Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2022 (40)** : 1–9.
ISSN-0037-8712 in press
doi:10.5269/bspm.51777

Some Fixed Point Results and Their Applications on Integral Type Contractive Condition in Fuzzy Metric Spaces

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ABSTRACT: In this paper, we have derived common fixed point results for weakly compatible self mappings satisfying E.A. property in fuzzy metric spaces. As consequences of our results we obtain some corollaries. Finally, we give an example and some applications to integral type contractions to verify the effectiveness and applicability of our main results.

Key Words: Fuzzy metric space, Common fixed point, Complete subspace, E.A. property, Integral type.

Contents

1. Introduction

The origin of fuzzy mathematics can be only attributed to the introduction of fuzzy sets in the innovatory paper of Zadeh [23] in 1965. This concept provides a new way to represent the vagueness in everyday life. In the course of this fuzzification process, like other concepts, the concept of a fuzzy metric was introduced in many ways. George and Veeramani [5] modified the concept of a fuzzy metric space introduced by Kramosil and Michalek [18]. They also induced Hausdorff topology for this variety of fuzzy metric spaces and showed that every metric space induces fuzzy metric spaces.
In 1988, M. Grabiec [4] proved the Banach contraction principle in the sense of fuzzy metric space introduced by Karmosil and Michalek, which is a milestone in developing fixed point theorems in fuzzy metric space.

Theorem 1.1. *[4] Let $(X, M, *)$ be a complete fuzzy metric spaces such that*

1. $\lim_{t\to\infty} M(x, y, t) = 1$;
2. $M(Fx, Fy, kt) \geq M(x, y, t)$,

for all $x, y \in X$, where $0 < k < 1$. Then F has a unique fixed point.

Jungck [16] introduced the concept of compatible maps and proved some common fixed point theorems. Mishra [19] was the first to employed this notion of compatible mapping in fuzzy metric space. Aamri and El Moutawakil [1] generalized the concept of non compatibility by defining the notion of (E.A) property and proved common fixed point theorems under strict contractive conditions. Jungck and Rohades [17] termed a pair of self-maps to be coincidentally commuting or equivalently weak-compatible, if they commute at their coincidence points. This concept is most universal among all the commutativity concepts in this field. Jungck showed that every pair of R-weakly commuting maps is compatible and each pair of compatible maps is weak-compatible but the reverse is not true always.

2010 *Mathematics Subject Classification:* 47H10, 54H25.
Submitted January 13, 2020. Published June 09, 2020

This idea of Gungck was further unified and extended by many authors in various spaces. Few of them are [2,3,6,7,8,9,10,11,12,13,14,15,20,22].

2. Preliminaries

Definition 2.1. *[21] A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a* ***continuous triangular norm*** *if for all $a, b, c, e \in [0,1]$ the following conditions are satisfied:*

1. *$*$ is commutative and associative;*
2. *$a * 1 = a$;*
3. *$*$ is continuous;*
4. *$a * b \leq c * e$, whenever $a \leq c$ and $b \leq e$.*

In short it is also called ***continuous t-norms****.*

Definition 2.2. *[18] The triplet $(X, M, *)$ is fuzzy metric space if X is an arbitrary set, $*$ is continuous t-norm, M is fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions:*

1. *$M(x, y, 0) = 0$;*
2. *$M(x, y, t) = 1$, for all $t > 0$ iff $x = y$;*
3. *$M(x, y, t) = M(y, x, t)$;*
4. *$M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $x, y, z \in X$ and $t, s > 0$;*
5. *$M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous;*
6. *$\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$.*

The triplet $M(x, y, t)$ can be taken as the degree of nearness between x and y with respect to $t \geq 0$.

Definition 2.3. *[4] A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be convergent to $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.*

Definition 2.4. *[4] A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called Cauchy sequence if $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ for all $t > 0$ and each $p > 0$.*

Definition 2.5. *[4] A fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence in X converges in X.*

Definition 2.6. *[16] Two mappings P and Q of a fuzzy metric space $(X, M, *)$ into itself are said to be compatible maps if*

$$\lim_{n\to\infty} M(PQx_n, QPx_n, t) = 1 \text{ for all } t > 0,$$

where x_n is a sequence in X such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Px_n = u \in X$.

Definition 2.7. *[1] Two self mappings S and T of a fuzzy metric space are said to satisfies E.A. property if there exist a sequence $\{x_n\} \in X$ such that*

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = x_0 \text{ for some } x_0 \in X.$$

In our result, we define a class L of all mappings $\Xi : [0, 1] \to [0, 1]$ satisfying the following conditions:

1. Ξ is increasing on $[0, 1]$;
2. $\Xi(t) > t$ for all $t \in (0, 1]$ and $\Xi(t) = t$ if and only if $t = 1$.

