,1$, we have $s_0^+(G)=n$ and $s_1^+(G)=2m$. Likewise for $\beta=\frac{1}{2}$, we have $s_{\frac{1}{2}}^+(G)=\sum\limits_{i=1}^{n}\sqrt{q_i}=IE(G)$, known as incidence energy of the graph G.

Motivating the definitions of $s_\beta(G)$ and $s_\beta^+(G)$, we put forward $S_\beta^\alpha(G)=\sum\limits_{i=1}^{n}\rho_i^{\beta,}$ for the sum of the β^{th} powers of the A_α eigenvalues of the graph G. If $\beta=0$, we get $S_0^\alpha(G)=n$ and for $\beta=1$, we have $S_1^\alpha(G)=Tr(A_\alpha(G))=2\alpha m$. To avoid trivialities, we assume $\beta\neq 0,1$. In particular for $\beta=\frac{1}{2}$, we obtain $S_{\frac{1}{2}}^\alpha(G)=\sum\limits_{i=1}^{n}\sqrt{\rho_i}=IE^{A_\alpha}(G)$. This quantity is similar to $LEL(G)$ and $IE(G)$ and is called A_α-energy-like invariant.

The *first general Zagreb index* [7] (also called the general zeroth-order Randić index) of a graph G is denoted by $Z_a(G)$ and is defined as $Z_a(G)=\sum\limits_{i=1}^{n} d_i^a$, where a is any real number other than 0 and 1. Also, for $a=2$, we have $Z_2(G)=\sum\limits_{i=1}^{n} d_i^2=M_1(G)$, which is known as the *first Zagreb index* [5] of G. For concepts and notations not defined here, we refer the reader to any standard text, such as [2,6,15].

The following inequalities play an important role in Sections 2 and 3.

Lemma 1.1 (Power mean inequality)**.** *If $q>p>0$, and $x_1,x_2,\dots,x_n$ are non negative real numbers, then*

$$\left(\frac{x_1^p+x_2^p+\cdots+x_n^p}{n}\right)^{\frac{1}{p}}\leq\left(\frac{x_1^q+x_2^q+\cdots+x_n^q}{n}\right)^{\frac{1}{q}},$$

with equality if and only if $x_1=x_2=\cdots=x_n$.

Lemma 1.2 (Jensen's inequality)**.** *Let f be a convex function on an interval $\mathcal{I}$ and let $x_1,x_2,\dots,x_n$ be points of $\mathcal{I}$ and let $a_1,a_2,\dots,a_n$ be real numbers satisfying $\sum\limits_{k=1}^{n}a_k=1$. Then*

$$f\left(\sum_{k=1}^{n}a_kx_k\right)\leq\sum_{k=1}^{n}a_kf(x_k)$$

with equality if and only if $x_1=x_2=\cdots=x_n$.

The following lemmas will be used in the sequel.

Lemma 1.3. *[10,14] Let G be a connected graph of order n and size m having vertex degree sequence $\{d_1,d_2,\dots,d_n\}$. Then*
(1). $\sum\limits_{i=1}^{n}\rho_i=2\alpha m.$
(2). $\sum\limits_{i=1}^{n}\rho_i^2=\alpha^2Z_2(G)+(1-\alpha)^2 2m.$
(3). $\sum\limits_{i=1}^{n}s_i^2=\alpha^2Z_2(G)+(1-\alpha)^2 2m-\frac{4\alpha^2m^2}{n}.$
(4). $\rho(G)\geq\dfrac{2m}{n}$, *equality holds if and only if G is degree regular graph.*
(5). $\rho(G)\geq\sqrt{\frac{Z_2(G)}{n}}$, *equality holds if and only if G is degree regular graph.*

Lemma 1.4. *[10] Let G be a connected graph of order n with diameter D. If A_α has exactly t distinct eigenvalues, then $D+1\leq t$.*

Lemma 1.5. *[10] Let G be a connected graph of order n with $\alpha\in\left[\frac{1}{2},1\right]$. Then A_α is a positive semidefinite matrix. If G has no isolated vertices then A_α is positive definite.*

From Lemma 1.5, for $\alpha \in \left[\frac{1}{2}, 1\right]$, we see that A_α is a positive semidefinite matrix, so that $\rho_i(G) \geq 0$ for $i = 1, 2, \ldots, n$. From now onwards, we assume that $\alpha \in \left[\frac{1}{2}, 1\right]$ unless otherwise stated.

