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Graded δ-Primary Structures

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ABSTRACT: Let G be a group with identity e, R a G-graded commutative ring with unity 1 and M a G-graded R-module. In this article, we unify the concepts of graded prime ideals and graded primary ideals into a new concept, namely, graded δ-primary ideals. Also, we unify the concepts of graded 2-absorbing ideals and graded 2-absorbing primary ideals into a new concept, namely, graded 2-absorbing δ-primary ideals. A number of results about graded prime, graded primary, graded 2-absorbing and graded 2-absorbing primary ideals are extended into these new structures. Finally, we extend the concept of graded δ-primary ideals into graded δ-primary submodules. A number of results about graded prime, graded primary submodules are extended into this new structure.

Key Words: Graded prime ideals, Graded primary ideals, Graded δ-primary ideals, Graded 2-absorbing ideals, Graded 2-absorbing primary ideals, Graded 2-absorbing δ-primary ideals, Expansion of graded submodules, Graded multiplication modules, Graded δ-primary submodules.

Contents

1. Introduction

Graded prime ideals and graded primary ideals are two of the most important structures in graded commutative algebra. Although they are different from each other in many aspects, they share quite a number of similar properties as well. However, these two structures have been treated rather differently, and all of their properties were proved separately. It is therefore natural to examine whether it is possible to have a unified approach for studying these two structures. In this article, we introduce and study the notion of graded δ-primary ideals where δ is a mapping that assigns to each graded ideal I a graded ideal $\delta(I)$ in the same graded ring. Such graded δ-primary ideals unify graded prime ideals and graded primary ideals under one frame. This approach clearly reveals how similar the two structures are and how they are related to each other.

The notion of graded 2-absorbing ideals which is a generalization of graded prime ideals has been introduced and studied in [8]. On the other hand, the notion of graded 2-absorbing primary ideals which is a generalization of graded primary ideals has been introduced and studied in [9]. In this article, we introduce and investigate the notion of graded 2-absorbing δ-primary ideals. Such graded 2-absorbing δ-primary ideals unify graded 2-absorbing ideals and graded 2-absorbing primary ideals under one frame. This approach clearly reveals how similar the two structures are and how they are related to each other. A number of results concerns graded prime, graded primary, graded 2-absorbing and graded 2-absorbing primary ideals are extended into these new structures. Finally, we extend the concept of graded δ-primary ideals into graded δ-primary submodules. A number of results about graded prime, graded primary submodules are extended into this new structure.

The article is organized as follows. In Section Two, we introduce and study graded δ-primary ideals. In Section Three, we introduce and investigate graded 2-absorbing δ-primary ideals. In Section Four, we introduce and examine graded δ-primary submodules.

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Throughout this article, G will be a group with identity e and R a commutative ring with a nonzero unity 1. We call R, G-graded if $R = \bigoplus_{g\in G} R_g$ with $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$ where R_g is an additive subgroup of R for all $g \in G$. The elements of R_g are called homogeneous of degree g. Consider $supp(R, G) = \{g \in G : R_g \neq 0\}$. If $x \in R$, then x can be written as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Moreover, R_e is a subring of R and $1 \in R_e$ and $h(R) = \bigcup_{g\in G} R_g$. An element $r \in R$ is said to be homogeneous zero divisor if $r \in h(R)$ and there exists a nonzero $s \in h(R)$ such that $rs = 0$.

Let I be an ideal of a graded ring R. Then I is said to be graded ideal if $I = \bigoplus_{g\in G}(I \cap R_g)$, i.e., for $x \in I$, $x = \sum_{g\in G} x_g$ where $x_g \in I$ for all $g \in G$. The following example shows that an ideal of a graded ring need not be graded.

Example 1.1. *Consider $R = \mathbb{Z}[i] = \{a + ib : a, b \in \mathbb{Z}, i^2 = -1\}$ (where $\mathbb{Z}$ is the ring of integers) and $G = \mathbb{Z}_2$ (the group of integers modulo 2). Then R is G-graded by $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Now, $I = \langle 1 + i\rangle$ is an ideal of R with $1 + i \in I$. If I is graded, then $1 \in I$, so $1 = a(1 + i)$ for some $a \in R$, i.e., $1 = (x + iy)(1 + i)$ for some $x, y \in \mathbb{Z}$. Thus $1 = x - y$ and $0 = x + y$, i.e., $2x = 1$ and hence $x = \frac{1}{2}$ a contradiction. So, I is not graded ideal of R.*

If R is a G-graded ring and I is an ideal of R, the R/I is G-graded by $(R/I)_g = (R_g + I)/I$ for all $g \in G$. This means that $x + I \in h(R/I)$ if and only if $x \in h(R)$. Assume that M is an R-module. Then M is said to be G-graded if $M = \bigoplus_{g\in G} M_g$ with $R_gM_h \subseteq M_{gh}$ for all $g, h \in G$ where M_g is an additive subgroup of M for all $g \in G$. The elements of M_g are called homogeneous of degree g. Also, we consider $supp(M, G) = \{g \in G : M_g \neq 0\}$. It is clear that M_g is an R_e-submodule of M for all $g \in G$. Moreover, $h(M) = \bigcup_{g\in G} M_g$.

