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Application of the Dual Space of Gelfand-Shilov Spaces of Beurling Type

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abstract: Using a previously obtained structure theorem of Gelfand-Shilov spaces Σ_α^β of Beurling type of ultradistributions, we prove that these ultradistributions can be represented as an initial values of solutions of the heat equation by describing the action of the Gauss-Weierstrass semigroup on the dual space $(\Sigma_\alpha^\beta)'$.

Key Words: Short-time Fourier transform, Tempered Ultradistributions, Structure Theorem, Gelfand-Shilov spaces.

Contents

1. Introduction

The theory of generalized functions devised by L. Schwartz was to provide a satisfactory framework for the Fourier transform (see [11]). They are objects which generalize functions, and they extend the concept of derivative to all integrable functions and beyond, and used to formulate generalized solutions of partial differential equations (see [5]).

Gelfand and Shilov have introduced other types of distributions called ultradistributions in the study of the uniqueness of the Cauchy problems of partial differential equations (see [3]). These spaces are invariant under Fourier transform, closed under differentiation and multiplication by polynomials, moreover, it contains Schwartz space of tempered distributions as a subspace. This makes the Gelfand Shilov spaces appropriate domains for quantum field theory. S. Pilipovic obtained structural theorems and defined the convolution for Gelfand-Shilov spaces of Roumieu and Beurling type (see [9], [10], [4]).

In this paper, we use the characterization of Gelfand-Shilov spaces of Beurling type of test functions of tempered ultradistribution in terms of their Fourier transform obtained in [2] and the structure theorem for functionals in dual space $(\Sigma_\alpha^\beta)'$ equipped with the weak topology, to study the action of Gauss-Weierstrass semigroup on the dual space $(\Sigma_\alpha^\beta)'$. Consequently, we prove that these ultradistributions can be represented as an initial values of solutions of the heat equation $u_t - Au = 0$.

Throughout the paper the symbols C^∞, C_0^∞, L^p, etc., denote the usual spaces of functions defined on $\mathbb{R}^n$, with complex values. We denote $|\cdot|$ the Euclidean norm on $\mathbb{R}^n$, while $\|\cdot\|_p$ indicates the p-norm in the space L^p, where $1 \leq p \leq \infty$. In general, we work on the Euclidean space $\mathbb{R}^n$ unless we indicate other than that as appropriate. The Fourier transform of a function f will be denoted by $\mathcal{F}(f)$ or $\widehat{f}$ and it will be defined as $\int_{\mathbb{R}^n} e^{-2\pi i x\xi} f(x)\,dx$. A Fréchet spaces are a locally convex topological vector spaces that are completely metrizable.

2. Preliminary definitions and results

In this section, we introduce basic notations and recalling some facts concerning Gelfand-Shilov spaces.

Remark 2.1. *For $\alpha > 1$, the function $|\bullet|^{1/\alpha} : [0,\infty) \to [0,\infty)$ has the following properties:*

1. $|\bullet|^{1/\alpha}$ *is increasing, continuous and concave,*
2. $|x|^{1/\alpha} \geq a + b\ln(1+x)$ *for some $a \in \mathbb{R}$ and some $b > 0$.*

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Remark 2.2. *Property 2 of Remark 2.1 implies that the function $e^{-N|x|^{1/\alpha}}$ is integrable for some positive constant N. In fact, if $N > \frac{n}{b}$ is an integer, then*

$$C_N = \int_{\mathbb{R}^n} e^{-N|x|^{1/\alpha}} dx < \infty, \text{ for all } \alpha > 1,$$

where b is the constant in Property 2 of Remark 2.1. Moreover, Property 1 in Remark 2.1 implies that $|\bullet|^{1/\alpha}$ is subadditive..

In the following theorem we state a symmetric characterization of the Gelfand-Shilov spaces Σ_α^β in terms of the Fourier transformations.

