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General Integral Transform - its Convergence and Consequences

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ABSTRACT: The objective of this study is to give a shape of general integral transform for studying its convergence in the general set-up, from which convergences of some well know integral transforms follow easily as well as these integral transforms appear as particular cases of the present integral transform; some more may appear as new but special cases of present transform which have been listed but not studied here.

Key Words: Integral transforms, Convergence.

Contents

1. Introduction

History reveals several naming integral transforms with their important properties and variety of important applications in physics, astronomy, engineering, differential and integral equations, economics, and planning and finance, of which Fourier and Laplace transforms are quite focussed everywhere and in wide range, others are also used in many fields in special cases. Regarding invention of integral transform, history tells that integral transform was first invented by the Swiss mathematician Leonhard Euler [1707-1783] within the context of second order differential equation problems [10]. There are a number of integral transforms viz. Fourier, Laplace, Mellin, Sumudu, Elzaki, Abel, CaderÓn-Zygmund, Bateman, Hankel , Hartley, Hilbert, Laplace-Carson, Laplace-Stieltjes, N, Wavelet, Linear-Canonical, Polynomial etc [available in internet history] for solving differential and integral equations and may be more, of these, the most well-known in the applied mathematics community are the Laplace transform and the Fourier transform. The origin and history of the former have been described in a series of articles by Deakin [[7],[8],[9],[11]] . Fourier transform has wide applications in signal and image processing, and in analyzing quantum mechanics phenomena, pattern recognition, it has enough applications in genetics, genome analysis, and in medical fields, bio-informatics. In spite of stupendous applications of Fourier transform, there is a little knowledge about how it came into play, and diffused into and widespread to several branches of study, perhaps because there are a few works dealing with the Fourier transform development over time. The prime source of history on the Fourier transform is an article from 1850 by the German mathematician Heinrich Friedrich Karl Ludwig Burkhardt (1861-1914) [12]; a second source is found in the remarks and quotations section of a book by Astro-Hungarian mathematician Soloman Bochner (1899-1982) [13]. As a limiting process of Fourier series, Fourier transform is obtained, which was first used by the French mathematician Jean Baptiste Joseph Fourier (1768-1830) in a manuscript submitted to the Institute of France in 1807 [16], and in a memoir deposited in the institute in 1811 [17]. Both works were then collected and expanded by the same author in his famous book about the Analytic Theory of Heat [18].

The Laplace transform is the name after the mathematician and astronomer Pierre-Simon Laplace, who used an integral transform, similar to the Fourier transform in his work on probability Theory [19].The details discussions on the relation between Laplace and Fourier transforms and their nature,

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behaviour, properties and their chronological development with applications to the modern age are available in history from internet [[12], [19], [21], [22], [27], [30]].

From 1744, Leonhard Euler investigated integrals of the form,

$$Z = \int X(x)e^{ax}dx \text{ and } Z = \int X(x)x^{A}dx,$$

as solutions of differential equations but did not pursue the matter very far [13]. It is also known from history about Euler's study that he introduced the definite integral form,

$$Y(u) = \int_a^b e^{K(u)Q(x)}P(x)dx.$$

In particular, this expression appeared in Euler's 1768 Institutiones Calculi Integralis, Vol -II [14] where he used it to solve the equation

$$L\frac{d^2y}{du^2} + M\frac{dy}{du} + Ny = U(u), \text{ if } U(u) = R(a)e^{K(u)Q(a)}.$$