Let N be an R-submodule of a graded R-module M. Then N is said to be graded R-submodule if $N = \bigoplus_{g\in G}(N \cap M_g)$, i.e., for $x \in N$, $x = \sum_{g\in G} x_g$ where $x_g \in N$ for all $g \in G$. The following example shows that an R-submodule of a graded R-module need not be graded.

Example 1.2. *Consider $R = \mathbb{Z}$, $M = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then R is G-graded by $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Also, M is G-graded by $M_0 = \mathbb{Z}$ and $M_1 = i\mathbb{Z}$. Similarly as in Example 1.1, $N = \langle 1+i\rangle$ is an R-submodule of M which is not graded.*

Lemma 1.3. *([14], Lemma 2.1) Let M be a graded R-module.*

1. *If I and J are graded ideals of R, then $I + J$ and $I \bigcap J$ are graded ideals of R.*
2. *If N and K are graded R-submodules of M, then $N + K$ and $N \bigcap K$ are graded R-submodules of M.*
3. *If N is a graded R-submodule of M, $r \in h(R)$, $x \in h(M)$ and I is a graded ideal of R, then Rx, IN and rN are graded R-submodules of M.*

If N is a graded R-submodule of M, then $(N :_R M) = \{r \in R : rM \subseteq N\}$ is a graded ideal of R, (see [10] and [11]).

2. Graded δ-Primary Ideals

In this section, we introduce the concept of graded δ-primary ideals of a graded commutative ring. Several results have been given.

Recall that a proper graded ideal P of a graded ring R is said to be graded prime if whenever $x, y \in h(R)$ such that $xy \in P$, then either $x \in P$ or $y \in P$. Also, P is said to be a graded primary ideal of R if whenever $x, y \in h(R)$ such that $xy \in P$, then either $x \in P$ or $y \in Grad(P)$, where $Grad(P)$ is the graded radical of P, and it is defined as follows:

$Grad(P) = \left\{ x = \sum_{g \in G} x_g \in R : \text{ for all } g \in G, \text{ there exists } n_g \in \mathbb{N} \text{ such that } x_g^{n_g} \in P \right\}$. And note that $Grad(P)$ is always a graded ideal (see [20]). Every graded prime ideal is graded primary, and the next example shows that the converse is not true in general.

Example 2.1. *Let $R = \mathbb{Z}[i]$ and $G = \mathbb{Z}_4$. Then R is G-graded by $R_0 = \mathbb{Z}$, $R_2 = i\mathbb{Z}$ and $R_1 = R_3 = \{0\}$. Let p be a prime number. Then $P = \langle p^2 \rangle$ is a graded primary ideal of R which is not graded prime.*

Definition 2.2. *A graded ideal expansion of a graded ring R is a function δ which assigns to every graded ideal I of a graded ring R to another graded ideal $\delta(I)$ of R such that the following hold:*

1. *$I \subseteq \delta(I)$, and*
2. *$I \subseteq J$ implies that $\delta(I) \subseteq \delta(J)$.*

Example 2.3.
1. *The identity function δ_0, where $\delta_0(I) = I$ for every graded ideal I of a graded ring R, is a graded ideal expansion.*
2. *The function δ_1 that assigns the largest graded ideal R to every graded ideal is a graded ideal expansion.*
3. *Define $\delta_{max}(I)$ to be the intersection of all graded maximal ideals containing I when I is a proper graded ideal of a graded ring R, and $\delta_{max}(R) = R$. Then δ_{max} is a graded ideal expansion.*

Example 2.4. *Let R be a graded ring. For each graded ideal I of R define $\delta_{\frac{1}{2}}(I) = Grad(I)$. Then $\delta_{\frac{1}{2}}$ is a graded ideal expansion.*

Definition 2.5. *Let R be a graded ring and δ a graded ideal expansion of R. Then a proper graded ideal P of R is said to be graded δ-primary if whenever $x, y \in h(R)$ such that $xy \in P$, then either $x \in P$ or $y \in \delta(P)$.*

Example 2.6.
1. *A graded ideal P of a graded ring R is graded δ_0-primary if and only if it is graded prime.*
2. *Every proper graded ideal P of a graded ring R is graded δ_1-primary.*
3. *A graded ideal P of a graded ring R is graded $\delta_{\frac{1}{2}}$-primary if and only if it is graded primary.*

Remark 2.1.
1. *Given two graded ideal expansions γ and ζ, define $\delta(I) = \gamma(I) \bigcap \zeta(I)$. Then δ is also a graded ideal expansion. In general, the intersection of any collection of graded ideal expansions is a graded ideal expansion.*
2. *If δ and ζ are two graded ideal expansions such that $\delta(I) \subseteq \zeta(I)$ for all graded ideal I, then every graded δ-primary ideal is graded ζ-primary. So, in particular, every graded prime ideal is graded δ-primary for all graded ideal expansion δ. On the other hand, the graded ideal P in the Example 2.1 is graded $\delta_{\frac{1}{2}}$-primary which is not graded prime.*

The next proposition shows that the set of all graded δ-primary ideals is a direct complete poset with respect to the inclusion order.