Theorem 2.3. *The space Σ_α^β can be described as a set as well as topologically by*

$$\Sigma_\alpha^\beta = \left\{ \begin{array}{c} \varphi : \mathbb{R}^n \to \mathbb{C} : \varphi \text{ is continuous and for all} \\ k = 0, 1, 2, ..., \; p_{k,0}(\varphi) < \infty, \pi_{k,0}(\varphi) < \infty. \end{array} \right\},$$

where $p_{k,0}(\varphi) = \left\| e^{k|x|^{1/\alpha}} \varphi \right\|_\infty$, $\pi_{k,0}(\varphi) = \left\| e^{k|\xi|^{1/\beta}} \widehat{\varphi} \right\|_\infty$.
The space Σ_α^β, equipped with the family of semi-norms

$$\mathcal{N} = \{p_{k,0}, \pi_{k,0} : k \in \mathbb{N}_0\},$$

is a Fréchet space.

The proof of Theorem 2.3 mimics the proof of Theorem 3.1 in [7] so we omit it. In the other hand, we can employ the above theorem to prove the following structure theorem for functionals $T \in (\Sigma_\alpha^\beta)'$.

Theorem 2.4. *If $T \in (\Sigma_\alpha^\beta)'$, then there exist two regular complex Borel measures μ_1 and μ_2 of finite total variation and $k \in \mathbb{N}_0$ such that*

$$T = e^{k|\bullet|^{1/\alpha}} \mu_1 + \mathcal{F}(e^{k|\bullet|^{1/\beta}} \mu_2) \tag{2.1}$$

in the sense of $(\Sigma_\alpha^\beta)'$.

The following Lemma will be useful in the proofs later.

Lemma 2.5. *([7]) Let $\varphi \in \Sigma_\alpha^\beta$. Then $\varphi(x+y) \in \Sigma_\alpha^\beta$ for each $y \in \mathbb{R}^n$.*

Proof: Fix $y \in \mathbb{R}^n$ and let $\varphi \in \Sigma_\alpha^\beta$. First, let us prove that

$$\left\| e^{k|x|^{1/\alpha}} \varphi(x+y) \right\|_\infty < \infty.$$

To do so, we use concavity property of $|\bullet|^{1/\alpha}$ as follows:

$$\begin{aligned} e^{k|x|^{1/\alpha}} |\varphi(x+y)| &= e^{k|x|^{1/\alpha}} e^{-2k|x+y|^{1/\alpha}} e^{2k|x+y|^{1/\alpha}} |\varphi(x+y)| \\ &\leq e^{k|x|^{1/\alpha}} e^{-k|x+y|^{1/\alpha}} \left\| e^{2k|x+y|^{1/\alpha}} \varphi(x+y) \right\|_\infty \\ &\leq Ce^{2k(\frac{|x|^{1/\alpha}}{2} - 2|x+y|^{1/\alpha})} \leq e^{2k(\frac{|x|^{1/\alpha}}{2} - \left|\frac{x+y}{2}\right|^{1/\alpha})} \\ &\leq Ce^{2k(-\frac{|y|^{1/\alpha}}{2})} \leq Ce^{-k|y|^{1/\alpha}} < \infty. \end{aligned}$$

This proves that $\left\| e^{k|x|^{1/\alpha}} \varphi(x+y) \right\|_\infty < \infty$. Similarly, $\left\| e^{k|x|^{1/\beta}} \widehat{\varphi}(x+y) \right\|_\infty < \infty$. This completes the proof of Lemma 2.5. □

Given two functionals T and S that are integrable functions, the classical definition of convolution of T and S is given by

$$\langle T * S, \phi \rangle = \langle T_x, \langle S_y, \phi(y+x) \rangle \rangle.$$

Using this definition, Definition 1.6.11, and results from Section 1.7 of [1], it is easy to show that if $T \in (\Sigma_\alpha^\beta)'$ and $\varphi \in \Sigma_\alpha^\beta$, then the functional $T * \varphi \in (\Sigma_\alpha^\beta)'$.

