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On Generalizations of Graded Multiplication Modules

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Abstract: Let G be a group with identity e, R be a G-graded ring with unity 1 and M be a G-graded R-module. In this article, we introduce the concept of graded quasi multiplication modules, where graded M is said to be graded quasi multiplication if for every graded weakly prime R-submodule N of M, $N = IM$ for some graded ideal I of R. Also, we introduce the concept of graded absorbing multiplication modules; M is said to be graded absorbing multiplication if M has no graded 2-absorbing R-submodules or for every graded 2-absorbing R-submodule N of M, $N = IM$ for some graded ideal I of R. We prove many results concerning graded weakly prime submodules and graded 2-absorbing submodules that will be useful in providing several properties of the two classes of graded quasi multiplication modules and graded absorbing multiplication modules.

Key Words: Graded multiplication modules, Graded weakly prime submodules, Graded quasi multiplication modules, Graded 2-absorbing submodules, Graded absorbing multiplication modules.

Contents

1. Introduction

A graded R-module M is said to be graded multiplication if for every graded R-submodule N of M, $N = IM$ for some graded deal I of R. In this case, it is known that $I = (N :_R M)$. Graded multiplication modules was firstly introduced and studied by Escoriza and Torrecillas [12], and further results were obtained by several authors, see for example [1,6,17]. In [10], Atani introduced the concept of graded prime submodules; a proper graded R-submodule N of a graded R-module M is said to be graded prime if whenever $r \in h(R)$ and $m \in h(M)$ such that $rm \in N$, then either $r \in (N : M)$ or $m \in N$. The set of all graded prime submodules of M is denoted by $GSpec(M)$. For more details see [3,4]. A graded R-module M is said to be graded weak multiplication if $GSpec(M) = \emptyset$ or for every graded prime R-submodule N of M, $N = IM$ for some graded deal I of R. Graded weak multiplication modules have been studied by several authors, for example, see [5].

In [11], S. E. Atani introduced the concept of graded weakly prime submodules over graded commutative rings; where a graded proper R-submodule N of a graded R-module M is said to be graded weakly prime R-submodule of M if whenever $r \in h(R)$ and $m \in h(M)$ such that $0 \neq rm \in N$, then either $m \in N$ or $r \in (N :_R M)$. One can easily see that every graded prime submodule is graded weakly prime. However, the converse is not true in general; for example $\{0\}$. In this article, we prove several results on graded weakly prime submodules that will be useful in studying the new concept of graded quasi multiplication modules introduced here; a graded R-module M is said to be graded quasi multiplication if for every graded weakly prime R-submodule N of M, $N = IM$ for some graded ideal I of R.

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A proper graded R-submodule N of a graded R-module M is said to be graded 2-absorbing if whenever $a, b \in h(R)$ and $m \in h(M)$ such that $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$. The set of all graded 2-absorbing R-submodules of M is denoted by $GABSpec(M)$. This concept has been first introduced and studied in [7], and then generalized into graded n-absorbing submodules in [16]. In this article, we follow a parallel study given in [2] to investigate the new class, introduced here, of graded absorbing multiplication modules, by first providing many interesting results on 2-absorbing submodules. A graded R-module M is said to be graded absorbing multiplication if $GABSpec(M) = \emptyset$ or for every graded 2-absorbing R-submodule N of M, $N = IM$ for some graded deal I of R.

The next example shows that the class of graded absorbing multiplication modules is different than the class of graded semiprime multiplication modules that was investigated In [2]. Recall that a proper graded R-submodule N of a graded R-module M is said to be graded semiprime if whenever I is a graded ideal of R and K is a graded R-submodule of M such that $I^n K \subseteq N$ for some positive integer n, then $IK \subseteq N$. A graded R-module M is said to be graded semiprime if $\{0\}$ is a graded semiprime R-submodule of M, and the set of all graded semiprime R-submodules of M is denoted by $GSSpec(M)$. Note that the concept of graded semiprime submodules has been first introduced by S. C. Lee and R. Varmazyar in [18], and also studied in [15].

Example 1.1. *Let $R = \mathbb{Z}$, $M = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then R is trivially graded by $R_0 = R$ and $R_1 = \{0\}$. Also, M is graded by $M_0 = \mathbb{Z}$ and $M_1 = i\mathbb{Z}$. Clearly, $N = 4\mathbb{Z} \bigoplus \langle 0 \rangle$ is a graded 2-absorbing R-submodule of M, but N is not graded semiprime since $2 \in h(R)$ and $(3, 0) \in h(M)$ such that $2^2(3, 0) \in N$ and $2(3, 0) \notin N$. On the other hand, $K = 30\mathbb{Z} \bigoplus \langle 0 \rangle$ is a graded semiprime R-submodule of M that is not graded 2-absorbing.*

Our article is organized as follows: In section two, we give some preliminaries that will be needed in the sequel, then in section three we investigate the graded weakly prime submodules. In section four we introduce and study the concept of graded quasi multiplication modules, and section five will be devoted to the study of graded 2-absorbing submodules. Finally, in Section Six we introduce and study the concept of graded absorbing multiplication modules.