Now we state and prove our main result.

3. Main Result

Theorem 3.1. *Let f and g be two weakly compatible self mappings of a fuzzy metric space $(X, M, *)$ with $t * t \geq t$ and for each $x \neq y \in X$, $t > 0$ satisfying the condition*

$$M(fx, fy, kt) \geq \min \left\{ \begin{array}{c} M(gx, gy, t), M(fx, gx, t), \\ M(gy, fy, t), M(gx, fy, t), \\ M(fx, gy, t), \frac{M(gx,fy,t).M(fx,gy,t)}{M(gx,gy,t)} \end{array} \right\}, \tag{3.1}$$

where $0 < k < 1$.
Further, assume that the following assertions hold:

i) f *and* g *satisfy the E.A property;*

ii) $f(X) \supset g(X)$*;*

iii) $f(X)$ *or* $g(X)$ *is a complete subspace of* X.

Then f *and* g *have a unique common fixed point in* X.

Proof. Since f and g satisfy E.A property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x_0 \text{ for some } x_0 \in X.$$

Suppose that gX is complete then $\lim_{n\to\infty} gx_n = ga$ for some $a \in X$. Therefore, from (i) $\lim_{n\to\infty} fx_n = ga$.
Claim that $fa = ga$. Suppose not, that is $fa \neq ga$. Then equation (3.1) implies that

$$M(fx_n, fa, kt) \geq \min \left\{ \begin{array}{c} M(gx_n, ga, t), M(fx_n, gx_n, t), \\ M(ga, fa, t), M(gx_n, fa, t), \\ M(fx_n, ga, t), \frac{M(gx_n,fa,t).M(fx_n,ga,t)}{M(gx_n,ga,t)} \end{array} \right\}.$$

Letting $\lim n \to \infty$

$$M(ga, fa, kt) \geq \min \left\{ \begin{array}{c} M(ga, ga, t), M(ga, ga, t), \\ M(ga, fa, t), M(ga, fa, t), \\ M(ga, ga, t), \frac{M(ga,fa,t).M(ga,ga,t)}{M(ga,ga,t)} \end{array} \right\},$$

implies

$$M(ga, fa, kt) \geq \min\Big(1, M(ga, fa, t)\Big). \tag{3.2}$$

Therefore $fa = ga$.
Next we show that fa is the common fixed point of f and g .
Suppose that $fa \neq ffa$. Since f and g are weakly compatible, then $fga = gfa$ and therefore $gga = ffa$.
Again from (3.1),

$$M(fa, ffa, kt) \geq \min \left\{ \begin{array}{c} M(fa, fga, t), M(fa, fa, t), \\ M(fga, ffa, t), M(fa, ffa, t), \\ M(fa, fga, t), \frac{M(fa,ffa,t).M(fga,ffa,t)}{M(fa,fga,t)} \end{array} \right\}.$$

On simplifying, we get

$$M(fa, ffa, kt) \geq \min\Big(1, M(fa, ffa, t)\Big), \tag{3.3}$$

which contradicts our assumption. Therefore $fa = ffa$. Hence fa is the common fixed point of f and g.
Next we claim that fa is the unique common fixed point of f and g.
Suppose not, therefore there exist $a \neq b \in X$ such that $fa = ga = a$ and $fb = gb = b$.
Consider,

$$M(a, b, kt) = M(fa, fb, kt) \geq \min \left\{ \begin{array}{c} M(ga, gb, t), M(fa, ga, t), \\ \\ M(gb, fb, t), M(ga, fb, t), \\ \\ M(fa, gb, t), \frac{M(ga, fb, t).M(fa, gb, t)}{M(ga, gb, t)} \end{array} \right\}$$

$$M(a, b, kt) \geq \min \left\{ \begin{array}{c} M(a, b, t), M(a, a, t), \\ \\ M(b, b, t), M(a, b, t), \\ \\ M(a, b, t), \frac{M(a, b, t).M(a, b, t)}{M(a, b, t)} \end{array} \right\}$$

This implies

$$M(a, b, kt) \geq \min\Big(1, M(a, b, t)\Big), \tag{3.4}$$

this contradicts our assumption and hence $a = b$. Therefore fa is the unique common fixed point of f and g . □