Theorem 2.6. *If $T \in (\Sigma_\alpha^\beta)'$ and $\varphi \in \Sigma_\alpha^\beta$, then the functional $T * \varphi \in (\Sigma_\alpha^\beta)'$ and given by $\langle T, \varphi(x - \cdot)\rangle$.*

We end this section with the definition of operator semigroup on a Banach space that we will use in application in the next section.

Definition 2.1. [8] Let $\mathfrak{B}$ be a Banach space. An operator semigroup on $\mathfrak{B}$ is a family $(T_t : t \in \mathbb{R}^+)$ of bounded linear operators on $\mathfrak{B}$ such that

i) $T_0 = I$,

ii) $T_s T_t = T_{s+t}$ for all $t, s \in \mathbb{R}^+$.

3. Applications

In this section, we study some applications on the structure theorem of Σ_α^β tempered ultradistributions stated in Theorem 2.4 by proving some results on a semi-group acting on the Fréchet space Σ_α^β and extend it to its dual $(\Sigma_\alpha^\beta)'$. We start this section by recalling a previously proved result which says that the convolution in Theorem 2.6 coincides with classical definition of convolution of two integrable functionals.

Theorem 3.1. *If $T \in (\Sigma_\alpha^\beta)'$ and $\varphi \in \Sigma_\alpha^\beta$, then the functional $T * \varphi$ defined by*

$$\langle T * \varphi, \phi\rangle = \langle T_y, (\varphi_z, \phi(x + y)\rangle$$

coincides with the functional given by integration against the function $\psi(x) = \langle T_y, \varphi(x - y)\rangle$.

Proof: Using (2.1) in Theorem 2.4, we can write for each x

$$\psi(x) = \langle T_y, \varphi(x - y)\rangle = \int_{\mathbb{R}^n} e^{k|y|^{1/\alpha}} \varphi(x + y) d\mu_1(y) + \int_{\mathbb{R}^n} e^{k|\xi|^{1/\beta}} e^{-2\pi i y.\xi} \mathcal{F}^{-1}(\varphi)(\xi) d\mu_2(\xi).$$

So,

$$\begin{aligned}
\langle T * \varphi, \phi\rangle &= \langle T_y, (\varphi_z, \phi(x + y)\rangle \\
&= \int_{\mathbb{R}^n} e^{k|y|^{1/\alpha}} (\int_{\mathbb{R}^n} \varphi(x - y)\phi(y) d\mu_1(y)) + \int_{\mathbb{R}^n} e^{k|\xi|^{1/\beta}} \mathcal{F}^{-1}(\varphi)(\xi)\widehat{\phi}(\xi) d\mu_2(\xi) \\
&= \int_{\mathbb{R}^n} e^{k|y|^{1/\alpha}} (\int_{\mathbb{R}^n} \varphi(x - y)\phi(y) d\mu_1(y)) + \int_{\mathbb{R}^n} e^{k|\xi|^{1/\beta}} \mathcal{F}(\overset{\curlyvee}{\varphi} * \phi)(\xi) d\mu_2(\xi) \\
&= \langle e^{k|\bullet|^{1/\alpha}} \mu_1(y), \langle \varphi(x - y), \phi(x)\rangle\rangle + \langle \mathcal{F}(e^{k|\bullet|^{1/\beta}} \mu_2)(y), \langle \varphi(x - y), \phi(x)\rangle\rangle \\
&= \langle e^{k|\bullet|^{1/\alpha}} \mu_1(y) + \mathcal{F}(e^{k|\bullet|^{1/\beta}} \mu_2)(y), \langle \varphi(x - y), \phi(x)\rangle\rangle \\
&= \langle T_y, \langle \varphi(x - y), \phi(x)\rangle\rangle
\end{aligned}$$

for all $\phi \in \Sigma_\alpha^\beta$. This completes the proof of Theorem 3.1.

□

Now we employ the above theorem to describe the action of the semi-group defined by the convolution kernel $t^{-n} T(\frac{x-y}{t})$, where $t > 0$ on $(\Sigma_\alpha^\beta)'$.