2. Preliminaries

Throughout this article, R is assumed to be a commutative ring with a nonzero unity 1. Let G be a group with identity e. A ring R is said to be G-graded ring if there exist additive subgroups R_g of R such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called homogeneous of degree g and R_e (the identity component of R) is a subring of R and $1 \in R_e$. For $x \in R$, x can be written uniquely as $\sum_{g \in G} x_g$ where x_g is the component of x in R_g. Also, we write $h(R) = \bigcup_{g \in G} R_g$ and $supp(R, G) = \{g \in G : R_g \neq 0\}$.

An ideal I of a G-graded ring R is called G-graded ideal if $I = \bigoplus_{g \in G} \left(I \bigcap R_g \right)$, i.e., if $x \in I$ and $x = \sum_{g \in G} x_g$, then $x_g \in I$ for all $g \in G$. An ideal of a G-graded ring need not be G-graded.

Let M be a left R-module. Then M is a G-graded R-module if there exist additive subgroups M_g of M indexed by the elements $g \in G$ such that $M = \bigoplus_{g \in G} M_g$ and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. The elements of M_g are called homogeneous of degree g. If $x \in M$, then x can be written uniquely as $\sum_{g \in G} x_g$, where x_g is the component of x in M_g. Clearly, M_g is R_e-submodule of M for all $g \in G$. Also, we write $h(M) = \bigcup_{g \in G} M_g$. and $supp(M, G) = \{g \in G : M_g \neq 0\}$.

Let M be a G-graded R-module and N be an R-submodule of M. Then N is called G-graded R-submodule if $N = \bigoplus_{g \in G} \left(N \bigcap M_g \right)$, i.e., if $x \in N$ and $x = \sum_{g \in G} x_g$, then $x_g \in N$ for all $g \in G$. Also, an

R-submodule of a G-graded R-module need not be G-graded.

For more details on graded rings and graded modules we refere the reader to ([19] and [9]). The following lemma can be found in ([13], Lemma 2.1).

Lemma 2.1. *Let R be a G-graded ring and M be a G-graded R-module.*

1. *If I and J are graded ideals of R, then $I+J$ and $I\bigcap J$ are graded ideals of R.*
2. *If N and K are graded R-submodules of M, then $N+K$ and $N\bigcap K$ are graded R-submodules of M.*
3. *If N is a graded R-submodule of M, $r\in h(R)$, $x\in h(M)$ and I is a graded ideal of R, then Rx, IN and rN are graded R-submodules of M. Moreover, $(N:_R M)=\{r\in R: rM\subseteq N\}$ is a graded ideal of R.*

Also it was shown in [14] that if N is a graded R-submodule of M, then $Ann(N)=\{r\in R: rN=0\}$ is a graded ideal of R.

3. Graded Weakly Prime Submodules

In this section, we study the graded weakly prime submodules.

Let M be a G-graded R-module. Then the subset $Z_G(M)$ of M is defined by

$$Z_G(M)=\{m\in h(M): rm=0 \text{ for some nonzero } r\in h(R)\}.$$

It is obvious that every graded prime R-submodule is graded weakly prime. However, the converse is not true in general; for example $\{0\}$. We introduce the following:

Theorem 3.1. *Let M be a G-graded R-module. If $Z_G(M)=\{0\}$, then every graded weakly prime R-submodule of M is graded prime.*

Proof. Let N be a graded weakly prime R-submodule of M. Let $s\in h(R)$ and $m\in h(M)$ such that $sm\in N$. If $sm\neq 0$, then since N is graded weakly prime, either $m\in N$ or $s\in(N:M)$. Suppose $sm=0$. Then since $Z_G(M)=\{0\}$, either $s=0\in(N:M)$ or $m=0\in N$. Hence, N is graded prime R-submodule of M. □

Let R be a graded ring and P be a graded maximal ideal of R. Then (R,P) is said to be a graded quasi local ring if P is the only graded maximal ideal in R.

