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Existence and Uniqueness Results for a Fractional Differential Equations with Nonlocal Boundary Conditions

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ABSTRACT: In this paper, we consider a boundary value problem of differential equations of fractional order involving the nonlocal boundary condition. We establish sufficient conditions for the existence of solution of the boundary value problem with the help of Schaefer's fixed point theorem. Our uniqueness result is based on contraction mapping principle. As an application, we give two examples that illustrate our results.

Key Words: Caputo derivative, Boundary Value Problem, Fixed point theorem.

Contents

1. Introduction

Fractional differential equations are being used in various fields of science and engineering such as control system, electrochemistry, electromagnetics, viscoelasticity, physics, biophysics, porous media, blood flow phenomena, electrical circuits, biology, fitting of experimental data etc. Due to these features, models of fractional order become more practical and realistic than the models of integer-order. The existence and uniqueness of boundary value problem for fractional differential equations have attracted attention of many authors, see ([1]-[9]). For some recent development on the topic, see [10,11,12,13], and the references therein. In papers [14,15], the authors consider the stability of fractional differential equations. Besides these cited works, few more contributions [16,17], have been made to the analytical and numerical study of the solutions of fractional integral equations via fixed point theorems.

In [18], Cabrera et al. study the existence and uniqueness of positive solutions to the following nonlinear fourth-order boundary value problem which describes the deflection of an elastic beam with the left extreme fixed and the right extreme is attached to a bearing device given by a known function.

$$\begin{cases} u^4(t) = f(t, u(t), (Hu)(t)), \quad t \in [0,1], \\ u(0) = u'(0) = u''(1) = 0, u'''(1) = g(u(1)), \end{cases} \tag{1.1}$$

where $f : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$, $g : [0,\infty) \to (-\infty, 0]$ are continuous functions.

Motivated by the problem in [18], we study the existence and uniqueness of solutions for the following nonlinear fractional boundary value problem with nonlocal boundary condition

$$\begin{cases} {}^cD^\alpha z(\xi) = w(\xi, z(\xi)), \quad \xi \in [0,1], \\ z(0) = z'(0) = z''(1) = 0, z'''(1) = g(z), \end{cases} \tag{1.2}$$

where $3 < \alpha \leq 4$ and ${}^cD^\alpha$ denotes the Caputo fractional derivative of order α, $w : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $g : C([0,1], \mathbb{R}) \to \mathbb{R}$ are continuous functions.

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In boundary value problem (1.2), the authors consider fractional order derivative but in BVP (1.1) cabrera et al. consider the ordinary derivative of fourth order. So derivative in BVP (1.1) becomes a particular case of derivative in BVP (1.2) for $\alpha = 4$. Also we consider the non local boundary conditions. As remarked by Byszewski [19,20], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena.

2. Preliminaries

Let us recall some basic definitions and results of fractional calculus.

Definition 2.1. ([21]) For a continuous function $f : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s)ds, \ n = [q] + 1$$

provided that $f^{(n)}(t)$ exists, where $[q]$ denotes the integer part of the real number q.

Definition 2.2. ([21]) The Riemann-Liouville fractional integral of order q for a continuous function $f(t)$ is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds, \ q > 0,$$

provided that such integral exists.

Lemma 2.1. *([22]) Let $q > 0$, then*

$$I^q \ ^cD^q u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1},$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n-1$, where n is the smallest integer greater than or equal to q.

3. Auxiliary Result

Here we establish supporting result for the main results of the next section.