Theorem 3.2. *Let $T \in \Sigma_\alpha^\beta$ and $\{P_t\}_{t \geq 0}$ be a semi-group defined by the convolution kernel $t^{-n} T(\frac{x-y}{t})$, where $t > 0$. Then, the action of P_t on $(\Sigma_\alpha^\beta)'$ is given by the integration against the function*

$$\rho(x) = \langle S_y, t^{-n} T(\frac{x - y}{t})\rangle, \tag{3.1}$$

where $S_y \in (\Sigma_\alpha^\beta)'$ and y indicates on which variable the functional S acts.

Proof: Using Lemma 2.5 and Theorem 3.1, it is enough to show that $T(\frac{\cdot}{t}) \in \Sigma_\alpha^\beta$ for each $t > 0$. Note that

$$\begin{aligned} \left| e^{k|x|^{1/\alpha}} T(\frac{x}{t}) \right| &\leq \left| e^{kt\left|\frac{x}{t}\right|^{1/\alpha}} T(\frac{x}{t}) \right| \\ &\leq \left| e^{([kt]+1)\left|\frac{x}{t}\right|^{1/\alpha}} T(\frac{x}{t}) \right| \\ &= \left| e^{m\left|\frac{x}{t}\right|^{1/\alpha}} T(\frac{x}{t}) \right| \\ &\leq \left\| e^{m|\bullet|^{1/\alpha}} T \right\|_\infty \end{aligned}$$

and

$$\left| e^{k|\xi|^{1/\beta}} \widehat{T}(\frac{x}{t})(\xi) \right| = \left| e^{k|\xi|^{1/\beta}} t\widehat{T}(t\xi) \right| = C_t \left| e^{k|\xi|^{1/\beta}} \widehat{T}(t\xi) \right|.$$

Now if $t \geq 1$, then $|\xi|^{1/\beta} \leq |t\xi|^{1/\beta}$ and therefore

$$\begin{aligned} \left| e^{k|\xi|^{1/\beta}} \widehat{T}(t\xi) \right| &\leq \left| e^{k|t\xi|^{1/\beta}} \widehat{T}(t\xi) \right| \\ &\leq \left\| e^{k|\bullet|^{1/\beta}} \widehat{T} \right\|_\infty . \end{aligned}$$

For $0 < t < 1$, we have

$$\begin{aligned} \left| e^{k|\xi|^{1/\beta}} \widehat{T}(t\xi) \right| &\leq \left| e^{kN|t\xi|^{1/\beta}} \widehat{T}(t\xi) \right| \\ &\leq \left\| e^{kN|\bullet|^{1/\beta}} \widehat{T} \right\|_\infty , \end{aligned}$$

where N is an integer such that $N \geq \frac{1}{t}$. This completes the proof of Theorem 3.2. □

Theorem 3.3. *Let B be a bounded subset of Σ_α^β. Then*

$$\varphi_t(x) = \langle t^{-n} T(\frac{x-y}{t}), \varphi(x) \rangle = \int_{\mathbb{R}^n} t^{-n} T(\frac{x-y}{t}) \varphi(y) dy \to \varphi$$

in Σ_α^β as $t \to 0^+$ uniformly on B.

Proof: We note that $\varphi_t \in \Sigma_\alpha^\beta \subset (\Sigma_\alpha^\beta)'$ for each $t > 0$. If $0 < t < 1$ and $z = \frac{x-y}{t}$, then for any $\delta > 0$, we can write

$$\begin{aligned} e^{k|y|^{1/\alpha}} |\varphi_t(x) - \varphi(y)| &= \int_{\mathbb{R}^n} e^{k|y|^{1/\alpha}} T(z) |\varphi(y+tz) - \varphi(y)| \, dz \\ &\leq I_1 + I_2 + I_3, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_{|y|\leq\delta} e^{k|y|^{1/\alpha}} T(z) |\varphi(y+tz) - \varphi(y)| \, dz, \\ I_2 &= \int_{|y|\geq\delta} e^{k|y|^{1/\alpha}} T(z) |\varphi(y+tz)| \, dz, \\ I_3 &= \int_{|y|\geq\delta} e^{k|y|^{1/\alpha}} T(z) |\varphi(y)| \, dz. \end{aligned}$$