Theorem 3.2. *Let (R,P) be a graded quasi local ring and M be a graded R-module. If $PM=0$, then every proper graded R-submodule of M is graded weakly prime.*

Proof. Let N be a proper graded R-submodule of M. Assume $s\in h(R)$ and $x\in h(M)$ such that $0\neq sx\in N$. We show that s is a unit. Suppose s is not unit. Then $s\in P$ and then $sx\in PM=0$, i.e., $sx=0$ which is a contradiction. So, s is a unit which implies that $x\in N$ and hence N is graded weakly prime. □

Theorem 3.3. *Let (R,P) be a graded quasi local domain and M be a graded R-module. If $P^2=0$, then every proper graded R-submodule of M is graded weakly prime.*

Proof. Let N be a proper graded R-submodule of M. Assume $s\in h(R)$ and $x\in h(M)$ such that $0\neq sx\in N$. We show that s is a unit. Suppose s is not unit. Then $s\in P$ and then $s^2\in P^2=0$, i.e., $s^2=0$ which implies that $s=0$ since R is a domain and so $sx=0$ which is a contradiction. Thus, s is a unit which implies that $x\in N$ and hence N is graded weakly prime. □

Theorem 3.4. *Let M be a graded R-module such that $Z_G(M)=\{0\}$. If N is a graded weakly prime R-submodule of M, then $(N:_R M)$ is a graded weakly prime ideal of R.*

Proof. By Lemma 2.1, $(N :_R M)$ is a graded ideal of R. Let $a, b \in h(R)$ such that $0 \neq ab \in (N :_R M)$. Then $abM \subseteq N$. Since N is a graded weakly prime R-submodule of M, $N \neq M$, i.e., there exists $m \in M - N$ and then there exists $g \in G$ such that $m_g \notin N$. Now, $abm_g \in abM \subseteq N$. If $abm_g = 0$, then $m_g \in Z_G(M) = \{0\}$, i.e., $m_g = 0 \in N$ which is a contradiction. So, $0 \neq abm_g \in N$ and since N is graded weakly prime, either $bm_g \in N$ or $a \in (N :_R M)$. If $bm_g \in N$, then since N is graded weakly prime and $m_g \notin N$, $a \in (N :_R M)$. Hence, $(N :_R M)$ is a graded weakly prime ideal of R. □

4. Graded Quasi Multiplication Modules

In this section, we introduce and study the concept of graded quasi multiplication modules.

Definition 4.1. *Let M be a graded R-module. Then M is said to be a graded quasi multiplication module if for every graded weakly prime R-submodule N of M, we have $N = IM$ for some graded ideal I of R.*

Theorem 4.2. *Let M be a graded R-module. Then M is a graded quasi multiplication R-module if and only if for every graded weakly prime R-submodule N of M, we have $N = (N :_R M)M$.*

Proof. Suppose M is a graded quasi multiplication R-module. Let N be a graded weakly prime R-submodule of M. Then $N = IM$ for some graded ideal I of R and then $I \subseteq (N :_R M)$ which implies that $N = IM \subseteq (N :_R M)M \subseteq N$. Thus, $N = (N :_R M)M$. The converse is clear by taking $I = (N :_R M)$ (see Lemma 2.1). □

Theorem 4.3. *Every graded quasi multiplication R-module is graded weak multiplication.*

Proof. Suppose M is a graded quasi multiplication R-module. Let N be a graded prime R-submodule of M. Then N is a graded weakly prime R-submodule of M and then $N = (N :_R M)M$. Since N is a graded prime R-submodule of M, $(N :_R M)$ is a graded prime ideal of R by ([10], Proposition 2.7) and hence M is a graded weak multiplication module. □

Theorem 4.4. *Let M be a graded weak multiplication R-module. If $Z_G(M) = \{0\}$, then M is a graded quasi multiplication R-module.*

Proof. Let N be a graded weakly prime R-submodule of M. Then by Theorem 3.1, N is a graded prime R-submodule of M and since M is graded weak multiplication, $N = IM$ for some graded prime ideal of R. Hence, M is a graded quasi multiplication R-module. □

Clearly, every graded multiplication R-module is graded quasi multiplication. However, the next example shows that the converse is not true in general.