Proposition 2.7. *Let $\{P_i : i \in \Delta\}$ be a directed collection of graded δ-primary ideals of a graded ring R, then $P = \bigcup_{i \in \Delta} P_i$ is a graded δ-primary ideal of R.*

Proof. Clearly, P is a graded ideal of R. Let $x, y \in h(R)$ such that $xy \in P$. Then $xy \in P_k$ for some $k \in \Delta$. Suppose that $x \notin P$. Then $x \notin P_k$ and then $y \in \delta(P_k)$ since P_k is graded δ-primary. So, $y \in \delta(P)$ since $\delta(P_k) \subseteq \delta(P)$. Hence, P is a graded δ-primary ideal of R. $\square$

Lemma 2.8. *Let P be a proper graded ideal of a G-graded ring R. Then P is graded δ-primary if and only if for every two graded ideals K_1 and K_2 of R such that $K_1K_2 \subseteq P$, either $K_1 \subseteq P$ or $K_2 \subseteq \delta(P)$.*

Proof. Suppose that P is a graded δ-primary ideal of R. Let K_1 and K_2 be two graded ideals of R such that $K_1K_2 \subseteq P$ and $K_1 \nsubseteq P$. Then there exists $x \in K_1$ such that $x \notin P$, and then there exists $g \in G$ such that $x_g \in K_1 - P$. Assume that $y \in K_2$. Then $y_h \in K_2$ for all $h \in G$ since K_2 is graded, and then for all $h \in G$, $x_gy_h \in K_1K_2 \subseteq P$. Since P is graded δ-primary and $x_g \notin P$, $y_h \in \delta(P)$ for all $h \in G$, which implies that $y \in \delta(P)$. Hence, $K_2 \subseteq P$. Conversely, let $r, s \in h(R)$ such that $rs \in P$. Then $I = \langle r\rangle$ and $J = \langle s\rangle$ are graded ideals of R such that $IJ \subseteq P$. Assume that $r \notin P$. Then $I \nsubseteq P$, and then by assumption $J \subseteq \delta(P)$, which implies that $s \in \delta(P)$. Hence, P is a graded δ-primary ideal of R. □

Lemma 2.9. *Let R be a G-graded ring, P a graded ideal of R and $T \subseteq h(R)$. Then $(P : T) = \{r \in R : rT \subseteq P\}$ is a graded ideal of R.*

Proof. Clearly, $(P : T)$ is an ideal of R. Let $r = \sum\limits_{g\in G} r_g \in (P : T)$. It is enough to prove that $r_gT \subseteq P$ for all $g \in G$. So, without loss of generality, we may assume that $r = \sum\limits_{i=1}^{m} r_{g_i}$ where $r_{g_i} \neq 0$ for $i = 1, 2, ..., m$ and $r_g = 0$ for all $g \notin \{g_1, ..., g_m\}$. As $r \in (P : T)$, $\sum\limits_{i=1}^{m} r_{g_i}T \subseteq P$. Let $x \in T$. Then since $T \subseteq h(R)$, for all $i = 1, 2, ..., m$, $r_{g_i}x \in h(R)$ such that $\sum\limits_{i=1}^{m} r_{g_i}x = rx \in P$. Since P is graded, $r_{g_i}x \in P$ for all $i = 1, 2, ..., m$ and then $r_{g_i}T \subseteq P$ for all $i = 1, 2, ..., m$. □

Lemma 2.10. *If P and K are two graded ideals of a G-graded ring R, then $(P : K)$ is a graded ideal of R.*

Proof. Let $r \in (P : K)$ and write $r = \sum\limits_{g\in G} r_g$. Let T be a homogeneous generating set for K. Note that $r \in (P : K)$ if and only if $rx \in P$ for all $x \in T$. Thus, $\left(\sum\limits_{g\in G} r_g\right) x \in P$. But $\left(\sum\limits_{g\in G} r_g\right) x = \sum\limits_{g\in G} r_gx$ is a homogeneous decomposition of rx. As P is graded, $r_gx \in P$ for all $g \in G$, $x \in T$. Hence, $r_g \in (P : K)$ for all $g \in G$. □

Theorem 2.11. *Let R be a graded ring.*

1. *If P is a graded δ-primary ideal of R and K is a graded ideal of R such that $K \nsubseteq \delta(P)$, then $(P : K) = P$.*
2. *For any graded δ-primary ideal P of R and any $T \subseteq h(R)$, $(P : T)$ is a graded δ-primary ideal of R.*

Proof.
1. Clearly, $P \subseteq (P : K)$. Also, it is obvious that $K(P : K) \subseteq P$. But $(P : K)$ is a graded ideal of R by Lemma 2.10 with $K \nsubseteq \delta(P)$, so by Lemma 2.8, $(P : K) \subseteq P$. Therefore, $(P : K) = P$.
2. By Lemma 2.9, $(P : T)$ is a graded ideal of R. Let $x, y \in h(R)$ such that $xy \in (P : T)$. Suppose that $x \notin (P : T)$. Then there exists $t \in T$ such that $xt \notin P$. But $xty = xyt \in P$. So, $y \in \delta(P) \subseteq \delta((P : T))$. Hence, $(P : T)$ is a graded δ-primary ideal of R.