Lemma 1.6. *[14] Let G be a connected graph of order n and size m, where $m \geq n$ and let $G^{'} = G - e$ be a connected graph obtained from G by deleting an edge. Then $\rho_i(A_\alpha(G)) \geq \rho_i(A_\alpha(G^{'}))$ holds for all $1 \leq i \leq n$.*

Lemma 1.7. *[10] The A_α eigenvalues of the complete graph K_n are $\{n-1, (\alpha n - 1)^{[n-1]}\}$, where $[j]$ means the multiplicity of λ.*

Lemma 1.8. *[14] Let G be a connected graph of order n having vertex degree sequence $[d_1, d_2, \ldots, d_n]$. Then $\rho(G) \geq \sqrt{\frac{Z_2(G)}{n}} \geq \frac{2m}{n}$, with equalities if and only if G is degree regular.*

Lemma 1.9. *[10] Let G be a graph with maximum degree $\triangle(G) = \triangle$. Then*

$$\rho(G) \geq \frac{1}{2}\left(\alpha(\triangle + 1) + \sqrt{\alpha^2(\triangle+1)^2 + 4\triangle(1 - 2\alpha)}\right).$$

If $\alpha \in [0, 1)$ and G is a connected graph, equality holds if and only if $G \cong K_{1,\triangle}$.

In Section 2, we obtain upper and lower bounds for $S^{\alpha}_{\beta}(G)$ in terms of different parameters related to graphs like maximum degree, number of edges, trace of A_α, clique number, independence number and other parameters. In Section 3, we obtain bounds for $IE^{A_\alpha}(G)$.

2. Bounds for $S^{\alpha}_{\beta}(G)$

Let G be a connected (n, m) graph with A_α eigenvalues $\rho_1(G) \geq \rho_2(G) \geq \cdots \geq \rho_n(G)$. For brevity, we use ρ_i instead of $\rho_i(G)$. For $1 \leq k \leq n-1$, let $M_k = \sum_{i=1}^{k} \rho_i$ and $m_k = \sum_{i=1}^{k} \rho_{n-i}$. If G is connected without isolated vertices and $\alpha \in [\frac{1}{2}, 1)$, then $M_k \geq \alpha \sum_{i=1}^{k} 1 = \alpha k$, for $1 \leq k \leq n-1$, which is a consequence of **Schur's theorem** stating that *the spectrum of any positive definite symmetric matrix majorizes its main diagonal.* This can be further improved as follows:

$$\frac{M_k}{k} = \frac{\sum_{i=1}^{k} \rho_i}{k} \geq \frac{\sum_{i=k+1}^{n} \rho_i}{n-k} = \frac{2\alpha m - M_k}{n-k} \tag{2.1}$$

which after simplification gives $M_k \geq \dfrac{2\alpha mk}{n}$. It can be easily verified that equality holds if and only if $G \cong K_n$. Similarly, we can show that $m_k \leq \dfrac{2\alpha mk}{n}$ with equality if and only if $G \cong K_n$.

Now, we have the following observation.

Lemma 2.1. *If G be a connected (n, m) graph having $m \geq n$ edges, then $\rho_2(G) = \rho_3(G) = \cdots = \rho_n(G)$ if and only if $G \cong K_n$.*

Proof. Suppose $\rho_2 = \rho_3 = \cdots = \rho_n$. Then $t = 2$ and from Lemma 1.4, $D = 1$.
Conversely, if $G \cong K_n$. Then $\rho_2 = \rho_3 = \cdots = \rho_n$ and the result follows. □

Lemma 2.2. *Let G be a connected (n, m) graph with $m \geq n$ edges. Then*

$$M_k \leq \frac{2\alpha mk + \left\{k(n-k)[n(\alpha^2 Z_2(G) + 2m(1-\alpha)^2) - (2\alpha m)^2]\right\}^{\frac{1}{2}}}{n} \tag{2.2}$$

with equality if and only if $G \cong K_n$.

Proof. Using *Cauchy-Schwartz's inequality* and Lemma 1.3, we have

$$\begin{aligned}(2\alpha m - M_k)^2 &= \left(\sum_{i=k+1}^{n} \rho_i\right)^2 \le (n-k)\left(\sum_{i=k+1}^{n} \rho_i^2\right) = (n-k)\left(\sum_{i=1}^{n}\rho_i^2 - \sum_{i=1}^{k}\rho_i^2\right)\\ &= (n-k)\left(\alpha^2 Z_2(G) + (1-\alpha)^2 2m - \sum_{i=1}^{k}\rho_i^2\right)\\ &\le (n-k)\left(\alpha^2 Z_2(G) + (1-\alpha)^2 2m - \frac{M_k^2}{k}\right).\end{aligned}$$

After making simplifications, we obtain

$$nM_k^2 - 4\alpha mkM_k + 4\alpha^2 m^2 - k(n-k)(\alpha^2 Z_2(G) + 2m(1-\alpha)^2) \le 0.$$

Hence, it follows that

$$M_k \le \frac{2\alpha mk + \sqrt{k(n-k)[n(\alpha^2 Z_2(G) + 2m(1-\alpha)^2) - 4\alpha^2 m^2]}}{n}$$

which is inequality (2.2).