Example 4.5. *Let $R = \mathbb{Z}$ (the ring of integers) and $M = \mathbb{Q}[i] = \{a + ib : a, b \in \mathbb{Q}\}$ where $\mathbb{Q}$ is the ring of rational numbers. Then M is an R-module. Suppose $G = \mathbb{Z}_2$ (the group of integers modulo 2). Then R is trivially G-graded by $R_0 = \mathbb{Z}$ and $R_1 = 0$. Also, M is G-graded by $M_0 = \mathbb{Q}$ and $M_1 = i\mathbb{Q}$. Let N be a nonzero graded prime R-submodule of M. Then $N \neq M$. Take $x \in M - N$ and $0 \neq y \in N$. Therefore, x_0 or $x_1 \notin N$ and $0 \neq y_0 \in N$ or $0 \neq y_1 \in N$. We may assume that $x_0 \notin N$ and $0 \neq y_0 \in N$. Now, $x_0 = \frac{a}{b}$ and $y_0 = \frac{r}{s}$ for some nonzero integers a, b, r and s. Hence, $rbx_0 = ra = \frac{r}{s}sa \in N$. Since N is graded prime and $x_0 \notin N$, $rb \in (N :_R M)$. So, $rbM \subseteq N$. Since $rbM = M$, we have $N = M$ which is a contradiction. Hence, $\{0\}$ is the only graded prime R-submodule of M and $\{0\} = \{0\}M$. This proves that M is a graded weak multiplication R-module. Since $Z_G(M) = \{0\}$, M is a graded quasi multiplication R-module by Theorem 4.4. On the other hand, $N = \mathbb{Z} \bigoplus \langle 0 \rangle$ is a graded R-submodule of M and $\{0\} \neq N \neq IM$ for every nonzero graded ideal I of R. Thus, M is not graded multiplication R-module.*

5. Graded 2-Absorbing Submodules

We start this section by introducing the concepts of graded 2-absorbing submodules and modules.

Definition 5.1 ([7]). *Let M be a graded R-module. A proper graded R-submodule N of M is said to be graded 2-absorbing if whenever $a, b \in h(R)$ and $m \in h(M)$ such that $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$.*

Definition 5.2. *A graded R-module M is said to be graded 2-absorbing if $\{0\}$ is a graded 2-absorbing R-submodule of M.*

Remark 5.1. *One can easily see that a graded prime submodule of a graded R-module M is graded 2-absorbing.*

We begin our results with following.

Proposition 5.3. *Let M be a G-graded R-module, J be a graded ideal of R and N be a graded 2-absorbing R-submodule of M. If $r \in h(R)$ and $m \in h(M)$ such that $Jrm \subseteq N$, then either $rm \in N$ or $Jr \subseteq (N :_R M)$ or $J_g m \subseteq N$ for some $g \in G$.*

Proof. Suppose that $rm \notin N$ and $Jr \nsubseteq (N :_R M)$. Then there exists $s \in J$ such that $sr \notin (N :_R M)$ and then since J is graded, there exists $g \in G$ and $s_g \in J \bigcap R_g$ such that $s_g r \notin (N :_R M)$. So, $s_g \in h(R)$ such that $s_g rm \in N$ which implies that $s_g m \in N$ since N is a graded 2-absorbing R-submodule of M. We show that $J_g m \subseteq N$. Let $x_g \in J_g$. Then $s_g + x_g \in J \bigcap R_g$ and then $s_g + x_g \in h(R)$ such that $(s_g + x_g)rm \in N$. Hence, either $(s_g + x_g)m \in N$ or $(s_g + x_g)r \in (N :_R M)$. If $(s_g + x_g)m \in N$, then by $s_g m \in N$, it follows that $x_g m \in N$. If $(s_g + x_g)r \in (N :_R M)$, then $x_g r \notin (N :_R M)$, but $x_g rm \in N$. Thus, $x_g m \in N$. So, we conclude that $J_g m \subseteq N$. □

Let M be a G-graded R-module and N be an R-submodule of M. Then M/N can be regarded as a G-graded R-module by taking $(M/N)_g = (M_g + N)/N$ for all $g \in G$ (see [19]).

Lemma 5.4. *Let M be a graded R-module, L a graded R-submodule of M, and N an R-submodules of M such that $L \subseteq N$. Then N is a graded R-submodule of M if and only if N/L is a graded R-submodule of M/L.*

Proof. Suppose that N is a graded R-submodule of M. Clearly, N/L is an R-submodule of M/L. Let $x + L \in N/L$. Then $x \in N$ and since N is graded, $x = \sum_{g \in G} x_g$ where $x_g \in N$ for all $g \in G$ and then $(x + L)_g = x_g + L \in N/L$ for all $g \in G$. Hence, N/L is a graded R-submodule of M/L. Conversely, let $x \in N$. Then $x = \sum_{g \in G} x_g$ where $x_g \in M_g$ for all $g \in G$ and then $(x_g + L) \in (M_g + L)/L = (M/L)_g$ for all $g \in G$ such that $\sum_{g \in G}(x + L)_g = \sum_{g \in G}(x_g + L) = \left(\sum_{g \in G} x_g\right) + L = x + L \in N/L$. Since N/L is graded, $x_g + L \in N/L$ for all $g \in G$ which implies that $x_g \in N$ for all $g \in G$. Hence, N is a graded R-submodule of M. □