Lemma 3.1. *Let $3 < \alpha \leq 4$. Then for $h \in C([0, 1], \mathbb{R})$, the solution of*

$$\begin{cases} ^cD^\alpha z(\xi) = h(\xi), \quad \xi \in [0, 1]; \\ z(0) = z'(0) = z''(1) = 0, z'''(1) = g(z), \end{cases} \tag{3.1}$$

is given by

$$z(\xi) = \int_0^1 K(\xi, s)h(s)ds + \frac{g(z)\xi^2(\xi - 3)}{6} \tag{3.2}$$

where

$$\begin{aligned} K(\xi, s) \quad &= \quad \frac{\xi^2(3-\xi)(1-s)^{\alpha-4}}{6\Gamma(\alpha-3)} - \frac{\xi^2(1-s)^{\alpha-3}}{2\Gamma(\alpha-2)} \\ &+ \quad \begin{cases} \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } 0 \leq s \leq \xi \leq 1 \\ 0, & \text{if } 0 \leq \xi \leq s \leq 1. \end{cases} \end{aligned}$$

Proof. In view of Lemma 2.1, (3.1) is equivalent to

$$z(\xi) = \frac{1}{\Gamma(\alpha)} \int_0^\xi (\xi - s)^{\alpha-1} h(s)ds - c_0 - c_1\xi - c_2\xi^2 - c_3\xi^3 \tag{3.3}$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, 3$.

From $z(0) = 0$, it follows $c_0 = 0$. Also $z'(0) = 0 \Rightarrow c_1 = 0$.

$$z''(1) = 0 \Rightarrow \frac{1}{\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-3} h(s)ds - 2c_2 - 6c_3 = 0.$$

$$z'''(1) = g(z) \Rightarrow \frac{1}{\Gamma(\alpha-3)} \int_0^1 (1-s)^{\alpha-4} h(s)ds - 6c_3 = g(z).$$

$$\Rightarrow c_3 = \frac{1}{6\Gamma(\alpha-3)} \int_0^1 (1-s)^{\alpha-4} h(s)ds - \frac{1}{6} g(z)$$

and

$$\begin{aligned} c_2 = & \frac{1}{2\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-3} h(s)ds + \frac{g(z)}{2} \\ & - \frac{1}{2\Gamma(\alpha-3)} \int_0^1 (1-s)^{\alpha-4} h(s)ds. \end{aligned}$$

On putting the values of c_i in (3.3), we obtain the solution (3.2). □

Let $Z = C([0,1], \mathbb{R})$, then obviously $(Z, \|.\|_Z)$ is a Banach space equipped with the norm

$$\|z\|_Z = \{\sup |z(\xi)| : \xi \in [0,1]\}.$$

Let us define the operator $W : Z \to Z$ as

$$W(z)(\xi) = \int_0^1 K(\xi, s) w(s, z(s))ds + \frac{g(z)\xi^2(\xi - 3)}{6} \tag{3.4}$$

Observe that the fixed point of W are the solution of (1.2).

4. Main results

In this section, we develop two different types of results for existence and uniqueness of the proposed nonlinear fractional differential equation (1.2) by using Banach contraction principle and Scheafer's fixed point theorem.

First result is based on Banach contraction principle.

Theorem 4.1. *Assume that*

(A) $\exists\ L > 0$ *such that* $|w(\xi, z_1) - w(\xi, z_2)| \leq L|z_1 - z_2|,\ \forall \xi \in [0,1]$ *and* $\forall z_1, z_2 \in \mathbb{R}$,

(B) $\exists\ K > 0$ *such that* $|g(z_1) - g(z_2)| \leq K|z_1 - z_2|,\ \forall z_1, z_2 \in Z$ *with*

$$\Lambda = L\left[\frac{1}{2\Gamma(\alpha-2)} + \frac{1}{2\Gamma(\alpha-1)} + \frac{1}{\Gamma(\alpha+1)}\right] + \frac{K}{2} < 1$$

then (1.2) *has a unique solution defined on [0,1].*

Proof. We shall prove W is a contraction.

