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An Interesting Integral Involving Product of Two Generalized Hypergeometric Function

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ABSTRACT: In this research note, an interesting integral involving hypergeometric function has been evaluated in terms of gamma function. It is further used to evaluate an integral involving product of two generalized hypergeometric functions. A few very interesting special cases have also been given.

Key Words: Hypergeometric Function, Generalized Hypergeometric Function, Watson Theorem, MacRobert's Definite Integral.

Contents

1. Introduction and Results Required

In order to justify our doing, we must quote Sylvester [11]: " *It seems to be expected of every pilgrim up the slopes of the mathematical parnassus, that he will at some point or other of his journey sit down and invent a definite integral or two towards the increase of the common stock.*"

It is well-known that the Gaussian hypergeometric function ${}_2F_1$ and the confluent hypergeometric function ${}_1F_1$ form the core of special functions. A large number of elementary functions can be expressed in terms of ${}_2F_1$ as its limiting or special cases.

Th natural generalization of the above mentioned functions is the generalized hypergeometric function with p numerator parameters and q denominator parameters denoted by ${}_pF_q$ and is defined in the following manner [3].

$$ {}_pF_q\left[\begin{matrix} a_1, & \ldots, & a_p \\ & & & ; & x \\ b_1, & \ldots, & b_q \end{matrix}\right] = {}_PF_q\left[a_1,\ldots,a_p;b_1,\ldots,b_q;x\right] = \sum_0^{\infty}\frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{i=1}^{q}(b_i)_n}\frac{x^n}{n!} \tag{1.1} $$

where $(a)_n$ is the well known Pochhammer symbol (or the raised or the shifted factorial, since $(1)_n = n!$) defined for $a\in\mathbb{C}$ by

$$ (a)_n := \begin{cases} a(a+1)...(a+n-1) & ; n\in\mathbb{N} \\ 1 & ; n=0 \end{cases} \tag{1.2} $$

or in terms of Gamma function

$$ (a)_n := \frac{\Gamma(a+n)}{\Gamma(a)}\ (a\in\mathbb{C}\backslash\mathbb{Z}_0^-) \tag{1.3} $$

For a complete detail about ${}_pF_q$ (including its convergence conditions and properties, we refer to the standard texts [1,3,7,8]).

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In the theory of hypergeometric and generalized functions classical summations theorems play an important role. For interesting results on the products of generalized hypergeometric functions by employing the classical summation theorems, we refer a paper by Bailey [2].

Here, we would like to mention classical Watson's summation theorem [1,2] viz.

$$3F_2\left[\begin{matrix} a, & b, & c \\ \frac{a+b+1}{2}, & 2c & \end{matrix} ;1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(c-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{b}{2}+\frac{1}{2}\right)} \tag{1.4}$$

provided $Re(2c-a-b) > -1$.
From (1.4), we shall first evaluate the following integral involving hypergeometric function which is also believed to be new.

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} e^{2ic\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}\ {}_2F_1\left[\begin{matrix} a, & b \\ \frac{a+b+1}{2} & \end{matrix} ;\ e^{i\theta}\cos\theta\right] d\theta \\ &= \frac{e^{\frac{i\pi c}{2}}\Gamma\left(\frac{1}{2}\right)\Gamma(c)\Gamma(c)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)}{\Gamma(2c)\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(c-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{b}{2}+\frac{1}{2}\right)} \end{aligned} \tag{1.5}$$

provided $Re(c) > 0$ and $Re(2c-a-b) > -1$.

Proof. Denoting the left-hand side of (1.5) by I, we have

$$I = \int_0^{\frac{\pi}{2}} e^{2ic\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}\ {}_2F_1\left[\begin{matrix} a, & b \\ \frac{a+b+1}{2} & \end{matrix} ;\ e^{i\theta}\cos\theta\right] d\theta.$$

Now, expressing ${}_2F_1$ as a series, change the order of integration, which is easily seen to be justified due to uniform convergence of the series involved in the process, we have

$$I = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(\frac{1}{2}(a+b+1)_n)n!} \int_0^{\frac{\pi}{2}} e^{i(2c+n)\theta}(\sin\theta)^{c-1}(\cos\theta)^{c+n-1}d\theta.$$

Evaluating the integral with the help of the following well known integral due to MacRobert [6]

$$\int_0^{\frac{\pi}{2}} e^{i(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1} = e^{i\frac{\pi\alpha}{2}}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

provided $Re(\alpha) > 0$ and $Re(\beta) > 0$, and using the relation (1.3), we have

$$I = e^{i\frac{\pi c}{2}}\frac{\Gamma(c)\Gamma(c)}{\Gamma(2c)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{\frac{1}{2}(a+b+1)_n(2c)_n n!}.$$

Summing up the series, we have

$$I = e^{i\frac{\pi c}{2}}\frac{\Gamma(c)\Gamma(c)}{\Gamma(2c)}\ {}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{a+b+1}{2}, & 2c & \end{matrix} ;1\right].$$

Finally, using the result (1.4), we easily arrive at the right-hand side of (1.5). □

It is not out of place to mention here that, recently good progress has been done in generalizing and extending the classical Watson's summation theorem (1.4). For this, we refer to the readers, interesting research paper by Rakha and Rathie [9] and Kim, et al. [5].