Assume that equality holds in (2.2). Then all above inequalities must be equalities. So $\rho_1 = \rho_2 = \cdots = \rho_k$ and $\rho_{k+1} = \rho_{k+2} = \cdots = \rho_n$, that is, G has exactly two distinct A_α eigenvalues. So, by Equation (2.1), $G \cong K_n$. Similarly it is easy to check equality other way round. □

Inequality (2.2) can also be written in terms of the trace of the matrix as

$$M_k \le \frac{kTr(A_\alpha) + \sqrt{k(n-k)[n(\alpha^2 Z_2(G) + (1-\alpha)^2 Tr(A^2)) - (Tr(A_\alpha))^2]}}{n}.$$

If we proceed similar to Lemma 2.2, we have

$$m_k \ge \frac{2\alpha mk + \left\{k(n-k)[n(\alpha^2 Z_2(G) + 2m(1-\alpha)^2) - (2\alpha m)^2]\right\}^{\frac{1}{2}}}{n} \tag{2.3}$$

with equality if and only if $G \cong K_n$.

If ρ_1 and ρ_n are respectively the largest and the smallest A_α eigenvalues, for $k = 1$, then Lemmas 2.2 and 2.3 imply that

$$\rho_1 \le \frac{2\alpha m + \left\{(n-1)[n(\alpha^2 Z_2(G) + 2m(1-\alpha)^2) - (2\alpha m)^2]\right\}^{\frac{1}{2}}}{n}$$

and

$$\rho_n \ge \frac{2\alpha m + \left\{(n-1)[n(\alpha^2 Z_2(G) + 2m(1-\alpha)^2) - (2\alpha m)^2]\right\}^{\frac{1}{2}}}{n}.$$

If $G - e$ is the graph obtained from G by deleting the edge e, using Lemma (1.6) and the fact that if $a \le b$, then $a^l \le b^l$ for each $l > 0$ and $a^l \ge b^l$ for each $l < 0$, we get

$$\begin{aligned} S_\beta^\alpha(G) &\ge S_\beta^\alpha(G-e), \quad if \beta > 0\\ S_\beta^\alpha(G) &\le S_\beta^\alpha(G-e), \quad if \beta < 0. \end{aligned} \tag{2.4}$$

As G is a spanning subgraph of K_n, using (3.4) and Lemma (1.7) , we have

$$\begin{aligned} S_\beta^\alpha(G) &\le (n-1)^\beta + (n-1)(\alpha n - 1)^\beta, \qquad if\ \beta > 0\\ S_\beta^\alpha(G) &\ge (n-1)^\beta + (n-1)(\alpha n - 1)^\beta, \qquad if\ \beta < 0, \end{aligned}$$

with equality occurring in both cases if and only if $G \cong K_n$.

If G is a connected bipartite graph of order n with partite sets of cardinality a and b, then G is the spanning subgraph of the complete bipartite graph $K_{a,b}$. For $n \geq 2$ and $m \geq n$, we have

$$S^{\alpha}_{\beta}(G) \leq x_1^{\beta} + x_2^{\beta} + (b-1)(a\alpha)^{\beta} + (a-1)(\alpha b)^{\beta}, \qquad if\ \beta > 0$$
$$S^{\alpha}_{\beta}(G) \geq x_1^{\beta} + x_2^{\beta} + (b-1)(a\alpha)^{\beta} + (a-1)(\alpha b)^{\beta}, \qquad if\ \beta < 0.$$

where $x_1 = \frac{1}{2}(\alpha n + \sqrt{(\alpha n)^2 + 4ab(1-2\alpha)})$ and $x_2 = \frac{1}{2}(\alpha n - \sqrt{(\alpha n)^2 + 4ab(1-2\alpha)})$, equality occurring in both cases if and only if $G \cong K_{a,b}$.

A *complete split graph*, denoted by $CS_{n-k,k}$, is the graph consisting of an independent set on k vertices and a clique on $n-k$ vertices, such that each vertex of the clique is connected to every vertex of the independent set. It is well known that $CS_{n-k,k} = K_{n-k} \bigtriangledown \overline{K}_k$. Using this information in Proposition 37 of [10], we can find A_α spectrum of $CS_{n-k,k}$.

For $\alpha \in [0,1]$, the eigenvalues of $A_\alpha(CS_{n-k,k})$ are

$$\left\{ \frac{n-k-1+\alpha n \pm \sqrt{\theta}}{2}, (\alpha(n-k))^{[k-1]}, (\alpha n - 1)^{[n-k-1]} \right\},$$

where $\theta = k^2(4\alpha - 3) + k(2n + 2 - 2\alpha n - 4\alpha) + n(\alpha - 1)(n\alpha - \alpha + 2) + 1$.