Joseph Louis Lagrange was an admirer of Euler and in his work on integrating probability density functions, he investigated expressions of the form:

$$\int Xe^{-ax}a^{x}dx \quad [[16], [17]].$$

These types of integrals seem first to have attracted Laplace's attention in 1782, in regard to using these integrals as solutions of equations [18] following the spirit of Euler. However, in 1785. Laplace took a critical step forward, not only looking for a solution in the form of an integral, but also started applying the transforms in the sense that was later to become popular. He used an integral of the form,

$$\int x^{s}\phi(x)dx,$$

akin to Melline transform, to transform difference equations [27]. In this light, another interesting transform draws attention from history due to K. Watugala [32], named as Sumudu transform which is similar to Laplace and Laplace-Carson transforms, very useful for solving ordinary linear differential equations in Control Engineering Problems. He introduced this in early 1990. "Sumudu" is a Sinhala word, meaning "smooth"; perhaps this name means that this transform is smooth in some sense in behavior. In 2008, Zafar Hayat Khan [23] introduced a very simple looking integral transform, known as N-transform. Interestingly, it converges to both Laplace and Sumudu transforms by changing variables. This transform inherits all the applied aspects of both Laplace and Sumudu transforms. Very recently, F.B.M. Belgacem [3] has renamed N-transform as Natural transform. Another integral transform introduced by Tarig M. Elzaki in 2011 [29] which has very deeper connection with Laplace and Sumudu transforms, known as Elzaki transform which is used for solving ordinary linear differential equations.

History depicts that Fourier, Laplace, Laplace-Carson, Melline, Sumudu, Elzaki and Normal transforms have many interconnections and can be changed from one to other by minor manipulations after changing variables. Further, these transforms have much similarities in regards to their properties and applications as each is used mostly in one or the other way to solve differential, integral and difference equations. Despite their similarities, they have individual importance in respect of their own nature, properties as well as applicability in practical problems and also have some major dissimilarities, Amidst the history of these integral transforms, another related integral transform draws attention to the researchers which is known as Polynomial integral transform, introduced by Benedict Bornes in 2016 [5]. Likewise Laplace and other integral transforms, this is also used to solve differential equations. This method transforms a linear differential equation into an algebraic equation, from which solution is obtained. This transform involves a polynomial kernel which ensures a rapid convergence of the solution of a differential equation. This brief historical development composed and presented here, is based on internet study for which sufficient references are being incorporated from internet, and cross-references, some of which have no direct access to the author and it is not the prime focus of the present study but a little passage to the ocean of integral transforms, and to gather information and knowledge about various forms of integral transforms with corresponding kernels.

The general integral transform of a function f in x transforms f to a new function g in u which has a known definition as under:

$$g(u) = T(f(x)) = \int_a^b K(x,u)f(x)dx, \tag{1.1}$$

where $K(x,u)$ is the kernel of the transformation. The convergence criterion of (1.1) is a very important study and it can be studied if the kernel $K(x,u)$ has a definite form. The present study is an attempt to express the kernel $K(x,u)$ with a definite shape in a general framework involving some parameters for studying its convergence in the general frame. With various choices of the parameters involved in the transform under study, it would yield some well-known integral transforms as special cases which are listed in the study and their convergences follow immediately from the convergence of the present transform under study. Possibly, some more integral transforms may be included as special cases of the present one. Also, some more integral transforms, may be new, not described and studied before (as not found in the history), come up under the present transform of study which are also listed for further study (not included here).

2. Main Result

Here we define a new generalized integral transform and study its convergence.

Definition-2.1 :

Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a piece-wise continuous function on every finite interval $[0,a]$, for $a > 0$ and be of exponential order $\sigma > 0$. Let us consider functions $\psi, \theta, \lambda : \mathbb{R}^+ \to \mathbb{R}$.

We define an integral transform F of the function f, named D-transform of f, as follows;

$$F(s,u) = D(f(x)) = \int_c^d k[\Psi(s)]^p e^{-q\Psi(s)\theta(x)}[\lambda(x)]^{s-1} f(xu)dx \tag{2.1}$$

where $k \in \mathbb{R}_0^+ = \mathbb{R}^+ \backslash \{0\}, p \in \mathbb{R}, q \in \mathbb{Z}, s, u \in \mathbb{R}$, u is a parameter, provided that this integral exists, where $c, d \in \mathbb{R}, \mathbb{R}_0^+ =$ set of positive real numbers.