□

Theorem 2.12. *Let R be a graded ring and δ a graded ideal expansion of R such that $\delta(K) \subseteq \delta_{\frac{1}{2}}(K)$ for all graded ideal K of R. If P is a graded δ-primary ideal of R, then $\delta(P) = \delta_{\frac{1}{2}}(P)$.*

Proof. Let $x \in \delta_{\frac{1}{2}}(P)$. Then $x_g \in \delta_{\frac{1}{2}}(P)$ for all $g \in G$ since $\delta_{\frac{1}{2}}(P)$ is graded. Assume that $g \in G$. Then there exists k_g which is the least positive integer k_g such that $x_g^{k_g} \in P$. If $k_g = 1$, then $x_g \in P$ for all $g \in G$, and then $x \in P \subseteq \delta(P)$. If $k_g > 1$, then $x_g^{k_g-1}x \in P$. But $x_g^{k_g-1} \notin P$, so $x_g \in \delta(P)$ for all $g \in G$. Hence, $\delta_{\frac{1}{2}}(P) \subseteq \delta(P)$. Therefore, $\delta(P) = \delta_{\frac{1}{2}}(P)$. □

For two G-graded rings R and S, a ring homomorphism $f : R \to S$ is said to be graded homomorphism if $f(R_g) \subseteq S_g$ for all $g \in G$, (see [17]).

Definition 2.13. *Let R be a graded ring and δ a graded ideal expansion of R.*

1. *δ is said to be graded intersection preserving if $\delta(K_1 \bigcap K_2) = \delta(K_1) \bigcap \delta(K_2)$ for all graded ideals K_1 and K_2 of R.*
2. *δ is said to be graded global if for every graded ring homomorphism*
   $f : R \to S$, $\delta(f^{-1}(K)) = f^{-1}(\delta(K))$ for all graded ideal K of S.

Remark 2.2. *The graded ideal expansions δ_0, δ_1 and $\delta_{\frac{1}{2}}$ are clearly both graded intersection preserving and graded global.*

Theorem 2.14. *δ_{max} is graded intersection preserving.*

Proof. Let K_1 and K_2 be two graded ideals of R. Suppose that
$X = \{K : K$ is a graded maximal ideal of R containing $K_1 \bigcap K_2\}$ and
$Y = \{K : K$ is a graded maximal ideal of R containing K_1 or $K_2\}$. Then $\bigcap X = \delta_{max}(K_1 \bigcap K_2)$ and $\bigcap Y = \delta_{max}(K_1) \bigcap \delta_{max}(K_2)$. Clearly, $Y \subseteq X$. Assume that $K \in X$. Then K is a graded maximal ideal of R such that $K_1 \bigcap K_2 \subseteq K$. So, K is a graded prime ideal of R, and hence either $K_1 \subseteq K$ or $K_2 \subseteq K$, which implies that $K \in Y$. Therefore, $X = Y$ and $\delta_{max}(K_1 \bigcap K_2) = \delta_{max}(K_1) \bigcap \delta_{max}(K_2)$. □

The next example shows that δ_{max} is not graded global.

Example 2.15. *Let $R = \mathbb{C}[X]$ and $G = \mathbb{Z}$. Then R is G-graded by $R_0 = \mathbb{C}$, $R_j = \mathbb{C}X^j$ for $j > 0$ and $R_j = \{0\}$ for $j < 0$. Suppose that $P = \langle X \rangle$. Then P is a graded prime ideal of R and the graded Jacobson radical of R is $\{0\}$ ($GJac(R) = \{0\}$). Consider R_P (the localization of R at P) and the graded natural homomorphism $f : R \to R_P$. Then $K = \left\{\frac{p}{r} : p, r \in h(R) \text{ such that } p \in P, r \notin P\right\}$ is the unique graded maximal ideal of R_P. Consider the graded ideal $I = \{0\}$ of R_P. Then $\delta_{max}(f^{-1}(I)) = \delta_{max}(\{0\}) = GJac(R) = \{0\} \neq P = f^{-1}(K) = f^{-1}(\delta_{max}(I))$.*

Theorem 2.16. *Let δ be a graded ideal expansion of R and $K_1, ..., K_n$ graded δ-primary ideals of R such that $\delta(K_i) = P$ for all $1 \leq i \leq n$. If δ is graded intersection preserving, then $K = \bigcap_{i=1}^{n} K_i$ is a graded δ-primary ideal of R.*

Proof. Let $a, b \in h(R)$ such that $ab \in K$. Suppose that $a \notin K$. Then $a \notin K_j$ for some j. But $ab \in K \subseteq K_j$, so $b \in \delta(K_j) = P = \bigcap_{i=1}^{n} \delta(K_i) = \delta\left(\bigcap_{i=1}^{n} K_i\right) = \delta(K)$. Hence, K is a graded δ-primary ideal of R. □

Theorem 2.17. *Let K be a proper graded ideal of R and δ is graded global. Then K is a graded δ-primary ideal of R if and only if $r + K \in \delta(\{0_{R/K}\})$ for all homogeneous zero divisor $r + K$ in R/K.*

Proof. Suppose that K is a graded δ-primary ideal of R. Let $r + K \in h(R/K)$ be a zero divisor. Then $r \in h(R)$ and there exists $s \in h(R) - K$ such that $rs \in K$, and then $r \in \delta(K)$, that is $r + K \in \delta(K)/K$. Consider the graded natural homomorphism $f : R \to R/K$. Since δ is graded global, $\delta(K) = \delta(f^{-1}(\{0_{R/K}\})) = f^{-1}(\delta(\{0_{R/K}\}))$, and since f is surjective, $\delta(K)/K = f(\delta(K)) = \delta(\{0_{R/K}\})$. Hence, $r+K \in \delta(\{0_{R/K}\})$. Conversely, let $x, y \in h(R)$ such that $xy \in K$ and $x \notin K$. Then $x+K, y+K \in h(R/K)$

such that $(x+K)(y+K) = xy+K = 0+K$, that is $y+K$ is a homogeneous zero divisor in R/K, and then by assumption, $y+K \in \delta(\{0_{R/K}\}) = \delta(K)/K$. So, there exists $s \in \delta(K)$ such that $y-s \in K \subseteq \delta(K)$, which implies that $y = (y-s)+s \in \delta(K)$. Hence, K is a graded δ-primary ideal of R. □