In case G is a connected graph on $n \geq 2$ vertices having independence number k, then

$$S^{\alpha}_{\beta}(G) \leq x_1^{\beta} + x_2^{\beta} + (k-1)(\alpha n - \alpha k)^{\beta} + (n-k-1)(\alpha n - 1)^{\beta}, \qquad if\ \beta > 0$$
$$S^{\alpha}_{\beta}(G) \geq x_1^{\beta} + x_2^{\beta} + (k-1)(\alpha n - \alpha k)^{\beta} + (n-k-1)(\alpha n - 1)^{\beta}, \qquad if\ \beta < 0,$$

where

$$x_1 = \frac{1}{2}\left[n - k - 1 + \alpha n + \{k^2(4\alpha - 3) + k(2n + 2 - 2\alpha n - 4\alpha) + n(\alpha - 1)(n\alpha - \alpha + 2) + 1\}^{\frac{1}{2}}\right]$$

and

$$x_2 = \frac{1}{2}\left[n - k - 1 + \alpha n - \{k^2(4\alpha - 3) + k(2n + 2 - 2\alpha n - 4\alpha) + n(\alpha - 1)(n\alpha - \alpha + 2) + 1\}^{\frac{1}{2}}\right],$$

equality occurring in both cases if and only if $G \cong CS_{n-k,k}$.

Further, if G is a degree regular graph on $n \geq 3$ vertices, then

$$S^{\alpha}_{\beta}(C_n) \leq S^{\alpha}_{\beta}(G) \leq (n-1)^{\beta} + (n-1)(\alpha n - 1)^{\beta}, \qquad if\ \beta > 0$$
$$S^{\alpha}_{\beta}(C_n) \geq S^{\alpha}_{\beta}(G) \geq (n-1)^{\beta} + (n-1)(\alpha n - 1)^{\beta}, \qquad if\ \beta < 0,$$

equality holds on the right if and only if $G \cong K_n$ and equality occurs on the left if and only if $G \cong C_n$.

Theorem 2.3. *Let G be a connected graph of order $n \geq 2$.*
(i) If $\beta < 0$ or $\beta > 1$, then

$$S^{\alpha}_{\beta}(G) \geq \left(\frac{2m}{n}\right)^{\beta} + \frac{(2m(\alpha n - 1))^{\beta}}{n^{\beta}(n-1)^{\beta-1}},$$

with equality if and only if $G \cong K_n$.
(ii) If $0 < \beta < 1$, then

$$S^{\alpha}_{\beta}(G) \leq \left(\frac{2m}{n}\right)^{\beta} + \frac{(2m(\alpha n - 1))^{\beta}}{n^{\beta}(n-1)^{\beta-1}},$$

with equality if and only if $G \cong K_n$.

Proof. For $\beta \neq 0, 1$ and $x > 0$, we see that x^β is concave up when $\beta < 0$ or $\beta > 1$. Thus, by *Jensen's inequality*, we have

$$\left(\sum_{i=2}^{n} \frac{1}{n-1}\rho_i\right)^{\beta} \leq \sum_{i=2}^{n} \frac{1}{n-1}\rho_i^{\beta},$$

which implies that $\sum_{i=2}^{n}\rho_i^{\beta} \geq \frac{1}{(n-1)^{\beta}}\left(\sum_{i=2}^{n}\rho_i\right)^{\beta}$ with equality if and only if $\rho_2=\rho_3=\cdots=\rho_n$. Now, using this observation in the definition of $S_{\beta}^{\alpha}(G)$, we have

$$S_{\beta}^{\alpha}(G) \geq \rho_1^{\beta} + \frac{1}{(n-1)^{\beta}}\left(\sum_{i=2}^{n}\rho_i\right)^{\beta} = \rho_1^{\beta} + \frac{(2\alpha m-\rho_1)^{\beta}}{(n-1)^{\beta-1}}.$$

Let $f(x)=x^{\beta}+\frac{(2\alpha m-x)^{\beta}}{(n-1)^{\beta-1}}$. By solving $f^{'}(x)\geq 0$, we see that $f(x)$ is increasing for $x\geq \frac{2\alpha m}{n}$. By Lemma 1.3, we have $\rho_1 \geq \frac{2m}{n} \geq \frac{2\alpha m}{n}$ and thus

$$S_{\beta}^{\alpha}(G) \geq f\left(\frac{2m}{n}\right) = \left(\frac{2m}{n}\right)^{\beta} + \frac{(2m(\alpha n-1))^{\beta}}{n^{\beta}(n-1)^{\beta-1}},$$

with equality if and only if $\rho_2=\rho_3=\cdots=\rho_n$ and $\rho_1=\frac{2m}{n}$. Therefore, G has exactly two distinct A_{α} eigenvalues and by Lemma 2.1, G is the complete graph K_n, proving part (i).
(ii) Suppose that $0<\beta<1$. Then, clearly x^{β} is concave down when $x>0$ or $0<\beta<1$. So,

$$\left(\sum_{i=2}^{n}\frac{1}{n-1}\rho_i\right)^{\beta} \geq \sum_{i=2}^{n}\frac{1}{n-1}\rho_i^{\beta},$$

with equality if and only if $\rho_2=\rho_3=\cdots=\rho_n$ and $f(x)$ is decreasing for $x\geq\frac{2\alpha m}{n}$. Now proceeding as in part (i), we obtain the required result. □