Let $z_1, z_2 \in Z$, then $\forall \xi \in [0,1]$,

$$\begin{aligned}
&|W(z_1)(\xi) - W(z_2)(\xi)| \\
&\le \int_0^1 |K(\xi,s)| \times |w(s,z_1(s)) - w(s,z_2(s))|ds \\
&+ \left|\frac{\xi^2(\xi-3)}{6}\right| \times |g(z_1) - g(z_2)| \\
&\le L||z_1 - z_2|| \int_0^1 |K(\xi,s)|ds + \left|\frac{\xi^2(\xi-3)}{6}\right| K||z_1 - z_2|| \\
&\le L||z_1 - z_2|| \left[\left|\frac{\xi^2(3-\xi)}{6\Gamma(\alpha-3)}\right| \int_0^1 (1-s)^{\alpha-4}ds \right.\\
&\left. + \left|\frac{\xi^2}{2\Gamma(\alpha-2)}\right| \int_0^1 (1-s)^{\alpha-3}ds\right] + \frac{L||z_1 - z_2||}{\Gamma(\alpha)} \\
&\times \int_0^\xi (\xi-s)^{\alpha-1}ds + \left|\frac{\xi^2(\xi-3)}{6}\right| K||z_1 - z_2|| \\
&= L||z_1 - z_2|| \left[\left|\frac{\xi^2(3-\xi)}{6\Gamma(\alpha-2)}\right| + \left|\frac{\xi^2}{2\Gamma(\alpha-1)}\right|\right] \\
&+ \frac{L||z_1 - z_2||\xi^\alpha}{\Gamma(\alpha+1)} + \left|\frac{\xi^2(\xi-3)}{6}\right| K||z_1 - z_2|| \\
&\le ||z_1 - z_2|| \left[L\left\{\frac{1}{2\Gamma(\alpha-2)} + \frac{1}{2\Gamma(\alpha-1)} + \frac{1}{\Gamma(\alpha+1)}\right\} + \frac{K}{2}\right]
\end{aligned}$$

Thus

$$||W(z_1) - W(z_2)|| \le \Lambda ||z_1 - z_2||.$$

As $\Lambda < 1$, therefore, W is a contraction. Hence, by Banach Fixed Point Theorem, W must have a unique fixed point i.e. (1.2) has a unique solution. □

Next result is based on Schaefer's fixed point theorem.

Theorem 4.2. *Assume that the following hypotheses hold*

(C) $\exists$ *a constant* $\mu > 0$ *such that* $|w(\xi,z)| \le \mu$ *for each* $\xi \in [0,1]$ *and* $z \in \mathbb{R}$.

(D) $\exists$ *a constant* $\lambda > 0$ *such that* $|g(z)| \le \lambda$ *for all* $z \in Z$

Then (1.2) *has at least one solution defined on* $[0,1]$.

Proof. We shall prove this result by Schaefer's fixed point theorem

Step I. W is Continuous.

Let $\{z_n\}$ be a sequence in Z such that $z_n \to z$. Then for each $\xi \in [0,1]$

$$\begin{aligned}
&|W(z_n)(\xi) - W(z)(\xi)| \\
&\le \int_0^1 |K(\xi,s)| \times |w(s,z_n(s)) - w(s,z(s))|ds \\
&+ \left|\frac{\xi^2(\xi-3)}{6}\right| \times |g(z_n) - g(z)| \\
&\le \int_0^1 |K(\xi,s)| \times \sup_{s\in[0,1]} |w(s,z_n(s)) - w(s,z(s))|ds \\
&+ \left|\frac{\xi^2(\xi-3)}{6}\right| \times |g(z_n) - g(z)|
\end{aligned}$$

Since w and g are continuous functions, therefore W is also continuous.