Lemma 2.18. *Let δ be a graded ideal expansion of R and S, $f : R \to S$ a graded ring homomorphism and K a graded δ-primary ideal of S. If δ is graded global, then $f^{-1}(K)$ is a graded δ-primary ideal of R.*

Proof. Let $x, y \in h(R)$ such that $xy \in f^{-1}(K)$. Suppose that $x \notin f^{-1}(K)$. Then $f(x) \notin K$. But $f(x)f(y) = f(xy) \in K$, so $f(y) \in \delta(K)$, and then $y \in f^{-1}(\delta(K)) = \delta(f^{-1}(K))$. Hence, $f^{-1}(K)$ is a graded δ-primary ideal of R. □

Theorem 2.19. *Let δ be a graded global ideal expansion of R and S, $f : R \to S$ a graded surjective ring homomorphism and K a graded ideal of R containing $Ker(f)$. Then K is a graded δ-primary ideal of R if and only if $f(K)$ is a graded δ-primary ideal of S.*

Proof. Since $Ker(f) \subseteq K$, it can easily proved that $f^{-1}(f(K)) = K$. Suppose that K is a graded δ-primary ideal of R. Let $a, b \in h(S)$ such that $ab \in f(K)$. Then there exist $x, y \in h(R)$ such that $f(x) = a$ and $f(y) = b$, and then $f(xy) = ab \in f(K)$, which implies that $xy \in K$. So, either $x \in K$ or $y \in \delta(K)$, and then either $a = f(x) \in f(K)$ or $b = f(y) \in f(\delta(K)) = f(\delta(f^{-1}(f(K))) = f(f^{-1}(\delta(f(K))) = \delta(f(K))$. Hence, $f(K)$ is a graded δ-primary ideal of S. The converse holds by Lemma 2.18. □

Remark 2.3. *From the proof of Theorem 2.19, one can see that if δ is a graded global ideal expansion, $f : R \to S$ a graded surjective ring homomorphism and K a graded ideal of R containing $Ker(f)$, then $f(\delta(K)) = \delta(f(K))$. On the other hand, if f is a ring isomorphism, then $f(\delta(K)) = \delta(f(K))$ for all graded ideal K of R.*

Corollary 2.20. *Let K_1 and K_2 be two graded ideals of R such that $K_1 \subseteq K_2$, and suppose that δ is graded global. Then K_2/K_1 is a graded δ-primary ideal of R/K_1 if and only if K_2 is a graded δ-primary ideal of R.*

3. Graded 2-Absorbing δ-Primary Ideals

In this section, we introduce the concept of graded 2-absorbing δ-primary ideals of a graded commutative ring. Interesting results have been discussed.

Recall that a proper graded ideal P of R is said to be graded 2-absorbing if whenever $x, y, z \in h(R)$ such that $xyz \in P$, then $xy \in P$ or $xz \in P$ or $yz \in P$. Several results on graded 2-absorbing ideals have been discussed in [8]. A proper graded ideal P of R is said to be graded 2-absorbing primary if whenever $x, y, z \in h(R)$ such $xyz \in P$, then $xy \in P$ or $xz \in Grad(P)$ or $yz \in Grad(P)$. See [9] for the properties of graded 2-absorbing primary ideals.

Definition 3.1. *Let R be a graded ring and δ a graded ideal expansion of R. Then a proper graded ideal P of R is said to be graded 2-absorbing δ-primary if whenever $x, y, z \in h(R)$ such that $xyz \in P$, then $xy \in P$ or $xz \in \delta(P)$ or $yz \in \delta(P)$.*

Remark 3.1.

1. *If P is a graded ideal of a graded ring R, then P is graded 2-absorbing δ_0-primary if and only if P is graded 2-absorbing.*
2. *If P is a graded ideal of a graded ring R, then P is graded 2-absorbing $\delta_{\frac{1}{2}}$-primary if and only if P is graded 2-absorbing primary.*
3. *If δ and ζ are two graded ideal expansions with $\delta(K) \subseteq \zeta(K)$ for all graded ideal K, then every graded 2-absorbing δ-primary ideal is graded 2-absorbing ζ-primary. In particular, every graded 2-absorbing ideal is graded 2-absorbing δ-primary for every δ.*

Theorem 3.2. *Let R be a graded ring and K a graded 2-absorbing δ-primary ideal of R. If $Grad(\delta(K)) = \delta(Grad(K))$, then $Grad(K)$ is a graded 2-absorbing δ-primary ideal of R.*

Proof. Let $x, y, z \in h(R)$ such that $xyz \in Grad(K)$. Then there exists a positive integer n such that $(xyz)^n \in K$. Suppose that $xz \notin Grad(K)$ and $yz \notin \delta(Grad(K)) = Grad(\delta(K))$. Then $x^n z^n \notin K$ and $y^n z^n \notin \delta(K)$ which implies that $x^n y^n \in \delta(K)$, and hence $xy \in Grad(\delta(K)) = \delta(Grad(K))$. Thus, $Grad(K)$ is a graded 2-absorbing δ-primary ideal of R. $\square$