We shall now study the existence of this integral by studying convergence taking $c = 0$ and $d \to \infty$ in the following :

Theorem A :

Let $f : \mathbb{R}^+ \to \mathbb{R}$ be such that
i) f is of exponential order $\sigma > 0$ where $\sigma = q\psi(c) > 0$ for $s = c \in \mathbb{R}$,
ii) f is piece-wise continuous on every finite interval $[0,a]$, for $a > 0$,
Then F exists for $k \in \mathbb{R}_0^+, p \in \mathbb{Z}, q \in \mathbb{N}, u > 0$ if
iii) θ is continuous, positive on $\mathbb{R}^+$ with bounded derivative;
iv) λ is continuous, increasing on $\mathbb{R}^+$ with $\lambda(x) = O(x^n), \forall x > a, n > 0$, and $\lambda(x) \geq Rx^t$ for $x < a, t \geq 1, R \in \mathbb{R}^+$, and
v) ψ is continuous and invertible for all $s \in \mathbb{R}^+$ with $\psi(0) > 0, \psi(s) > \psi(c) > 0$

Proof : We prove this theorem by cases as follows :

Let $a \in \mathbb{R}_0^+$ = set of positive real numbers.

Case 1 : Let us assume that $\lambda(x) \neq 1$.
Here $I = \int_0^\infty k[\psi(s)]^p e^{-q\psi(s)\theta(x)}[\lambda(x)]^{s-1} f(xu)dx = k[\psi(s)]^p\{I_1 + I_2\}$
where $I_1 = \int_0^a e^{-q\psi(s)\theta(x)}[\lambda(x)]^{s-1} f(xu)dx$, $I_2 = \int_a^\infty e^{-q\psi(s)\theta(x)}[\lambda(x)]^{s-1} f(xu)dx$.

Now $k, p, \psi(s) < +\infty \implies k[\psi(s)]^p < +\infty$.
If I_1 and I_2 are finite then I is finite, and so I and hence F exists.
Let us consider I_1 first. Note the following sub-cases :-

Sub-case i): $s \geq 1$

λ is continuous for all $x \in \mathbb{R}^+$, exponential function is continuous, so the integrand is continuous at $x = 0$, provided that f is continuous at $x = 0$ [precisely right continuous at $x = 0$];

Sub-case ii): $s < 1$

$\lambda(0) \neq 0 \implies \frac{f(xu)}{[\lambda(x)]^{1-s}}$ is finite and the integrand is continuous at $x = 0$;
$\lambda(0) = 0 \implies$ the integrand has an infinite discontinuity at $x = 0$ whenever $f(0) \neq 0$; otherwise $\frac{f(xu)}{[\lambda(x)]^{1-s}}$ being of $\frac{0}{0}$ form, can be made finite in limit as $x \to 0$ using La Hospital rule.

For Sub-case i), when $s \geq 1$:
f is piece-wise continuous on every finite interval $[0, a]$, for $a > 0$; $e^{-q\psi(s)\theta(x)}$ and λ are both continuous on $\mathbb{R}^+$ and hence piece-wise continuous on every finite interval $[0, a]$, $a > 0$. Therefore, the integrand is piece-wise continuous and hence integrable on every finite interval $[0, a]$, $a > 0$. So I_1 exists.

For Sub-case ii), when $s < 1$

Let $g(x) = e^{-q\psi(s)\theta(x)}[\lambda(x)]^{s-1} f(xu) = \frac{e^{-q\psi(s)\theta(x)} f(xu)}{[\lambda(x)]^{1-s}}$ and $h(x) = \frac{1}{[\lambda(x)]^{1-s}}$.