Step II. Bounded sets of Z are mapped into bounded sets of Z under the mapping W.
Now, for $z \in B_\epsilon$ and $\forall \xi \in [0,1]$,

$$\begin{aligned}
&|W(z)(\xi)| \leq \int_0^1 |K(\xi,s)| \times |w(s,z(s))| ds + \left|\frac{\xi^2(\xi-3)}{6}\right| \times |g(z)| \\
&\leq \mu \int_0^1 |K(\xi,s)| ds + \left|\frac{\xi^2(\xi-3)}{6}\right| \lambda \\
&\leq \mu \left|\frac{\xi^2(3-\xi)}{6\Gamma(\alpha-3)}\right| \int_0^1 (1-s)^{\alpha-4} ds + \mu \left|\frac{\xi^2}{2\Gamma(\alpha-2)}\right| \int_0^1 (1-s)^{\alpha-3} ds \\
&+ \mu \int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \lambda \left|\frac{\xi^2(\xi-3)}{6}\right|
\end{aligned}$$

Thus,

$$||W(z)|| \leq \mu \left[\frac{1}{2\Gamma(\alpha-2)} + \frac{1}{2\Gamma(\alpha-1)} + \frac{1}{\Gamma(\alpha+1)}\right] + \frac{\lambda}{2}$$

i.e.

$$\|W(z)\| < \infty$$

Step III. $W(B_\epsilon)$ is equi-continuous
Let $z \in B_\epsilon$ and $\xi_1, \xi_2 \in [0,1]$ with $\xi_1 < \xi_2$, then

$$\begin{aligned}
&|W(z)(\xi_2) - W(z)(\xi_1)| \\
&\leq \mu \int_0^1 |K(\xi_2,s) - K(\xi_1,s)| ds + \lambda \left|\frac{\xi_2^2(\xi_2-3)}{6} - \frac{\xi_1^2(\xi_1-3)}{6}\right| \\
&\leq \mu \frac{|3(\xi_2^2-\xi_1^2) - (\xi_2^3-\xi_1^3)|}{6\Gamma(\alpha-3)} \int_0^1 (1-s)^{\alpha-4} ds \\
&+ \mu \frac{|\xi_2^2-\xi_1^2|}{2\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-3} ds \\
&+ \frac{\mu}{\Gamma(\alpha)} \left[\int_0^{\xi_1} [(\xi_2-s)^{\alpha-1} - (\xi_1-s)^{\alpha-1}] ds + \int_{\xi_1}^{\xi_2} (\xi_2-s)^{\alpha-1} ds\right] \\
&+ \lambda \frac{|(\xi_2^3-\xi_1^3) - 3(\xi_2^2-\xi_1^2)|}{6} \\
&\leq \mu \frac{|3(\xi_2^2-\xi_1^2) - (\xi_2^3-\xi_1^3)|}{6\Gamma(\alpha-2)} + \frac{\mu|\xi_2^2-\xi_1^2|}{2\Gamma(\alpha-1)} \\
&+ \frac{\mu}{\Gamma(\alpha+1)} [(\xi_2-\xi_1)^\alpha + (\xi_2^\alpha - \xi_1^\alpha)] + \frac{\mu(\xi_2-\xi_1)^\alpha}{\Gamma(\alpha+1)} \\
&+ \lambda \frac{|(\xi_2^3-\xi_1^3) - 3(\xi_2^2-\xi_1^2)|}{6}
\end{aligned}$$

Now the right-hand side approaches to zero when ξ_1 approaches to ξ_2.

Combining Steps I to III and by the consequence of Arzelá-Ascoli theorem, W is completely continuous operator.

Step IV. Let $\Theta = \{z \in Z : z = \theta W(z) \text{ for some } 0 < \theta < 1\}$.
We will show that the set Θ is bounded.
Let $z \in \Theta \Rightarrow z(\xi) = \theta W(z)(\xi)$ for some $0 < \theta < 1$.