Remark For the finite integral involving hypergeometric function, see a paper by Brychkov [4].

In this research note, an interesting integral involving product of two generalized hypergeometric function has been evaluated in terms of gamma function. The integral is evaluated with the help of the known integral (1.5). A few very interesting special cases have also been given.

2. main integral formula

In this section, we shall evaluate the integral involving product of two generalized hypergeometric function given in the following theorem.

Theorem 2.1. *For $Re(c) > 0$ and $Re(2c - a - b) > -1$, the following result holds true.*

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} e^{2ci\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}\ {}_2F_1\left[\begin{matrix} a, & b \\ \frac{a+b+1}{2} & \end{matrix}\ ;\ e^{i\theta}\cos\theta\right] \\
&\times\ {}_2F_2\left[\begin{matrix} c-\frac{a}{2}+\frac{1}{2}, & c-\frac{b}{2}+\frac{1}{2} \\ c, & c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2} \end{matrix}\ ;\ -4ie^{2i\theta}\sin\theta\cos\theta\right] d\theta \\
&= \frac{e^{\frac{i\pi c}{2}+1}\sqrt{\pi}\Gamma(c)\Gamma(c)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)}{\Gamma(2c)\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(c-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{b}{2}+\frac{1}{2}\right)}
\end{aligned} \tag{2.1}$$

Proof. In order to evaluate the integral (2.1), we proceed as follows. Denoting the left-hand side of (2.1) by I, we have

$$\begin{aligned}
I = &\int_0^{\frac{\pi}{2}} e^{2ci\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}\ {}_2F_1\left[\begin{matrix} a, & b \\ \frac{a+b+1}{2} & \end{matrix}\ ;\ e^{i\theta}\cos\theta\right] \\
&\times\ {}_2F_2\left[\begin{matrix} c-\frac{a}{2}+\frac{1}{2}, & c-\frac{b}{2}+\frac{1}{2} \\ c, & c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2} \end{matrix}\ ;\ -4ie^{2i\theta}\sin\theta\cos\theta\right] d\theta
\end{aligned}$$

Express ${}_2F_2$ as a series, interchanging the order of integration and summation, which is easily seen to be justified due to the uniform convergence of the series involved in the process, we have

$$\begin{aligned}
I = &\sum_{n=0}^{\infty} \frac{\left(c-\frac{a}{2}+\frac{1}{2}\right)_n \left(c-\frac{b}{2}+\frac{1}{2}\right)_n 2^{2n}(-i)^n}{(c)_n\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right) n!} \\
&\times \int_0^{\frac{\pi}{2}} e^{i(2c+2n)\theta}(\sin\theta)^{c+n-1}(\cos\theta)^{c+n-1}\ {}_2F_1\left[\begin{matrix} a, & b \\ \frac{a+b+1}{2} & \end{matrix}\ ;\ e^{i\theta}\cos\theta\right] d\theta
\end{aligned}$$

Evaluating the integral with the help of the result (1.5) and making use of the result (1.2), we have after some simplification.

$$I = \frac{e^{i\frac{\pi c}{2}}\Gamma(c)\Gamma(c)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)}{\Gamma(2c)\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(c-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{b}{2}+\frac{1}{2}\right)}\sum_{n=0}^{\infty}\frac{1}{n!}$$

Finally, noting that $\sum_{n=0}^{\infty}\frac{1}{n!} = e$, we easily arrive at the right-hand side of (2.1). This completes the proof of (2.1). □

3. special cases

In this section, we shall mention a few very interesting special cases of our main integral (2.1) in the form of following corollaries.

Corollary 3.1. *In (2.1), if we let $b = -2n$ and replace a by $a + 2n$, where n is zero or a positive integer. Then we get the following result:*

$$\int_0^{\frac{\pi}{2}} e^{2ci\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}\ {}_2F_1\left[\begin{matrix} -2n, & a+2n \\ \frac{a+1}{2} & \end{matrix} ;\ e^{i\theta}\cos\theta\right]$$
$$\times\ {}_2F_2\left[\begin{matrix} c+n+\frac{1}{2}, & c-\frac{a}{2}+\frac{1}{2}-n \\ c, & c-\frac{a}{2}+\frac{1}{2} \end{matrix} ;\ -4ie^{2i\theta}\sin\theta\cos\theta\right] d\theta$$
$$= \frac{e^{\frac{i\pi c}{2}+1}\Gamma(c)\Gamma(c)}{\Gamma(2c)} \frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}+\frac{a}{2}-c\right)_n}{\left(c+\frac{1}{2}\right)_n\left(\frac{a+1}{2}\right)_n} \tag{3.1}$$