Remark 3.2. *The converse of Theorem 3.2 is not true in general (see [22], Example 2.3). Here, in ([22], Example 2.3), $Grad(I)$ is a graded 2-absorbing δ_0-primary but I is not a graded 2-absorbing δ_0-primary.*

Proposition 3.3. *Let $\{P_i : i \in \Delta\}$ be a directed collection of graded 2-absorbing δ-primary ideals of R, then $P = \bigcup_{i\in\Delta} P_i$ is a graded 2-absorbing δ-primary ideal of R.*

Proof. Let $x, y, z \in h(R)$ such that $xyz \in P$. Then $xyz \in P_i$ for some $i \in \Delta$. Suppose that $xy \notin P$ and $xz \notin \delta(P)$. Then $xz \notin P$ which implies $xy \notin P_i$ and $xz \notin P_i$. If $xz \in \delta(P_i)$, then $xz \in \delta(P)$. Assume that $xz \notin \delta(P_i)$. Then $yz \in \delta(P_i) \subseteq \delta(P)$. Hence, P is a graded 2-absorbing δ-primary ideal of R. $\square$

Graded semiprime ideals have been introduced in [15], a proper graded ideal P of R is said to be graded semiprime if whenever $a, b \in h(R)$ such that $a^k b \in P$ for some positive integer k, then $ab \in P$. Graded semiprime ideals have been also studied in [2]. We introduce the following.

Theorem 3.4. *Let R be a graded ring and δ a graded ideal expansion of R such that $\delta(P)$ is a graded semiprime ideal of R for every graded ideal P of R. Then $\delta_{\frac{1}{2}}(P) \subseteq \delta(P)$ for every graded 2-absorbing δ-primary ideal P of R.*

Proof. Let P be a graded 2-absorbing δ-primary ideal of R and $x \in \delta_{\frac{1}{2}}(P)$. Then there exists a positive integer n which is the least integer n such that $x^n \in P$. If $n = 1$, then $x \in P \subseteq \delta(P)$. Suppose that $n > 1$. Then $x^{n-2}xx \in P$. Since P is graded 2-absorbing δ-primary and $x^{n-1} \notin P$, $x^{n-2}x \in \delta(P)$ or $x^2 \in \delta(P)$, and since $\delta(P)$ is graded semiprime, $x \in \delta(P)$. Hence, $\delta_{\frac{1}{2}}(P) \subseteq \delta(P)$. $\square$

Theorem 3.5. *Let δ be a graded ideal expansion of R and $K_1, ..., K_n$ graded 2-absorbing δ-primary ideals of R such that $\delta(K_i) = P$ for all $1 \leq i \leq n$. If δ is graded intersection preserving, then $K = \bigcap_{i=1}^{n} K_i$ is a graded 2-absorbing δ-primary ideal of R.*

Proof. Let $a, b, c \in h(R)$ such that $abc \in K$, $ab \notin K$ and $ac \notin \delta(K)$. Then $ab \notin K_i$ and $ac \notin \delta(K_r)$ for some i and r. Since $\delta(K_i) = \delta(K_r)$, $ac \notin \delta(K_i)$, and then $bc \in \delta(K_i)$. But $\delta(K_j) = P = \bigcap_{i=1}^{n} \delta(K_i) = \delta\left(\bigcap_{i=1}^{n} K_i\right) = \delta(K)$, which implies that $bc \in \delta(K)$. Hence, K is a graded 2-absorbing δ-primary ideal of R. $\square$

Theorem 3.6. *Let δ be a graded global ideal expansion of R and P a proper graded ideal of R. Then P is a graded 2-absorbing δ-primary ideal of R if and only if $\{0_{R/P}\}$ is a graded 2-absorbing δ-primary ideal of R/P.*

Proof. Suppose that P is a graded 2-absorbing δ-primary ideal of R. Let $x + P, y + P, z + P \in h(R/P)$ such that $(x + P)(y + P)(z + P) = 0 + P$. Assume that $(x + P)(y + P) \neq 0 + P$ and $(x + P)(z + P) \notin \delta(\{0_{R/P}\})$. Consider the graded natural ring homomorphism $f : R \to R/P$. Since δ is global, $\delta(P) = \delta(f^{-1}(\{0_{R/P}\})) = f^{-1}(\delta(\{0_{R/P}\}))$, and then since f is surjective, $\delta(P)/P = f(\delta(P)) = \delta(\{0_{R/P}\})$. Now, $x, y, z \in h(R)$ such that $xyz \in P$, $xy \notin P$ and $xz \notin \delta(P)$, and then $yz \in \delta(P)$. Hence, $(y + P)(z + P) \in \delta(\{0_{R/P}\})$ which implies that $\{0_{R/P}\}$ is a graded 2-absorbing δ-primary ideal of R/P. Conversely, let $a, b, c \in h(R)$ such that $abc \in P$. Assume that $ab \notin P$ and $ac \notin \delta(P)$. Then $(a + P)(b + P) \neq 0 + P$ and $(a + P)(c + P) \notin \delta(P)/P = \delta(\{0_{R/P}\})$, and then $(b + P)(c + P) \in \delta(0_{R/P}) = \delta(P)/P$, and hence $ac \in \delta(P)$. Therefore, P is a graded 2-absorbing δ-primary ideal of R. $\square$