Theorem 5.5. *Let M be a graded R-module, L a graded R-submodule of M, and N an R-submodules of M such that $L \subseteq N$. Then N is a graded 2-absorbing R-submodule of M if and only if N/L is a graded 2-absorbing R-submodule of M/L.*

Proof. Suppose that N is a graded 2-absorbing R-submodule of M. By Lemma 5.4, N/L is a graded R-submodule of M/L. Let $x, y \in h(R)$ and $m + L \in h(M/L)$ such that $xy(m + L) \in N/L$. Then $m \in h(M)$ such that $xym \in N$. Since N is graded 2-absorbing, either $xm \in N$ or $ym \in N$ or $xy \in (N :_R M)$ and then either $x(m + L) \in N/L$ or $y(m + L) \in N/L$ or $xy \in (N/L :_R M/L)$. Hence, N/L is a graded 2-absorbing R-submodule of M/L. Conversely, by Lemma 5.4, N is a graded R-submodule of M. Let $x, y \in h(R)$ and $m \in h(M)$ such that $xym \in N$. Then $m + L \in h(M/L)$ such that $xy(m + L) \in N/L$.

Since N/L is graded 2-absorbing, either $x(m+L) \in N/L$ or $y(m+L) \in N/L$ or $xy \in (N/L :_R M/L)$ and then either $xm \in N$ or $ym \in N$ or $xy \in (N :_R M)$. Hence, N is a graded 2-absorbing R-submodule of M. □

When L is a graded 2-absorbing R-submodule of M, we get the following

Corollary 5.6. *Let M be a graded R-module and L be a graded R-submodule of M. Then L is a graded 2-absorbing R-submodule of M if and only if M/L is a graded 2-absorbing R-module.*

Now, we study the behavior of the graded 2-absorbing property with respect to graded homomorphisms.

Proposition 5.7. *Let M and M' be a graded R-modules and let $f : M \to M'$ be a graded R-homomorphism.*

1. *If K' is a graded $2-$absorbing R-submodule of M' then $f^{-1}(K')$ is a graded $2-$absorbing R-submodule of M.*
2. *If f is surjective and K is a graded $2-$absorbing R-submodule of M with $Ker(f) \subseteq K$ then $f(K)$ is a graded $2-$absorbing R-submodule of M'.*

Proof. (i) Let $a, b \in h(R)$ and $m \in h(M)$ such that $abm \in f^{-1}(K')$. Then $f(abm) = abf(m) \in K'$. This implies that $af(m) \in K'$, $bf(m) \in K'$, or $ab \in (K' :_R M')$. Since $(K' :_R M') \subseteq (f^{-1}(K') :_R M)$, we get $am \in f^{-1}(K')$, $bm \in f^{-1}(K')$, or $ab \in (f^{-1}(K') :_R M)$

(ii) Let $a, b \in h(R)$ and $m' \in h(M')$ such that $abm' \in f(K)$. Then $f(abm) = f(k)$ for some $k \in K$. Choose $m \in f^{-1}(m') \cap K$. Then $abm \in k + Ker(f) \subseteq K$. Hence $am \in K$, $bm \in K$, or $ab \in (K :_R M) \subseteq (f(K) :_R M')$. So, $am' \in f(K)$, $bm' \in f(K)$, or $ab \in (f(K) :_R M')$. □

Corollary 5.8. *Let M and M' be a graded R-modules and let $f : M \to M'$ be a graded R-homomorphism.*

1. *If M' is a graded $2-$absorbing R-module then $Ker(f)$ is a graded $2-$absorbing R-submodule of M.*
2. *If f is surjective and M is a graded $2-$absorbing R-module then M' is a graded $2-$absorbing R-module.*

Corollary 5.9. *Let L be a graded R-submodule of M, then for every graded $2-$absorbing R-submodule K of M such that $L \nsubseteq K$ we have $K \cap L$ is graded $2-$absorbing R-Submodule of L.*

6. Graded Absorbing Multiplication Modules

Definition 6.1. *Let M be a graded R-module. Then M is said to be graded absorbing multiplication if $GABSpec(M) = \emptyset$ or for every graded 2-absorbing R-submodule N of M, $N = IM$ for some graded ideal I of R.*

Remark 6.1. *It is easy to prove that if M is a graded absorbing multiplication R-module, then $N = (N :_R M)M$ for every graded 2-absorbing R-submodule of M.*