Corollary 3.2. *In (2.1), if we let $b = -2n-1$ and replace a by $a+2n+1$, where n is zero or a positive integer. Then we get the following result:*

$$\int_0^{\frac{\pi}{2}} e^{2ci\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}\ {}_2F_1\left[\begin{matrix} -2n-1, & a+2n+1 \\ \frac{a+1}{2} & \end{matrix} ;\ e^{i\theta}\cos\theta\right]$$
$$\times\ {}_2F_2\left[\begin{matrix} c+n+1, & c-\frac{a}{2}-n \\ c, & c-\frac{a}{2}+\frac{1}{2} \end{matrix} ;\ -4ie^{2i\theta}\sin\theta\cos\theta\right] d\theta = 0 \tag{3.2}$$

Corollary 3.3. *In (2.1), if we let $a = b = \frac{1}{2}$ and making use of the known result [7, p.473, equ.(75)]*

$${}_2F_1\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ 1 & \end{matrix} ;\ x\right] = \frac{2}{\pi}K(\sqrt{x}) \tag{3.3}$$

where $K(k)$ is the well-known Elliptic function of the first kind defined by

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1-k^2 sin^2 t}} \tag{3.4}$$

then, we get the following result:

$$\int_0^{\frac{\pi}{2}} e^{2ci\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}\ K(\sqrt{e^{i\theta}\cos\theta})$$
$$\times\ {}_2F_2\left[\begin{matrix} c+\frac{1}{4}, & c+\frac{1}{4} \\ c, & c \end{matrix} ;\ -4ie^{2i\theta}\sin\theta\cos\theta\right] d\theta$$
$$= e^{\frac{i\pi c}{2}+1}\pi^{\frac{3}{2}} \frac{\Gamma^3(c)\Gamma\left(c+\frac{1}{2}\right)}{\Gamma(2c)\Gamma^2\left(\frac{3}{4}\right)\Gamma^2\left(c+\frac{1}{4}\right)} \tag{3.5}$$

provided $Re(c) > 0$.

Corollary 3.4. *In (2.1), if we let $a = b = 1$ and making use of the known result [7, p.476, equ.(147)]*

$${}_2F_1\left[\begin{matrix} 1, & 1 \\ \frac{3}{2} & \end{matrix} ;\ x\right] = \frac{sin^{-1}(\sqrt{x})}{\sqrt{x(1-x)}} \tag{3.6}$$

then, we get the following result:

$$\int_0^{\frac{\pi}{2}} e^{2ci\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}\sqrt{e^{i\theta}\cos\theta(1-e^{i\theta}\cos\theta)}$$
$$\times\ sin^{-1}(\sqrt{e^{i\theta}\cos\theta})\ {}_1F_1\left[\begin{matrix} c \\ c-\frac{1}{2}\end{matrix} ;\ -4ie^{2i\theta}\sin\theta\cos\theta\right]d\theta$$
$$=\frac{\pi e^{\frac{i\pi c}{2}+1}}{2}\frac{\Gamma\left(c-\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)}{\Gamma(2c)} \tag{3.7}$$

provided $Re(c) > \frac{1}{2}$.

Corollary 3.5. *In (2.1), if we set $b = -a$ and making use of the known result [4, p.459, equ.(83)]*

$${}_2F_1\left[\begin{matrix} a, & -a \\ \frac{1}{2} & \end{matrix} ;\ x\right] = cos(2asin^{-1}\sqrt{x}) \tag{3.8}$$

then, we get the following result:

$$\int_0^{\frac{\pi}{2}} e^{2ci\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}\ cos(2asin^{-1}\sqrt{e^{i\cos\theta}})$$
$$\times\ {}_2F_2\left[\begin{matrix} c-\frac{a}{2}+\frac{1}{2}, & c+\frac{a}{2}+\frac{1}{2} \\ c, & c+\frac{1}{2}\end{matrix} ;\ -4ie^{2i\theta}\sin\theta\cos\theta\right]d\theta$$
$$=\frac{\pi e^{\frac{i\pi c}{2}+1}\Gamma^2(c)\Gamma^2\left(c+\frac{1}{2}\right)}{\Gamma(2c)\Gamma\left(\frac{1}{2}-\frac{a}{2}\right)\Gamma\left(\frac{1}{2}+\frac{a}{2}\right)\Gamma\left(c+\frac{1}{2}-\frac{a}{2}\right)\Gamma\left(c+\frac{1}{2}+\frac{a}{2}\right)} \tag{3.9}$$

provided $Re(c) > 0$.
Similarly, other result c obtained.

Conflict of Interests

The authors declare that they have no any conflict of interests.

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All authors contributed equally in this paper. They read and approved the final manuscript.

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