Now

$$\begin{aligned}
|z(\xi)| = |\theta W(z)(\xi)| &\leq \int_0^1 |K(\xi,s)| \times |w(s,z(s))| ds \\
&+ \left|\frac{\xi^2(\xi-3)}{6}\right| \times |g(z)| \\
&\leq \mu \int_0^1 |K(\xi,s)| ds + \left|\frac{\xi^2(\xi-3)}{6}\right| \lambda \\
&\leq \mu \left|\frac{\xi^2(3-\xi)}{6\Gamma(\alpha-3)}\right| \int_0^1 (1-s)^{\alpha-4} ds \\
&+ \mu \left|\frac{\xi^2}{2\Gamma(\alpha-2)}\right| \int_0^1 (1-s)^{\alpha-3} ds \\
&+ \mu \int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \lambda \left|\frac{\xi^2(\xi-3)}{6}\right|
\end{aligned}$$

Thus,

$$||z|| \leq \mu \left[\frac{1}{2\Gamma(\alpha-2)} + \frac{1}{2\Gamma(\alpha-1)} + \frac{1}{\Gamma(\alpha+1)}\right] + \frac{\lambda}{2}$$

i.e.

$$||z|| < \infty$$

which implies that Θ is a bounded set. By Schaefer's fixed point theorem, W must have at least one fixed point which is a solution of (1.2). □

5. Examples

In this section, we discuss some examples to illustrate our results.

Example 5.1.

$$\begin{cases} {}^cD^{\frac{16}{5}} z(\xi) = \frac{1}{(\xi+8)^2} \frac{|z(\xi)|}{1+|z(\xi)|}, \ \xi \in [0,1] \\ z(0) = z'(0) = z''(1) = 0, z'''(1) = \Sigma_{i=1}^n c_i z(\xi_i) \end{cases} \tag{5.1}$$

where $0 < \xi_1 < \xi_2 < ... < \xi_n < 1, \ c_i, \ i = 1, 2, ..., n$ *are given positive constants with* $\Sigma_{i=1}^n c_i < 1.9648$

Here $\alpha = \frac{16}{5}, \ w(\xi, z) = \frac{1}{(\xi+8)^2} \frac{|z|}{1+|z|}$ *and* $g(z) = \Sigma_{i=1}^n c_i z(\xi_i)$. *As* $|w(\xi,z_1) - w(\xi,z_2)| \leq \frac{1}{64}|z_1 - z_2|$. *Also* $|g(z_1) - g(z_2)| \leq \Sigma_{i=1}^n c_i |z_1 - z_2|$, *therefore (A) and (B) are satisfied with* $L = \frac{1}{64}$ *and* $K = \Sigma_{i=1}^n c_i < 1.9648$. *Further,*

$$\begin{aligned}
\Lambda &= L\left[\frac{1}{2\Gamma(\alpha-2)} + \frac{1}{2\Gamma(\alpha-1)} + \frac{1}{\Gamma(\alpha+1)}\right] + \frac{K}{2} \\
&= \frac{1}{64}\left[\frac{1}{2\Gamma(\frac{6}{5})} + \frac{1}{2\Gamma(\frac{11}{5})} + \frac{1}{\Gamma(\frac{21}{5})}\right] + \frac{K}{2} \\
&= 0.0176 + \frac{K}{2} < 1.
\end{aligned}$$

Thus, by Theorem 4.1, we deduce that (5.1) *has a unique solution.*

Example 5.2.

$$\begin{cases} {}^cD^{\frac{17}{5}} z(\xi) = \frac{e^{-2\xi}}{7+\sin z(\xi)}, \ \xi \in [0,1], \\ z(0) = z'(0) = z''(1) = 0, z'''(1) = \cos z \end{cases} \tag{5.2}$$

Here $w(\xi, z) = \frac{e^{-2\xi}}{7+\sin z}$ *and* $g(z) = \cos z$. *Clearly* $|w(\xi,z)| \leq \frac{1}{6}$ *and* $|g(z)| \leq 1$, *i.e. (C) and (D) are satisfied with* $\mu = \frac{1}{6}$ *and* $\lambda = 1$.

Therefore, it follows from Theorem 4.2, there exists at least one solution of (5.2).

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