Lemma 3.7. *Let δ be a graded ideal expansion of R and S, $f : R \to S$ a graded ring homomorphism and K a graded 2-absorbing δ-primary ideal of S. If δ is graded global, then $f^{-1}(K)$ is a graded 2-absorbing δ-primary ideal of R.*

Proof. Let $x, y, z \in h(R)$ such that $xyz \in f^{-1}(K)$. Suppose that $xy \notin f^{-1}(K)$ and $xz \notin \delta(f^{-1}(K)) = f^{-1}(\delta(K))$. Then $f(x), f(y), f(z) \in h(S)$ such that $f(x)f(y)f(z) \in K$, $f(x)f(y) \notin K$ and $f(x)f(z) \notin \delta(K)$, and then $f(y)f(z) \in \delta(K)$ which implies that $yz \in f^{-1}(\delta(K)) = \delta(f^{-1}(K))$. Hence, $f^{-1}(K)$ is a graded 2-absorbing δ-primary ideal of R. □

Theorem 3.8. *Let δ be a graded global ideal expansion of R and S, $f : R \to S$ a graded surjective ring homomorphism and K a graded ideal of R containing $Ker(f)$. Then K is a graded 2-absorbing δ-primary ideal of R if and only if $f(K)$ is a graded 2-absorbing δ-primary ideal of S.*

Proof. Suppose that K is a graded 2-absorbing δ-primary ideal of R. Let $x, y, z \in h(S)$ such that $xyz \in f(K)$, $xy \notin f(K)$ and $xz \notin \delta(f(K))$. Then there exist $a, b, c \in h(R)$ such that $x = f(a)$, $y = f(b)$ and $z = f(c)$. Thus $abc \in f^{-1}(f(K)) = K$, $ab \notin K$ and $ac \notin f^{-1}(\delta(f((K)))$. Since $f^{-1}(\delta(f(K))) = \delta(f^{-1}(f(K))) = \delta(K)$, so $f(\delta(K)) = \delta(f(K))$. Hence $bc \in \delta(K) = f^{-1}(\delta(f(K))$ which implies that $yz \in \delta(f(K))$. So, $f(K)$ is a graded 2-absorbing δ-primary ideal of S. The converse holds by Lemma 3.7. □

Corollary 3.9. *Let K_1 and K_2 be two graded ideals of R such that $K_1 \subseteq K_2$, and suppose that δ is graded global ideal expansion of R and R/K_1. Then K_2/K_1 is a graded 2-absorbing δ-primary ideal of R/K_1 if and only if K_2 is a graded 2-absorbing δ-primary ideal of R.*

4. Graded δ-Primary Submodules

In this section, we extend the concept of graded δ-primary ideals into graded δ-primary submodules. A number of results about graded prime, graded primary submodules are extended into this new structure.

Graded prime submodules have been introduced by S. E. Atani in [10]. A proper graded R-submodule N of M is said to be graded prime if whenever $r \in h(R)$ and $m \in h(M)$ such that $rm \in N$, then either $m \in N$ or $r \in (N :_R M)$. Graded primary submodules have been also introduced by S. E. Atani and F. Farzalipour in [11]. A proper graded R-submodule N of M is said to be graded primary if whenever $r \in h(R)$ and $m \in h(M)$ such that $rm \in N$, then either $m \in N$ or $r \in Gr((N :_R M))$. In recent years, graded prime and graded primary submodules have attracted an excellent deal of attentions, for example, see [4], [5] and [19].

Definition 4.1. *Let R be a graded ring, δ a graded ideal expansion of R and M a graded R-module. A proper graded R-submodule N of M is said to be graded δ-primary if whenever $r \in h(R)$ and $m \in h(M)$ such that $rm \in N$, then either $m \in N$ or $r \in \delta((N :_R M))$.*

Example 4.2.
1. *A graded R-submodule is graded δ_0-primary if and only it is graded prime.*
2. *Let R be a graded ring. Then a graded R-submodule is graded $\delta_{\frac{1}{2}}$-primary if and only if it is graded primary.*

Lemma 4.3. *If N is a graded δ-primary R-submodule of M, then $(N :_R M)$ is a graded δ-primary ideal of R.*

Proof. Let $x, y \in h(R)$ such that $xy \in (N :_R M)$. Suppose that $x \notin (N :_R M)$. Then $xyM \subseteq N$ and $xM \nsubseteq N$, and then there exists $m \in M$ such that $xym \in N$ and $xm \notin N$. Since $xym \in N$ and N is graded, $xym_g \in N$ for all $g \in G$, and since $xm \notin N$, there exists $h \in G$ such that $xm_h \notin N$. So, $y \in h(R)$ and $xm_h \in h(M)$ such that $yxm_h \in N$. Since N is graded δ-primary and $xm_h \notin N$, $y \in \delta((N :_R M))$. Hence, $(N :_R M)$ is a graded δ-primary ideal of R. □

Lemma 4.4. *Let N be a proper graded R-submodule of M. Then N is graded δ-primary if and only if whenever K is a graded ideal of R and L is a graded R-submodule of M such that $KL \subseteq N$, then either $L \subseteq N$ or $K \subseteq \delta((N :_R M))$.*