Proposition 6.2. *Let M be a graded R-module and J be an ideal of R with $J \subseteq (0 :_R M)$. Then M is a graded absorbing multiplication R-module if and only if M is a graded absorbing multiplication as an R/J-module.*

Proof. It is easy to see that N is a graded 2-absorbing R-submodule of M if and only if N is a graded 2-absorbing R/J-submodule of M. So, the result holds by the fact that $(N :_R M) = (N :_{R/J} M)$. □

Theorem 6.3. *Let M be a graded absorbing multiplication R-module. If J is an ideal of R and L is a nonzero graded R-submodule of M such that $J \subseteq (L :_R M)$, then M/L is a graded absorbing multiplication R/J-module.*

Proof. Let N/L be a graded 2-absorbing submodule of M/L. Then by Proposition 5.5, N is a graded 2-absorbing submodule of M and then $N = (N :_R M)M$, where $J \subseteq (L :_R M) \subseteq (N :_R M) = I$. Clearly, $N/L = (I/J)(M/L)$. Hence, M/L is a graded absorbing multiplication R/J-module. □

Corollary 6.4. *Let M be a graded absorbing multiplication R-module. Then M/L is a graded absorbing multiplication R-module for every graded R-submodule L of M.*

Proof. Apply Proposition 6.3 with $J = \{0\}$. □

Proposition 6.5. *Let M and M' be a graded R-modules and let $f : M \to M'$ be a surjective graded R-homomorphism. If M is absorbing multiplication then so is M'.*

Proof. Suppose that $GABspec(M') \neq \emptyset$, and let N' be a graded $2-$absorbing R-submodule of M'. then $f^{-1}(N')$ is a graded $2-$absorbing R-submodule of M. Hence $f^{-1}(N') = IM$ for some graded ideal I of R. Therefore $N' = f(f^{-1}(N')) = f(IM) = IM'$ as desired. □

Let $S \subseteq h(R)$ be a multiplicative set and M be a graded R-module. Then $S^{-1}M$ is a graded $S^{-1}R$-module with

$$(S^{-1}M)_g = \left\{ \frac{m}{s}, m \in M_h, s \in S \cap R_{hg^{-1}} \right\}$$

$$(S^{-1}R)_g = \left\{ \frac{a}{s}, m \in R_h, s \in S \cap R_{hg^{-1}} \right\}$$

Theorem 6.6. *Let M be a graded R-module, N be a graded $2-$absorbing R-submodule of M. Then $S^{-1}N$ is a graded $2-$absorbing R-submodule of $S^{-1}M$*

Proof. Let $a, b \in h(S^{-1}R)$ and $m \in h(S^{-1}M)$ such that $abm \in S^{-1}N$. Then $a = \frac{\alpha}{s}$, $b = \frac{\beta}{t}$, and $m = \frac{k}{r}$, for some elements $\alpha, \beta \in h(R)$, $s, r, t \in S$, and $k \in h(M)$. Therefore $\frac{\alpha\beta k}{str} = \frac{n}{u}$ for some $n \in N$ and $u \in S$. Hence there exists $v \in S$ such that $vu\alpha\beta k = vstsn \in N$, which implies that $vu\alpha k \in N$, $\beta k \in N$, or $vu\alpha\beta \in (N :_R M)$. This yields $\frac{uv\alpha k}{uvsr} \in S^{-1}N$, $\frac{\beta k}{tr} \in S^{-1}N$, or $\frac{uv\alpha\beta}{uvst} \in S^{-1}(N :_R M)$, i.e. $am \in S^{-1}N$, $bm \in S^{-1}N$, or $ab \in S^{-1}(N :_R M)$. Now we only need to prove that $S^{-1}(N :_R M) \subseteq \left(S^{-1}N :_{S^{-1}R} S^{-1}M\right)$. Indeed, let $\frac{x}{s} \in S^{-1}(N :_R M)$ with $x \in (N :_R M)$. Then $\frac{x}{s}\frac{m}{t} = \frac{xm}{st} \in S^{-1}N$ for all $m \in M$, $s \in S$. So $\frac{x}{s}\left(S^{-1}M\right) \subseteq S^{-1}N$. Thus $\frac{x}{s} \in \left(S^{-1}N :_{S^{-1}R} S^{-1}M\right)$. This completes the proof. □

Corollary 6.7. *If M is a graded $2-$absorbing R-module, then $S^{-1}M$ is a graded $2-$absorbing $S^{-1}R$-module.*

Let R and S be two G-graded rings. A homomorphism $f : R \to S$ is said to be graded homomorphism if $f(R_g) \subseteq S_g$ for all $g \in G$. One can prove that if I is a graded ideal of R and J is a graded ideal of S, then $f(I)$ is a graded ideal of S and $f^{-1}(J)$ is a graded ideal of R (see [19]).