Using similar arguments as in Theorem 2.3 and Lemma 1.8, we have the following.
(i) If $\beta<0$ or $\beta>1$, then

$$S_{\beta}^{\alpha}(G) \geq \left(\frac{Z_2(G)}{n}\right)^{\frac{\beta}{2}} + \frac{(2m\alpha\sqrt{n}-Z_2(G))^{\beta}}{n^{\frac{\beta}{2}}(n-1)^{\beta-1}},$$

with equality if and only if $G\cong K_n$.
(ii) If $0<\beta<1$, then

$$S_{\beta}^{\alpha}(G) \leq \left(\frac{Z_2(G)}{n}\right)^{\frac{\beta}{2}} + \frac{(2m\alpha\sqrt{n}-Z_2(G))^{\beta}}{n^{\frac{\beta}{2}}(n-1)^{\beta-1}},$$

with equality if and only if $G\cong K_n$.

Theorem 2.4. *Let G be a graph of order $n\geq 2$ and $1\leq k\leq n-1$ be a positive integer.*
(i) If $0<\beta<1$, then

$$S_{\beta}^{\alpha}(G) \leq k^{1-\beta}\left(\frac{2\alpha mk}{n}\right)^{\beta} + (n-k)^{1-\beta}\left(2\alpha m\left(\frac{n-k}{n}\right)\right)^{\beta},$$

with equality if and only if $G\cong K_1$.
(ii) If $\beta>1$, then

$$S_{\beta}^{\alpha}(G) \geq k^{1-\beta}\left(\frac{2\alpha mk}{n}\right)^{\beta} + (n-k)^{1-\beta}\left(2\alpha m\left(\frac{n-k}{n}\right)\right)^{\beta},$$

with equality if and only if $G \cong K_1$.
(iii)If $\beta < 0$, *then*

$$S_\beta^\alpha(G) \leq k^{1-\beta}\left(\frac{2\alpha mk+\sqrt{\theta}}{n}\right)^\beta + (n-k)^\beta\left(\frac{2\alpha mk-\sqrt{\theta}}{n}\right)^\beta,$$

where $\theta = k(n-k)(n(\alpha^2 Z_2(G) + 2(1-\alpha)^2 m) - (2\alpha m)^2)$.

Proof. By *power mean inequality* with $0 < \beta < 1$, we have

$$\left(\frac{\sum_{i=1}^k \rho_i^\beta}{k}\right)^{\frac{1}{\beta}} \leq \frac{M_k}{k},$$

that is, $\sum\limits_{i=1}^{k} \rho_i^\beta \leq k^{1-\beta} M_k^\beta$ with equality if and only if $\rho_1 = \rho_2 = \cdots = \rho_k$.
Similarly, $\sum\limits_{i=k+1}^{n} \rho_i^\beta \leq (n-k)^{1-\beta}(2\alpha m - M_k)^\beta$, with equality if and only if $\rho_{k+1} = \rho_{k+2} = \cdots = \rho_n$.
Thus, by the definition of $S_\beta^\alpha(G)$, we have

$$S_\beta^\alpha(G) = \sum_{i=1}^{k}\rho_i^\beta + \sum_{i=k+1}^{n}\rho_i^\beta \leq k^{1-\beta}M_k^\beta + (n-k)^{1-\beta}(2\alpha m - M_k)^\beta.$$

Consider the function

$$f(x) = k^{1-\beta}x^\beta + (n-k)^{1-\beta}(2\alpha m - x)^\beta, \quad x \geq \frac{2\alpha mk}{n}.$$

We see that

$$f^{'}(x) = \beta\left(\frac{x}{k}\right)^{\beta-1} - \left(\frac{2\alpha m - x}{n-k}\right)^{\beta-1} \leq 0$$

provided $0 < \beta < 1$ and $x \geq \dfrac{2\alpha mk}{k}$. Thus $f(x)$ is a decreasing function on $x \geq \dfrac{2\alpha mk}{k}$. Therefore, by equation (2.1), $M_k \geq \dfrac{2\alpha mk}{n}$ and we have

$$S_\beta^\alpha(G) = f(M_k) \leq f\left(\frac{2\alpha mk}{n}\right) = k^{1-\beta}\left(\frac{2\alpha mk}{n}\right)^\beta + (n-k)\left(2\alpha m - \frac{2\alpha mk}{n}\right)^\beta,$$

proving part (i).