Now, $\frac{g(x)}{h(x)} = \frac{\frac{e^{-q\psi(s)\theta(x)} f(xu)}{[\lambda(x)]^{1-s}}}{\frac{1}{[\lambda(x)]^{1-s}}}$ [assuming $\lambda(x) \neq 0$ for any $x \in \mathbb{R}_0^+$]

$$\implies \lim_{x\to 0} \frac{g(x)}{h(x)} = \lim_{x\to 0} e^{-q\psi(s)\theta(x)} f(xu).$$

Now θ is continuous and positive on $\mathbb{R}^+$

$$\implies \lim_{x\to 0} \theta(x)$$

is finite. f is piece-wise continuous on every finite interval $[0, a]$, for $a > 0 \implies f(0+) = f(0) \neq 0$ and $f(0+)$ is finite for $u \in \mathbb{R}$.
Also, q and $\psi(s)$ are finite. So,

$$\lim_{x\to 0} e^{-q\psi(s)\theta(x)} f(xu) \neq 0$$

and finite for $s \in \mathbb{R}^+$

$$\implies \lim_{x\to 0} \frac{g(x)}{h(x)} \neq 0$$

and finite for all $s \in \mathbb{R}^+$.

Again, for all $x < a, \lambda(x) \geq Rx^t$, for $t \geq 1$
$\implies [\lambda(x)]^{1-s} \geq [Rx^t]^{1-s}$ (as $1-s>0$)$= R^{1-s}x^{t(1-s)}$
$\implies \frac{1}{[\lambda(x)]^{1-s}} \leq \frac{1}{R^{1-s}}.\frac{1}{x^{t(1-s)}} \implies \int_0^a \frac{dx}{[\lambda(x)]^{1-s}} \leq \int_0^a \frac{dx}{R^{1-s}x^{t(1-s)}} = \frac{1}{R^{1-s}}\int_0^a \frac{dx}{x^{t(1-s)}}$.

Now $\int_0^a \frac{dx}{x^{t(1-s)}} < +\infty$ if $t(1-s) < 1$ i.e. $s > 1 - \frac{1}{t} \geq 0$ for $t \geq 1$, and so $\int_0^a \frac{dx}{x^{t(1-s)}} < +\infty$ if $s > 1-\frac{1}{t}$ for $t \geq 1$ (as R^{1-s} is finite for $1-s>0$).
Consequently, I_1 exists finitely for $0 < s < 1$.

Next, we consider I_2.
f is of exponential order $\sigma > 0 \implies \exists M > 0 \ni |f(x)| \leq Me^{\sigma x}, \forall x > a$.
Here, we take $\sigma = q\psi(c)$ for some $s = c \in \mathbb{R}^+$ and choosing $\psi(c) > 0$, we have $\sigma > 0$ as $q \in \mathbb{N}$.
Also, $|f(x)| \leq Me^{q\psi(c)\theta(x)}$, $\forall x > a$.
So, for $u > 0$,

$$|f(xu)| \leq Me^{q\psi(c)\theta(xu)},\ for\ xu > a,\ x > a. \tag{2.2}$$

Now,

$$I_2 = \int_a^\infty e^{-q\psi(s)\theta(x)}[\lambda(x)]^{s-1}f(xu)dx = \lim_{\epsilon\to\infty}\int_a^\epsilon e^{-q\psi(s)\theta(x)}[\lambda(x)]^{s-1}f(xu)dx \tag{2.3}$$

Now, $|\int_a^\epsilon e^{-q\psi(s)\theta(x)}[\lambda(x)]^{s-1}f(xu)dx| \leq \int_a^\epsilon e^{-q\psi(s)\theta(x)}[\lambda(x)]^{s-1}|f(xu)|dx$

$\leq \int_a^\epsilon e^{-q\psi(s)\theta(x)}[\lambda(x)]^{s-1}M.e^{q\psi(c)\theta(xu)}dx$ [from (2.2)]

$$= M\int_a^\epsilon e^{-q[\psi(s)\theta(x)-\psi(c)\theta(xu)]}[\lambda(x)]^{s-1}dx \tag{2.4}$$

Now, $\psi(s)\theta(x) - \psi(c)\theta(xu) > \psi(c)\theta(x) - \psi(c)\theta(xu)$, if $\psi(s) > \psi(c)$

$= \psi(c)[\theta(x) - \theta(xu)] = \psi(c)\frac{\theta(x)-\theta(xu)}{(1-u)x}.|x - xu|$, for $u > 1$ or $u < 1$