Theorem 3.2. *Let f and g be two weakly compatible self mappings on a fuzzy metric space $(X, M, *)$ with $t * t \geq t$ such that for each $x \neq y \in X$, $t > 0$ satisfying*

$$M(fx, fy, kt) \geq \Xi(S(x, y, t)), \tag{3.5}$$

where $0 < k < 1$, $\Xi \in L$ and

$$S(x, y, t) = \min \left\{ \begin{array}{c} M(gx, gy, t), M(fx, gx, t), \\ \\ M(gy, fy, t), M(gx, fy, t), \\ \\ M(fx, gy, t), \frac{M(gx, fy, t).M(fx, gy, t)}{M(gx, gy, t)} \end{array} \right\}. \tag{3.6}$$

Further, assume that following assertions hold:

i) f and g satisfy the E.A property;

ii) $f(X) \supset g(X)$;

iii) $f(X)$ or $g(X)$ is a complete subspace of X.

Then f and g have a unique common fixed point in X.

Proof. On the line of Theorem 3.1, we define a sequence $\{x_n\} \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x_0 \text{ for some } x_0 \in X.$$

Here assert that gX is complete, then for some $a \in X$, $\lim_{n\to\infty} gx_n = ga$. Hence from (i), $\lim_{n\to\infty} fx_n = ga$.
We have to prove that $fa = ga$. Suppose on contrary that $fa \neq ga$. Then $M(ga, fa, t) > 1$.
Eq. (3.5) implies that

$$M(fx_n, fa, kt) \geq \Xi(S(x_n, a, t)), \tag{3.7}$$

where (from eq. 3.6)

$$S(x_n, a, t) = \min \left\{ \begin{array}{c} M(gx_n, ga, t), M(fx_n, gx_n, t), \\ \\ M(ga, fa, t), M(gx_n, fa, t), \\ \\ M(fx_n, ga, t), \frac{M(gx_n, fa, t).M(fx_n, ga, t)}{M(gx_n, ga, t)} \end{array} \right\}.$$

Letting $\lim n \to \infty$ in above equality, we get

$$S(a, a, t) \geq \min \left\{ \begin{array}{c} M(ga, ga, t), M(ga, ga, t), \\ \\ M(ga, fa, t), M(ga, fa, t), \\ \\ M(ga, ga, t), \frac{M(ga, fa, t).M(ga, ga, t)}{M(ga, ga, t)} \end{array} \right\},$$

implies that

$$S(a, a, t) \geq \min \Big(1, M(ga, fa, t)\Big).$$

Since $M(ga, fa, t) > 1$, then

$$S(a, a, t) \geq 1.$$

Hence from eq. (3.5) and using the fact that $\Xi \in L$, we have

$$M(ga, fa, kt) \geq \Xi(1) = 1.$$

This is a contradiction and therefore $fa = ga$.
Next assert that fa is the common fixed point of f and g.
Suppose it is not true, that is, $fa \neq ffa$. Then weakly compatible property of maps f and g implies that $fga = gfa$ and therefore $gga = ffa$.
Again consider eq. (3.5),

$$M(fa, ffa, kt) \geq \Xi(S(a, fa, t)), \tag{3.8}$$

where

$$S(a, fa, t) = \min \left\{ \begin{array}{c} M(fa, fga, t), M(fa, fa, t), \\ \\ M(fga, ffa, t), M(fa, ffa, t), \\ \\ M(fa, fga, t), \frac{M(fa, ffa, t).M(fga, ffa, t)}{M(fa, fga, t)} \end{array} \right\}. \tag{3.9}$$

Since $M(fa, ffa, t) > 1$, eq. (3.9) implies that

$$S(a, fa, t) = \min\Big(1, M(fa, ffa, t)\Big) = 1.$$

Thus from eq. (3.8), and using the fact that $\Xi \in L$, we get

$$M(fa, ffa, kt) \geq 1.$$

Therefore $fa = ffa$. Hence fa is the common fixed point of f and g .
For uniqueness of fa, assume that there exist $a \neq b \in X$ such that $fa = ga = a$ and $fb = gb = b$.
On using these values in, and after simplifying eq.(3.5) and eq.(3.6), we obtain

$$M(a, b, kt) \geq 1.$$

Thus, we arrive at contradiction. Therefore fa is the unique common fixed point of f and g . □

On following above theorems, we state two consequence result.