Suppose equality holds, that is, $\rho_1 = \rho_2 = \cdots = \rho_k, \rho_{k+1} = \rho_{k+2} = \cdots = \rho_n$ and $M_k = \dfrac{2\alpha mk}{n}$. From this, we have $\rho_1 = \rho_2 = \cdots = \rho_n = \dfrac{2\alpha m}{n}$, which happens if $G \cong K_1$. Conversely, we can easily verify that equality occurs if $G \cong K_1$.
(ii) For $\beta > 1$, using power mean inequality as in part (i), we obtain

$$S_\beta^\alpha(G) \geq k^{1-\beta}M_k^\beta + (n-k)^{1-\beta}(2\alpha m - M_k)^\beta.$$

Also, $f(x) = k^{1-\beta}x^\beta + (n-k)^{1-\beta}(2\alpha m - x)^\beta$ is an increasing function on $x \geq \frac{2\alpha mk}{n}$ for $\beta > 1$. Now proceeding similarly as in (i) we can establish (ii). Also, the equality can be discussed similar to (i).
(iii) We note that $f(x) = k^{1-\beta}x^\beta + (n-k)^{1-\beta}(2\alpha m - x)^\beta$ is an increasing function on $x \geq \dfrac{2\alpha mk}{n}$ as $\beta < 0$. From Equation (2.1) and Lemma 2.2, we have

$$\frac{2\alpha mk}{n} \leq x \leq \frac{2\alpha mk + \sqrt{\theta}}{n},$$

where $\theta = k(n-k)(n(\alpha^2 Z_2(G) + 2(1-\alpha)^2 m) - (2\alpha m)^2)$. Hence

$$S_\beta^\alpha(G) \leq f\left(\frac{2\alpha mk + \sqrt{\theta}}{n}\right) = k^{1-\beta}\left(\frac{2\alpha mk + \sqrt{\theta}}{n}\right)^\beta + (n-k)^\beta \left(\frac{2\alpha mk - \sqrt{\theta}}{n}\right)^\beta .$$

□

For a connected graph G of order $n \geq 3$, let $D = \prod_{i=1}^{n} \rho_i$, where $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ are the eigenvalues of A_α.

Theorem 2.5. *Let G be a connected (n,m) graph with $n \geq 3$. If $\beta < 0$ or $\beta > 1$, then*

$$S_\beta^\alpha(G) \geq \left(\frac{2m}{n}\right)^\beta + (n-1)D^{\frac{\beta}{n-1}}\left(\frac{2m}{n}\right)^{\frac{-\beta}{n-1}},$$

with equality if and only if $G \cong K_n$.

Proof. From the definition of $S_\beta^\alpha(G)$, we have $S_\beta^\alpha(G) = \rho_i^\beta + \sum_{i=2}^{n} \rho_i^\beta$. Applying *arithmetic-geometric mean inequality* to the second term of the R.H.S, we have

$$S_\beta^\alpha(G) \geq \rho_i^\beta + (n-1)\left(\prod_{i=2}^{n} \rho_i^\beta\right)^{\dfrac{1}{n-1}} = \rho_i^\beta + (n-1)\left(\frac{D}{\rho_1}\right)^{\dfrac{\beta}{n-1}},$$

with equality if and only if $\rho_2 = \rho_3 = \cdots = \rho_n$. Consider the function

$$f(x) = x^\beta + (n-1)D^{\dfrac{\beta}{n-1}} x^{\dfrac{-\beta}{n-1}}.$$

After differentiation, we have

$$f'(x) = \beta x^{\dfrac{-\beta}{n-1}-1}\left(x^{\dfrac{n\beta}{n-1}} - D^{\dfrac{\beta}{n-1}}\right).$$

For $\beta < 0$ or $\beta > 1$, we can easily verify that $f(x)$ is an increasing function for $x \geq D^{\frac{1}{n}}$. Therefore, by Lemma 1.3 and using arithmetic-geometric inequality, we have

$$\rho_1 \geq \frac{2m}{n} \geq \frac{2\alpha m}{n} = \frac{\sum_{i=1}^{n} \rho_i}{n} \geq \left(\prod_{i=1}^{n}\right)^{\dfrac{1}{n}} = D^{\dfrac{1}{n}}.$$

So, this implies that

$$S_\beta^\alpha(G) \geq f\left(\frac{2m}{n}\right) = \left(\frac{2m}{n}\right)^\beta + (n-1)D^{\dfrac{\beta}{n-1}}\left(\frac{2m}{n}\right)^{\dfrac{-\beta}{n-1}}.$$

Equality occurs if and only if $\rho_1 = \dfrac{2m}{n}$ and $\rho_2 = \rho_3 = \cdots = \rho_n$. That is, if and only if G is degree regular with two distinct A_α eigenvalues. So, by Lemma 2.1, $G \cong K_n$. □