Lemma 6.8. *Let R and S be two G-graded rings. Suppose that $f : R \to S$ is a graded homomorphism. If f is surjective, then $f(R_g) = S_g$ for all $g \in G$.*

Proof. Let $g \in G$. Since f is graded homomorphism, $f(R_g) \subseteq S_g$. Let $s_g \in S_g$. If $s_g = 0$, then $s_g = f(0_R) \in f(R_g)$. Suppose that $s_g \neq 0$. Since f is surjective, there exists $r \in R - \{0\}$ such that $f(r) = s_g$. Assume that $r = \sum_{i=1}^{n} r_{g_i}$ where $r_{g_i} \in R_{g_i} - \{0\}$, $g_i \neq g_j$ for $i \neq j$. Then $s_g = f(r) = \sum_{i=1}^{n} f(r_{g_i}) = \sum_{i=1}^{k} f(r_{g_{t_i}})$ where $1 \leq t_i \leq n$ and $f(r_{g_{t_i}}) \neq 0$ for all $1 \leq i \leq k$. Since $f(r_{g_{t_i}}) \in S_{g_{t_i}}$, $s_g \in S_g \bigcap \sum_{i=1}^{k} S_{g_{t_i}}$. Hence, $g = g_{t_1} = ... = g_{t_n}$ and hence $k = 1$ and $f(r_{g_{t_i}}) = f(r_g) = s_g$. So, $S_g \subseteq f(R_g)$ and hence $f(R_g) = S_g$. □

Lemma 6.9. *Let R and S be two G-graded rings. Suppose that $f : R \to S$ is a surjective graded homomorphism and M is a graded S-module. Then M is graded 2-absorbing as an R-module if and only if M is graded 2-absorbing as an S-module.*

Proof. Suppose M is graded 2-absorbing as an R-module, and let $s_1, s_2 \in h(S)$ and $m \in h(M)$ such that $s_1 s_2 m = 0$. Choose $g_1, g_2 \in G$ such that $s_1 \in S_{g_1} = f(R_{g_1})$ and $s_2 \in S_{g_2} = f(R_{g_2})$ (by Lemma 6.8) and then there exist $r_1 \in R_{g_1}$ and $r_2 \in R_{g_2}$ such that $f(r_1) = s_1$ and $f(r_2) = s_2$. So, $r_1, r_2 \in h(R)$ such that $r_1 r_2 m = f(r_1 r_2)m = f(r_1)f(r_2)m = s_1 s_2 m = 0$. Since M is graded 2-absorbing as an R-module, either $r_1 m = 0$ or $r_2 m = 0$ or $r_1 r_2 \in (0 :_R M)$ and then either $s_1 m = 0$ or $s_2 m = 0$ or $s_1 s_2 \in (0 :_S M)$. Hence, M is graded 2-absorbing as an S-module. For the converse, assume that M is graded 2-absorbing as an S-module, and let $r_1, r_2 \in h(R)$ and $m \in M$ such that $r_1 r_2 m = 0$. Then $f(r_1)f(r_2)m = f(r_1 r_2)m = r_1 r_2 m = 0$. This implies that $f(r_1)m = 0$, $f(r_2)m = 0$, or $f(r_1)f(r_2) \in (0 :_S M)$. Thus $r_1 m = 0$, $r_2 m = 0$, or $r_1 r_2 \in (0 :_R M)$. Therefore M is graded 2-absorbing as an R-module. □

Lemma 6.10. *Let R and S be two G-graded rings, M be a graded S-module, and K be a graded S-submodule of M. If there exists a surjective graded homomorphism from R to S, then K is a graded 2-absorbing R-submodule of M if and only if K is a graded 2-absorbing S-submodule of M.*

Proof. By Lemma 5.6, K is a graded 2-absorbing R-submodule of M if and only if M/K is a graded 2-absorbing R-module. Then by Lemma 6.9, this is true if and only if M/K is a graded 2-absorbing S-module. Then again, by Lemma 5.6, M/K is a graded 2-absorbing S-module if and only if K is a graded 2-absorbing S-submodule of M. □

Proposition 6.11. *Let R and S be two G-graded rings. Suppose that $f : R \to S$ is a surjective graded homomorphism and M is a graded S-module. Then M is a graded absorbing multiplication S-module if and only if M is a graded absorbing multiplication R-module.*