Corollary 3.3. *Let f and g be two non compatible self mappings of a fuzzy metric space $(X, M, *)$ with $t * t \geq t$ satisfying the condition: for each $x \neq y \in X$, $t > 0$*

$$M(fx, fy, kt) \geq \min \left\{ \begin{array}{c} M(gx, gy, t), M(fx, gx, t), \\ \\ M(gy, fy, t), M(gx, fy, t), \\ \\ M(fx, gy, t), \frac{M(gx,fy,t).M(fx,gy,t)}{M(gx,gy,t)} \end{array} \right\},$$

where $0 < k < 1$. Further, assume that following assertions hold:

i) $f(X) \supset g(X)$*;*

ii) $f(X)$ *or* $g(X)$ *is a complete subspace of* X*.*

Then f and g have a unique common fixed point in X. □

Corollary 3.4. *Let f and g be two non compatible self mappings of a fuzzy metric space $(X, M, *)$ with $t * t \geq t$ such that for each $x \neq y \in X$, $t > 0$ satisfying*

$$M(fx, fy, kt) \geq \Xi(S(x, y, t)),$$

where $0 < k < 1$, $\Xi \in L$ and

$$S(x, y, t) = \min \left\{ \begin{array}{c} M(gx, gy, t), M(fx, gx, t), \\ \\ M(gy, fy, t), M(gx, fy, t), \\ \\ M(fx, gy, t), \frac{M(gx,fy,t).M(fx,gy,t)}{M(gx,gy,t)} \end{array} \right\},$$

Further, assume that following assertions hold:

i) $f(X) \supset g(X)$*;*

ii) $f(X)$ *or* $g(X)$ *is a complete subspace of* X*.*

Then f and g have a unique common fixed point in X. □

4. Example and applications to integral type contractions

In this section, we give an example and some applications based on our results.

Let us define $\Psi : [0, \infty) \to [0, \infty)$, as $\Psi(t) = \int_0^t \varphi(t)\, dt \quad \forall \quad t > 0$, be a non-decreasing and continuous function. Moreover for each $\epsilon > 0$, $\varphi(\epsilon) > 0$. Also implies that $\varphi(t) = 0$ if and only if f $t = 0$.

Example 4.1. *Let $X = [1, +\infty)$. Define $f, g : X \to X$ by $fx = x^2$ and $gx = 2x - 1$ for all $x \in X$. Let fuzzy metric be defined by*

$$M(x, y, t) = \frac{t}{t + |x - y|},$$

then

1. *f and g satisfy the E.A property for the sequence $x_n = 1 + \frac{1}{n}$, $\forall\ n = 1, 2...$;*
2. *f and g are weakly compatible ;*
3. *f and g satisfy $\forall\ x \neq y$.*

$$M(fx, fy, kt) \geq \min \left\{ \begin{array}{c} M(gx, gy, t), M(fx, gx, t), \\ M(gy, fy, t), M(gx, fy, t), \\ M(fx, gy, t), \frac{M(gx,fy,t).M(fx,gy,t)}{M(gx,gy,t)} \end{array} \right\}.$$

Clearly $f1 = g1 = 1$, that is 1 is the common fixed point of f and g. □

Theorem 4.1. *Let f and g be two weakly compatible self mappings of a fuzzy metric space $(X, M, *)$ with $t * t \geq t$ such that for each $x \neq y \in X$, $t > 0$ and for $0 < k < 1$*

i - f and g satisfy the E.A property;

ii - $\int_0^{M(fx,fy,kt)} \varphi(t)\, dt \geq \int_0^{S(x,y,t)} \varphi(t)\, dt$;

where $S(x, y, t)$ is given by eq. (3.6) and $\varphi \in \Psi$.

iii - $f(X) \supset g(X)$;

iv - $f(X)$ or $g(X)$ is a complete subspace of X.

Then f and g have a unique common fixed point in X.

Proof. Here if we set $\varphi(t) = 1$, then we get the proof of result by using Theorem 3.1. □

Theorem 4.2. *Let f and g be two weakly compatible self mappings of a fuzzy metric space $(X, M, *)$ with $t * t \geq t$ such that for each $x \neq y \in X$, $t > 0$ and for $0 < k < 1$*

i - f and g satisfy the E.A property;

ii - $\int_0^{M(fx,fy,kt)} \varphi(t)\, dt \geq \Xi\left(\int_0^{S(x,y,t)} \varphi(t)\, dt \right)$;

where $S(x, y, t)$ is given by eq. (3.6), $\Xi \in L$ and $\varphi \in \Psi$.

iii - $f(X) \supset g(X)$;

iv - $f(X)$ or $g(X)$ is a complete subspace of X.

Then f and g have a unique common fixed point in X.

Proof. Let us take $\varphi(t) = 1$, then we get the proof of result by using Theorem 3.2. □

5. Conclusion

In this article, we have given some common fixed point theorems for weakly compatible self mappings satisfying E.A. property in fuzzy metric spaces. Based on the results in this paper, interesting researches may be prospective. In the future study, one can think of establishing some new fixed point results in different spaces.

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