We begin estimating I_1. For each $0 < t < 1$ and $z \in \mathbb{R}^n$, there exists $C > 0$ such that

$$e^{k|y|^{1/\alpha}} |\varphi(y+tz) - \varphi(y)| \leq Ct|z| .$$

We note that property 2 in Remark 2.1 implies that there exist $N \in \mathbb{N}$ and $C > 0$ such that $|z| \le Ce^{N|z|^{1/\alpha}}$. Substituting this into I_1, we obtain the estimate

$$\begin{aligned} I_1 &\le \int_{|y|\le\delta} Ct\,|z|\,T(z)dz \\ &\le C\int_{|y|\le\delta} te^{N|z|^{1/\alpha}}T(z)dz \\ &\le C\delta t\left\|e^{N|\bullet|^{1/\alpha}}T\right\|_\infty . \end{aligned} \tag{3.2}$$

Next, we estimate I_2. Using the subadditivity of $|\bullet|^{1/\alpha}$ and $0 < t < 1$, we obtain

$$\begin{aligned} I_2 &\le \int_{|y|\ge\delta} e^{k|y|^{1/\alpha}}T(z)\,|\varphi(y+tz)|\,dz \\ &= \int_{|y|\ge\delta} e^{k|y+tz-tz|^{1/\alpha}}T(z)\,|\varphi(y+tz)|\,dz \\ &\le \int_{|y|\ge\delta} e^{k|tz|^{1/\alpha}}T(z)\left|e^{k|y+tz|^{1/\alpha}}\varphi(y+tz)\right|dz \\ &\le \left\|e^{N|\bullet|^{1/\alpha}}\varphi\right\|_\infty \int_{|z|\ge\delta} e^{k|z|^{1/\alpha}}T(z)dz \\ &\le C\int_{|z|\ge\delta} e^{k|z|^{1/\alpha}}T(z)dz. \end{aligned} \tag{3.3}$$

Finally, let us estimate I_3. We have

$$\begin{aligned} I_3 &= \int_{|z|\ge\delta} e^{k|y|^{1/\alpha}}T(z)\,|\varphi(y)|\,dz \\ &\le \left\|e^{k|\bullet|^{1/\alpha}}\varphi\right\|_\infty \int_{|z|\ge\delta} e^{k|z|^{1/\alpha}}T(z)dz. \end{aligned} \tag{3.4}$$

Therefore, if we choose δ to be sufficiently large and t sufficiently small then the estimates in (3.2), (3.3) and (3.4) imply that $\left\|e^{k|y|^{1/\alpha}}\left(\varphi_t(x)-\varphi(y)\right)\right\|_\infty$ converges to 0 as $t \to 0^+$.

Now to prove that $\left\|e^{k|\xi|^{1/\beta}}\mathcal{F}\left(\varphi_t(x)-\varphi(y)\right)(\xi)\right\|_\infty$ converges to 0 as $t \to 0^+$, we consider

$$\begin{aligned} e^{k|\xi|^{1/\beta}}\left|\mathcal{F}\left(\varphi_t(x)-\varphi(y)\right)(\xi)\right| &= e^{k|\xi|^{1/\beta}}\left|\mathcal{F}\Big(\int_{\mathbb{R}^n} t^{-n}T\Big(\frac{x-y}{t}\Big)\varphi(y)dy\Big)(\xi)-\mathcal{F}(\varphi(y))(\xi)\right| \\ &= e^{k|\xi|^{1/\beta}}\left|\mathcal{F}\Big(\int_{\mathbb{R}^n} t^{-n}T\Big(\frac{x-y}{t}\Big)\varphi(y)dy\Big)(\xi)-\mathcal{F}(\varphi(y))(\xi)\right| \\ &= e^{k|\xi|^{1/\beta}}\widehat{\varphi}(\xi)\,|\mathcal{F}(T)(t\xi)-1| \\ &\le \left\|e^{k|\bullet|^{1/\beta}}\widehat{\varphi}\right\|_\infty |\mathcal{F}(T)(t\xi)-1| \le C\,|\mathcal{F}(T)(t\xi)-1|\,. \end{aligned}$$