$= \psi(c).|1-u|x[\theta'(x) + \delta]$, $\delta \to 0$ whenever $u \to 1$

$> \psi(c).|1-u|(h+\delta)x$ [θ has bounded derivative (from (iii)) $\implies \theta'$ is bounded $\implies \exists h, h \in \mathbb{R}^+ \ni h < \theta'(x) < h_1, \forall x \in \mathbb{R}^+$]

$= Ax$, where

$$A = \psi(c).|1-u|(h+\delta) > 0 \tag{2.5}$$

$$\implies e^{-q[\psi(s)\theta(x)-\psi(c)\theta(xu)]} < e^{-qAx} = e^{-Bx},\ [where\ B = qA > 0] \tag{2.6}$$

Also, for $x > a$,

$$\lambda(x) = O(x^n)\ (n > 0) \implies \exists m > 0 \ni \lambda(x) \leq mx^n,\ (as\ \lambda(x) > 0) \tag{2.7}$$

In view of (2.6) and (2.7), (2.4) reduces to

$$|\int_a^\epsilon e^{-q\psi(s)\theta(x)}[\lambda(x)]^{s-1}f(xu)dx| < M\int_a^\epsilon e^{-Bx}[mx^n]^{s-1}dx = Mm^{s-1}\int_a^\epsilon e^{-Bx}x^{n(s-1)}dx \tag{2.8}$$

Also, we can write from the series expansion of e^{Bx} as:

$e^{Bx} = 1 + Bx + \frac{B^2x^2}{2!} + ... > \frac{B^rx^r}{r!},\ r \geq 0$

$\implies e^{-Bx} < \frac{r!}{B^r}.\frac{1}{x^r}\ (for\ B > 0,\ r \geq 0)$.

With this (2.8) becomes

$|\int_a^{\epsilon} e^{-q\psi(s)\theta(x)}[\lambda(x)]^{s-1}f(xu)dx| < Mm^{s-1}|\int_a^{\epsilon}|\frac{r!}{B^r}.\frac{1}{x^r}.x^{n(s-1)}dx|$

$= \frac{M.m^{s-1}r!}{B^r}\int_a^{\epsilon}\frac{dx}{x^{r-n(s-1)}} = H\int_a^{\epsilon}\frac{dx}{x^{r-n(s-1)}}$, where $H = \frac{M.m^{s-1}r!}{B^r} > 0$

$$\implies I_2 \leq |I_2| = |\int_a^{\infty} e^{-q\psi(s)\theta(x)}[\lambda(x)]^{s-1}f(xu)dx| = \lim_{\epsilon\to\infty}|\int_a^{\epsilon} e^{-q\psi(s)\theta(x)}[\lambda(x)]^{s-1}f(xu)dx|$$

$$< H\lim_{\epsilon\to\infty}\int_a^{\epsilon}\frac{dx}{x^{r-n(s-1)}}.$$

Now, $\int_a^{\epsilon}\frac{dx}{x^{r-n(s-1)}}$ is convergent if $r-n(s-1) > 1$ i.e. $r > 1+n(s-1) > 0$ for any

$s > 1-\frac{1}{n} > 0$, for $n > 0$ and it is finite.

So I_2 exists finitely for $\psi(s)\psi(c) > 0$ and $r > 1+s(n-1)$ for any $s \in \mathbb{R}^+, s > 1-\frac{1}{n} > 0, n > 0$ i.e for any s with $0 < s < 1, n > 0$.

Consequently, I exists finitely and hence F exists finitely for $\psi(s)\psi(c) > 0, s, u \in \mathbb{R}^+, B = qA = q\psi(c)|1-u| > 0$ when $\lambda(x) \neq 1$.

Case-2 : Let us assume $\lambda(x) = 1$.