Theorem 2.6. *Let G be a graph of order $n \geq 2$ and $1 \leq k \leq n-1$ be a positive integer.*
(i) If $\beta < 0,\ 0 < \beta < 1$, then

$$S^{\alpha}_{\beta}(G) \geq \frac{(2\alpha m)^{2-\beta}}{(\alpha^2 Z_2(G) + (1-\alpha)^2 2\alpha m)^{1-\beta}}.$$

(ii) If $1 < \beta < 2, \beta > 2$, then

$$S^{\alpha}_{\beta}(G) \leq \frac{(2\alpha m)^{2-\beta}}{(\alpha^2 Z_2(G) + (1-\alpha)^2 2\alpha m)^{1-\beta}}.$$

Proof. Let $a_1, a_2, \ldots, a_n$ be positive real numbers and let k be a real number with $k \neq 0, \frac{1}{2}, 1$. It is clear that, $k < 0$ or $k > 0$, so that $\frac{2k-1}{k} > 0$. By Hölder's inequality, we have

$$\sum_{i=1}^{n} a_i^k = \sum_{i=1}^{n} a_i^{\frac{k}{2k-1}} a_i^{\frac{2k(k-1)}{2k-1}} \leq \left(\sum_{i=1}^{n}\left(a_i^{\frac{k}{2k-1}}\right)^{\frac{2k-1}{k}}\right)^{\frac{k}{2k-1}} \left(\sum_{i=1}^{n}\left(a_i^{\frac{2k(k-1)}{2k-1}}\right)^{\frac{2k-1}{k-1}}\right)^{\frac{k-1}{2k-1}}$$
$$= \left(\sum_{i=1}^{n} a_i\right)^{\frac{k}{2k-1}} \left(\sum_{i=1}^{n} a_i^{2t}\right)^{\frac{k-1}{2k-1}},$$

which implies that

$$\sum_{i=1}^{n} a_i \geq \frac{\left(\sum\limits_{i=1}^{n} a_i^k\right)^{\frac{2k-1}{k}}}{\left(\sum\limits_{i=1}^{n} a_i^{2k}\right)^{\frac{k-1}{k}}},$$

with equality if and only if $a_1 = a_2 \cdots = a_n$. Now, letting $a = \rho_i$ and $k = \frac{1}{\alpha}$, it implies that

$$S^{\alpha}_{\beta}(G) = \sum_{i=1}^{n} \rho_i^{\beta} \geq \frac{\left(\sum\limits_{i=1}^{n} \rho_i\right)^{2-\beta}}{\left(\sum\limits_{i=1}^{n} \rho_i^2\right)^{1-\beta}} = \frac{(2\alpha m)^{2-\beta}}{(\alpha^2 Z_2(G) + (1-\alpha)^2 2\alpha m)^{1-\beta}},$$

for each $\beta < 0$ or $0 < \beta < 1$. Similarly, if $1 < \beta < 2$ or $\beta > 2$, then $\frac{1}{2} < k < 1$ or $0 < k < \frac{1}{2}$. Taking $p = \frac{2k-1}{k}, q = \frac{2k-1}{k-1}$ and noting that $p > 0, q < 0$ if $\frac{1}{2} < k < 1$; and $p < 0, q > 0$ if $0 < k < \frac{1}{2}$. In each of these cases Hölders inequality gets reversed and the second part follows. □

3. Bounds for IE^{A_α} energy-like invariant

The graph invariant $S^{\alpha}_{\frac{1}{2}}(G) = \sum\limits_{i=1}^{n} \sqrt{\rho_i} = IE^{A_\alpha}(G)$ is called $A_\alpha-$energy-like invariant. From Theorem 2.3, we observe that

$$IE^{A_\alpha}(G) \leq \sqrt{\frac{2m}{n}} + \sqrt{\left(\frac{2m(\alpha n - 1)(n-1)}{n}\right)},$$

with equality if and only if $G \cong K_n$.

Also, we have

$$IE^{A_\alpha}(G) \leq \left(\frac{Z_2(G)}{n}\right)^{\frac{1}{4}} + \frac{\sqrt{(2m\alpha\sqrt{n} - Z_2(G))(n-1)}}{n^{\frac{1}{4}}},$$

with equality if and only if $G \cong K_n$.

From Theorem 2.4 part (i), we have

$$IE^{A_\alpha}(G) \leq (\sqrt{k}+n-k)\sqrt{\left(\frac{2\alpha m}{n}\right)},$$

with equality if and only if $G \cong K_1$.

If $G-e$ is the connected graph obtained from G by the deletion of an edge e, then

$$IE^{A_\alpha}(G) \geq IE^{A_\alpha}(G-e).$$

Further, we have

$$IE^{A_\alpha}(G) \leq \sqrt{(n-1)} + (n-1)\sqrt{\alpha n - 1},$$

with equality occurring in both cases if and only if $G \cong K_n$.