Proof. Suppose M is a graded absorbing multiplication S-module, and let K be a graded 2-absorbing R-submodule of M. Then by Lemma 6.10, K is a graded 2-absorbing S-submodule of M. Since M is graded absorbing multiplication as an S-module, it follows that $K = JM$ for some graded ideal J of S. So, $I = f^{-1}(J)$ is a graded ideal of R such that $f(I) = f(f^{-1}(J)) \bigcap f(R) = J$, and hence $IM = f(I)M = JM = K$. Thus, M is a graded absorbing multiplication R-module. For the converse, assume that M is a graded absorbing multiplication R-module, and let K be a graded 2-absorbing S-submodule of M. By Lemma 6.10, we obtain that K is a graded 2-absorbing R-submodule of M. Hence $K = IM$ for some graded ideal I of R. So we have $K = f(I)M$. Therefore M is a graded absorbing multiplication S-module. □

Definition 6.12. *Let M be a graded R-module. We define the torsion set of M with respect to the homogeneous elements of R to be*
$HT(M) = \{m \in M : rm = 0$ *for some nonzero* $r \in h(R)\}$.

Lemma 6.13. *If M is a graded R-module over an integral domain R, then $HT(M)$ is a graded R-submodule of M.*

Proof. Let $m, n \in HT(M)$. Then there exist $r, s \in h(R) - \{0\}$ such that $rm = sn = 0$. Since $r, s \in h(R)$, there exist $g, h \in G$ such that $r \in R_g$ and $r \in R_h$ and then $rs \in R_g R_h \subseteq R_{gh} \subseteq h(R)$. Since R is an integral domain, $rs \in h(R) - \{0\}$ such that $rs(m - n) = rsm - rsn = s(rm) - r(sn) = 0$ which implies that $m - n \in HT(M)$. Let $t \in R$. Then $r(tm) = t(rm) = 0$ which implies that $tm \in HT(M)$. Hence, $HT(M)$ is an R-submodule of M. We show that $HT(M)$ is graded. Let $m \in HT(M)$. Then there exists a nonzero $r \in h(R)$ such that $rm = 0$. Assume that $m = \sum\limits_{g \in G} m_g$ where $m_g \in M_g$ for all $g \in G$. Since $r \in h(R)$, $r \in R_h$ for some $h \in G$ and then $rm_g \in R_h M_g \subseteq M_{gh}$ for all $g \in G$. So, $rm_g \in h(M)$ for all $g \in G$ such that $\sum\limits_{g \in G} rm_g = r\left(\sum\limits_{g \in G} m_g\right) = rm = 0 \in \{0\}$ and since $\{0\}$ is graded R-submodule,

$rm_g \in \{0\}$ for all $g \in G$, i.e., $rm_g = 0$ for all $g \in G$ which implies that $m_g \in HT(M)$ for all $g \in G$. Hence, $HT(M)$ is a graded R-submodule of M. □

Lemma 6.14. *Let M be a graded R-module over an integral domain R. If $HT(M) \neq M$, then $HT(M)$ is a graded prime R-submodule of M with $(HT(M) :_R M) = \{0\}$.*

Proof. By Lemma 6.13, $HT(M)$ is a graded R-submodule of M. Let $r \in h(R)$ and $m \in h(M)$ such that $rm \in HT(M)$. Then there exists a nonzero $s \in h(R)$ such that $s(rm) = 0$. If $r = 0$, then $r \in (HT(M) :_R M)$. Suppose that $r \neq 0$, then $sr \in h(R) - \{0\}$ such that $sr(m) = s(rm) = 0$ which implies that $m \in HT(M)$. Hence, $HT(M)$ is a graded prime R-submodule of M. We show that $(HT(M) :_R M) = \{0\}$. Let $r \in (HT(M) :_R M)$. Then $rM \subseteq HT(M)$. Since $HT(M) \neq M$, there exists $m \in M$ such that $m \notin HT(M)$ and then $rm \in rM \subseteq HT(M)$ which implies that there exists a nonzero $s \in h(R)$ such that $s(rm) = 0$. Since $m \notin HT(M)$, $sr = 0$ and since $s \neq 0$, $r = 0$. Hence, $(HT(M) :_R M) = \{0\}$. □

Proposition 6.15. *Let M be a graded R-module over an integral domain R. If M is a graded absorbing multiplication R-module, then either $HT(M) = \{0\}$ or $HT(M) = M$.*

Proof. Suppose that $HT(M) \neq M$. Then by Lemma 6.14, $HT(M)$ is a graded prime R-submodule of M with $(HT(M) :_R M) = \{0\}$ and then $HT(M)$ is a graded 2-absorbing R-submodule of M and since M is graded absorbing multiplication, $HT(M) = (HT(M) :_R M)M = \{0\}$. □

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