Then $F(s,u) = \int_0^{\infty} k[\psi(s)]^p e^{-q\psi(s)\theta(x)}f(xu)dx = k[\psi(s)]^p\int_0^{\infty} e^{-q\psi(s)\theta(x)}f(xu)dx$

$= k[\psi(s)]^p[I_3+I_4]$ where $I_3 = \int_0^a e^{-q\psi(s)\theta(x)}f(xu)dx$ and $I_4 = \int_a^{\infty} e^{-q\psi(s)\theta(x)}f(xu)dx$.

First we shall consider I_3.

Here, we note that f is piece-wise continuous on every finite interval $[0,a]$, for $a > 0$, and $e^{-q\psi(s)\theta(x)}$ is continuous and hence piece-wise continuous on every finite interval $[0,a]$, for $a > 0$. Therefore, the integrand is piece-wise continuous on every finite interval $[0,a]$, for $a > 0$. So I_3 exists finitely.

Next, we shall consider I_4. Here, as in Case-1,

$$I_4 \leq |I_4| = \int_a^{\infty} e^{-q\psi(s)\theta(x)}f(xu)dx < H\lim_{\epsilon\to\infty}\int_a^{\epsilon} e^{-Bx}dx$$

[following from I_2 of Case-1]

$$= H\lim_{\epsilon\to\infty}[\frac{e^{-Bx}}{-B}]_a^{\epsilon} = H\lim_{\epsilon\to\infty}[\frac{e^{-B\epsilon}-e^{-Ba}}{-B}] = \frac{H}{B}e^{-Ba} < +\infty$$

$\implies I_4$ exists finitely if $\psi(c) > 0$ with $\psi(s) > \psi(c)$ for any $s \in \mathbb{R}^+$.

As $k[\psi(s)]^p$ is finite for any $s \in \mathbb{R}^+$, hence F exists finitely for $\psi(s) > \psi(c) > 0, s, u \in \mathbb{R}^+$ and u is a parameter.
Hence, in any case (2.1) is finite and it exists. This finishes the proof.
Now, we note that if we assign different values to $\psi, \theta, \lambda, k, p, q$ and u then (2.1) yields some well-known integral transforms which are listed below in tabular form, whose convergenes also follow very easily from Theorem-A, as discussed below briefly in appendix:

3. Appendix

Table 1

Sl	Choice of components	Transforms	Name
1	$\theta(x) = x, \psi(s) = s, \lambda(x) = 1,$ $p = 0, q = 1 = k, u = 1$	$F(s) = \int_0^\infty e^{-sx} f(x)dx$	Laplace Transform [Laplace]
2	$\theta(x) = x, \psi(s) = s, \lambda(x) = 1,$ $p = 1 = q = k, u = 1$	$F(s) = \int_0^\infty se^{-sx} f(x)dx$	Laplace-Carson Transform [Laplace-Carson]
3	$\theta(x) = x, \psi(s) = \frac{1}{s}, \lambda(x) = 1,$ $p = 1 = q = k, u = 1$	$F(s) = \int_0^\infty \frac{1}{s} e^{-\frac{x}{s}} f(x)dx$	Sumudu Transform [Watugala]
4	$\theta(x) = x, \psi(s) = \frac{1}{s}, \lambda(x) = 1,$ $p = -1, q = 1 = k, u = 1$	$F(s) = \int_0^\infty se^{-\frac{x}{s}} f(x)dx$	Elzaki Transform [Elzaki]
5	$\theta(x) = x, \psi(s) = s, \lambda(x) = 1,$ $p = 0, q = 1 = k$	$F(s) = \int_0^\infty e^{-sx} f(xu)dx$	Natural Transform [F. B. M. Beglacem, before it is N. Transform due to Z. H. Khan]
6	$\theta(x) = x, \psi(s) = 1, \lambda(x) = x,$ $p = 1 = k, q = 0, u = 1$	$F(s) = \int_0^\infty x^{s-1} f(x)dx$	Mellin Transform [Mellin]
7	$\theta(x) = x, \psi(s) = \frac{1}{s^2}, \lambda(x) = 1,$ $p = \frac{1}{2}, q = 1 = k, u = 1$	$F(s) = \frac{1}{s} \int_0^\infty se^{-\frac{x}{s^2}} f(x)dx$	Integral Transform [Artion K Shuri and Akli Fundo]
8	$\theta(x) = x, \psi(s) = \frac{1}{s}, \lambda(x) = 1,$ $p = \alpha, q = 1 = k, u = 1$	$F(s) = G(f)$ $= s^\alpha \int_0^\infty e^{-\frac{x}{s^2}} f(x)dx$	Laplace-typed integral Transform [H. Kim]
9	$\theta(x) = x, \psi(s) = \frac{1}{s}, \lambda(x) = 1,$ $p = 2, q = 1 = k, u = 1$	$G_{-2}(f) = \frac{1}{s^2} \int_0^\infty e^{-\frac{x}{s}} f(x)dx$	Integral Transform [H. Kim]
10	$\theta(x) = x = lnx, \psi(s) = s,$ $\lambda(x) = 1, p = 0, q = 1 = k, u = 1$	$F(s) = \int_1^\infty x^{-(s+1)} f(lnx)dx$	Polynomial integral Transform [Benedict Barnes]