Now by uniform continuity of $\mathcal{F}(T)(t\xi)$, we observe that $\mathcal{F}(T)(t\xi) \to \mathcal{F}(T)(0) = \left\|t^{-n}T(\frac{\cdot}{t})\right\|_1 = 1$, which implies that $C\,|\mathcal{F}(T)(t\xi)-1| \to 0$ as $t \to 0^+$ uniformly on compact subsets of $\mathbb{R}^n$. Thus

$$\left\|e^{k|\bullet|^{1/\beta}}\mathcal{F}(\varphi_t(x)-\varphi(y))\right\|_\infty$$

converges to 0 uniformly on B. This completes the proof of Theorem 3.3. $\square$

Example 3.4. *Consider the heat kernel*

$$E(x,t) = \begin{cases} (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}, & \text{for } t > 0, \\ 0 & , \text{for } t < 0. \end{cases}$$

It is known that $\|E(\cdot,t)\|_1 = 1$ *for* $t > 0$ *(see [6]). Also, consider the Gauss-Weierstrass semigroups* $\{T_t\}_{t\geq 0}$ *defined by the integration with respect to the heat kernel*

$$T_{\sqrt{t}}(\varphi)(x) = \langle E(x-y,t), \varphi(y)\rangle = \langle t^{-n/2} T(\frac{x-y}{\sqrt{t}}), \varphi(y)\rangle.$$

This semigroup generated by the Laplacian on $\mathbb{R}^n$ *and the function* $u(x,t) = T_{\sqrt{t}}(\varphi)(x)$ *is a solution of the equation* $u_t - \triangle u = 0$ *with* $u(x,0) = \varphi(x)$ *for an appropriate* φ*. That is, the convolution*

$$u(x,t) = E * \varphi$$

is the solution to the heat equation and

$$u(x,0) = \varphi(x) = \lim_{t\to 0^+} T_{\sqrt{t}}(\varphi)(x)$$

and the convergence is uniform on bounded subsets of $\mathbb{R}^n$*. Now it is clear that* $E(x,t) \in \Sigma_\alpha^\beta$ *for all* $|\bullet|^{1/\alpha}, |\bullet|^{1/\beta}$ *satisfying the properties in Remark 2.1 since* $E(x,t)$ *is exponentially decreasing and using Theorem 3.2. Moreover, Theorem 3.2 implies that the action of* $T_{\sqrt{t}}$ *on* $L \in (\Sigma_\alpha^\beta)'$ *for all such* $|\bullet|^{1/\alpha}, |\bullet|^{1/\beta}$ *can be defined by the integral against the function* $\rho(x)$ *given in (3.1) and by using Theorem 3.1, we conclude that this is equivalent to*

$$T_{\sqrt{t}}(T) = \langle L_y, \langle t^{-n/2} T(\frac{x-y}{\sqrt{t}}), \phi(x)\rangle\rangle$$

which implies that $\lim_{t\to 0^+} T_{\sqrt{t}}(T) = T$ *in the sense of* $(\Sigma_\alpha^\beta)'$ *and this is equivalent to*

$$\langle t^{-n/2} T(\frac{x-y}{\sqrt{t}}), \phi(x)\rangle\rangle \to \varphi \text{ in } (\Sigma_\alpha^\beta)' \text{as } t \to 0^+.$$

As a result, the $(\Sigma_\alpha^\beta)'$ *tempered ultradistributions can be realized as boundary values of the equation* $u_t - Au = 0$.

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