Also

$$IE^{A_\alpha}(G) \leq \sqrt{x_1} + \sqrt{x_2} + (b-1)\sqrt{(a\alpha)} + (a-1)\sqrt{(\alpha b)},$$

where $x_1 = \frac{1}{2}(\alpha(a+b) + \sqrt{\alpha^2(a+b)^2 + 4ab(1-2\alpha)})$ and $x_2 = \frac{1}{2}(\alpha(a+b) - \sqrt{\alpha^2(a+b)^2 + 4ab(1-2\alpha)})$, equality occurs if and only if $G \cong K_{a,b}$.

If G has independence number k, then

$$IE^{A_\alpha}(G) \leq \sqrt{x_1} + \sqrt{x_2} + (k-1)\sqrt{(\alpha n - \alpha k)} + (n-k-1)\sqrt{(\alpha n - 1)},$$

where $x_1 = \frac{1}{2}(n-k-1+\alpha n + \{k^2(4\alpha-3) + k(2n+2-2\alpha n - 4\alpha) + n(\alpha-1)(n\alpha - \alpha + 2) + 1\}^{\frac{1}{2}})$ and $x_2 = \frac{1}{2}(n-k-1+\alpha n - \{k^2(4\alpha-3) + k(2n+2-2\alpha n - 4\alpha) + n(\alpha-1)(n\alpha - \alpha + 2) + 1\}^{\frac{1}{2}})$, equality occurring in both cases if and only if $G \cong CS_{n-k,k}$.

If G is a degree regular graph on $n \geq 3$ vertices, then

$$\sqrt{(n-1)} + (n-1)\sqrt{(\alpha n - 1)} \geq IE^{A_\alpha}(G) \geq IE^{A_\alpha}(C_n),$$

equality occurs if and only if $G \cong C_n$.

From Lemma 1.9, if $B = \frac{1}{2}\left(\alpha(\triangle+1) + \sqrt{\alpha^2(\triangle+1)^2 + 4\triangle(1-2\alpha)}\right)$, then we can easily see that $\rho(G) \geq B \geq \dfrac{2\alpha m}{n}$. From second inequality of (2.4), it follows that

$$IE^{A_\alpha}(G) \leq \sqrt{B} + \sqrt{(n-1)(2\alpha m - B)},$$

where equality holds if and only if $G \cong K_{1,\triangle}$.

Theorem 3.1. *Let G be a connected graph (n,m) graph, where $n \geq 2$. Then*

$$IE^{A_\alpha}(G) \leq \left\{2\alpha m + (n-1)\left((\alpha n-1)(n-2) + 2(n-1)\sqrt{(n-1)(\alpha n-1)}\right)\right\}^{\frac{1}{2}}$$

where equality holds if and only if $G \cong K_n$.

Proof. Let G be a connected graph of order $n \geq 2$ having A_α eigenvalues $\rho_1, \rho_2, \dots, \rho_n$. Now

$$\left(IE^{A_\alpha}(G)\right)^2 = \left(\sum_{i=1}^{n}\sqrt{\rho_i}\right)^2 = \sum_{i=1}^{n}\rho_i + 2\sum_{i\neq j}\sqrt{\rho_i}\sqrt{\rho_j}. \tag{3.1}$$

As we know G is a connected spanning subgraph of K_n, thus by Lemma 1.6 and noting that α lies in $[\frac{1}{2}, 1)$, we have

$$\rho_1(G) \leq \rho_1(K_n) = n-1, \qquad \rho_i(G) \leq \rho_i(K_n) = \alpha n - 1, \qquad i = 2, 3, \dots, n.$$

Evaluating the second term of (3.1), we have

$$\sum_{i\neq j}\sqrt{\rho_i}\sqrt{\rho_j}=\sqrt{\rho_1}(\sqrt{\rho_2}+\sqrt{\rho_3}+\cdots+\sqrt{\rho_n})+\sqrt{\rho_2}(\sqrt{\rho_3}+\sqrt{\rho_4}+\cdots+\sqrt{\rho_n})+\cdots+\sqrt{\rho_{n-1}}\sqrt{\rho_n}$$
$$\leq (n-1)(\sqrt{(n-1)(\alpha n-1)})+(n-2)(\alpha n-1)+\cdots+(\alpha n-1)$$
$$=(n-1)\sqrt{(n-1)(\alpha n-1)}+(\alpha n-1)\left(\frac{(n-1)(n-2)}{2}\right).$$

Hence, from equation (3.1), we obtain

$$IE^{A_\alpha}(G)\leq\left\{2\alpha m+(n-1)\left((\alpha n-1)(n-2)+2(n-1)\sqrt{(n-1)(\alpha n-1)}\right)\right\}^{\frac{1}{2}}.$$

Equality occurs if and only if $\rho_1(G)=\rho_1(K_n)=n-1$ and $\rho_i(G)=\rho_i(K_n)=\alpha n-1$ for $i=2,3,\ldots,n$. That is, if and only if $G\cong K_n$. □

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