Proof. Suppose that N is a graded δ-primary R-submodule of M. Let K be a graded ideal of R and L a graded R-submodule of M such that $KL \subseteq N$. Assume that $L \nsubseteq N$. Then there exists $x \in L$ such that $x \notin N$, and then there exists $g \in G$ such that $x_g \in L - N$. Let $r \in K$. Then for all $h \in G$, $r_h \in K$ with $r_h x_g \in N$. Since N is graded δ-primary and $x_g \notin N$, $r_h \in \delta((N :_R M))$ for all $h \in G$, which implies that $r \in \delta((N :_R M))$. Hence, $K \subseteq \delta((N :_R M))$. Conversely, let $r \in h(R)$ and $m \in h(M)$ such that $rm \in N$ with $m \notin N$. Then $K = \langle r \rangle$ is a graded ideal of R and $L = \langle m \rangle$ is a graded R-submodule of M such that $KL \subseteq N$ and $L \nsubseteq N$, and then $K \subseteq \delta((N :_R M))$ by assumption, which implies that $r \in \delta((N :_R M))$. Hence, N is a graded δ-primary R-submodule of M. $\square$

Graded multiplication modules have been introduced by Escoriza and Torrecillas in [12]. A graded R-module M is said to be graded multiplication if for every graded R-submodule N of M there exits a graded ideal I of R such that $N = IM$. Graded multiplication modules have been widely studied by several authors, for example, see [1], [3], [6] and [7].

Theorem 4.5. *Let M be a graded multiplication R-module and N a proper graded R-submodule of M. Then N is a graded δ-primary R-submodule of M if and only if $(N :_R M)$ is a graded δ-primary ideal of R.*

Proof. Suppose that $(N :_R M)$ is a graded δ-primary ideal of R. We will use Lemma 4.4 to prove that N is a graded δ-primary R-submodule of M. Let K be a graded ideal of R and L a graded R-submodule of M such that $KL \subseteq N$. Since M is graded multiplication, $L = IM$ for some graded ideal I of R, and then $KIM \subseteq N$, which implies that $KI \subseteq (N :_R M)$, and then by Lemma 2.8, either $I \subseteq (N :_R M)$ or $K \subseteq \delta((N :_R M)$. If $I \subseteq (N :_R M)$, then $IM \subseteq N$ that is $L \subseteq N$. Hence, N is a graded δ-primary R-submodule of M. The converse holds by Lemma 4.3. $\square$

Lemma 4.6. *If N is a graded R-submodule of M and I a graded ideal of R, then $(N :_M I) = \{m \in M : Im \subseteq N\}$ is a graded R-submodule of M.*

Proof. Clearly, $(N :_M I)$ is an R-submodule of M. Let $m \in (N :_M I)$. Then $m = \sum_{g\in G} m_g$ where $m_g \in M_g$ for all $g \in G$ and $Im \subseteq N$. Suppose that X is the homogeneous generating set for I. Then $rm \in N$ for all $r \in X$ and $rm = \sum_{g\in G} rm_g$ is a homogeneous decomposition of an element in N. Thus, $rm_g \in N$ for all $g \in G$. Of course, this is true for all $r \in X$. So, $Im_g \subseteq N$ for all $g \in G$ that is $m_g \in (N :_M I)$ for all $g \in G$. Hence, $(N :_M I)$ is a graded R-submodule of M. $\square$

Theorem 4.7. *If N is a graded δ-primary R-submodule of M, then $(N :_M I) = N$ for every graded ideal I of R with $I \nsubseteq \delta((N :_R M))$.*

Proof. Let I be a graded ideal of R such that $I \nsubseteq \delta((N :_R M))$. Clearly, $N \subseteq (N :_M I)$. On the other hand, $I(N :_M I) \subseteq N$, and then by Lemma 4.4, $(N :_M I) \subseteq N$. Hence, $(N :_M I) = N$. $\square$

Theorem 4.8. *Let $N_1, ..., N_k$ be graded δ-primary R-submodules of M such that $\delta((N_i :_R M)) = P$ for all $1 \leq i \leq k$. If δ is graded intersection preserving, then $N = \bigcap_{i=1}^{k} N_i$ is a graded δ-primary R-submodule of M.*

Proof. Let $r \in h(R)$ and $m \in h(M)$ such that $rm \in N$, and $m \notin N$. Then there exists i such that $rm \in N_i$ and $m \notin N_i$, and then $r \in \delta((N_i :_R M)) = P$. Since δ is graded intersection preserving and $(N :_R M) = (\bigcap_{i=1}^{k} N_i :_R M) = \bigcap_{i=1}^{k}(N_i :_R M)$, $\delta((N :_R M)) = \bigcap_{i=1}^{k} \delta(N_i :_R M) = P$. Hence, $r \in \delta((N :_R M))$, and thus N is a graded δ-primary R-submodule of M. $\square$

Theorem 4.9. *Let R be a graded ring, δ a graded global ideal expansion of R, M a graded multiplication R-module, N a proper graded R-submodule of M and $K = (N :_R M)$. Then N is a graded δ-primary R-submodule of M if and only if $r + K \in \delta(\{0_{R/K}\})$ for all homogeneous zero divisor $r + K$ in R/K.*

Proof. Apply Theorem 4.5 and Theorem 2.17. □

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