Note: 1

In Table 1, Sl. (10) can be easily obtained from (1) by replacing x by lnx.

Note: 2

Clearly, Sl.(1) to (5) and (7) to (10) satisfy conditions of Theorem-A, so their convergences follow immediately from the Theorem-A. For convergence of Sl.(6), we note the following:
θ and ψ clearly satisfy conditions of Theorem-A. Only it remains to verify for λ.
Here $\lambda(x) = x$. Let $a \in \mathbb{R}_0^+$. Now, $x > a \implies \frac{1}{x} < \frac{1}{a} \implies \frac{1}{x^{n-1}} < \frac{1}{a}(n > 1)$
$\implies \frac{x}{x^n} < \frac{1}{a}(n > 1) \implies x < \frac{1}{a}.x^n(n > 1)$
Now, choose $K > 0$ such that $\frac{1}{a} < K$. So $x < Kx^n(n > 1 > 0) \implies x = O(x^n)$ for $x > a, n > 0$.
For, $x < a, x < a < ax^t(t \geq 1), x > 0$.
For $x, ax^t \in \mathbb{R}^+, x < ax^t, \exists N \in \mathbb{N} \ni Nx > ax^t$[by Archimedean Property] $\implies x > (\frac{a}{N})x^t$.

Choose $R = \frac{a}{N} > 0$ then $x > Rx^t(t \geq 1)$.
Thus $\lambda(x) = x \implies x = O(x^n)$ for $x > a, (n > 0)$ and $x > Rx^t$ for $x < a, (t \geq 1)$.
This shows that λ also satisfies the condition of Theorem-A. Therefore, convergence of Mellin's transform follows.

Note: 3

As is revealed from history that with different choices of $\theta, \psi, \lambda, p, q, k, u$, some different integral transforms may appear which have not yet been studied. For example, some may be listed in Table-2, which could be studied in respect of properties and applications further (not studied in the present context):

Table 2

Sl	Choice of components	Integral Transforms
1	$\theta(x)=x, \psi(s)=s, \lambda(x)=1,$ $p=-1, q=1=k$	$F(s,u)=\int_0^\infty \frac{1}{s}e^{-sx}f(xu)dx; [F(s)=\int_0^\infty \frac{1}{s}e^{-sx}f(x)dx.[u=1]]$
2	$\theta(x)=x, \psi(s)=\frac{1}{s}, \lambda(x)=1,$ $p=0, q=1=k$	$F(s,u)=\int_0^\infty e^{-\frac{x}{s}}f(xu)dx; [F(s)=\int_0^\infty e^{-\frac{x}{s}}f(x)dx.[u=1]]$
3	$\theta(x)=x, \psi(s)=\frac{1}{s}, \lambda(x)=x,$ $p=1=q=k$	$F(s,u)=\int_0^\infty se^{-\frac{x}{s}}x^{s-1}f(xu)dx; [F(s)=\int_0^\infty se^{-\frac{x}{s}}x^{s-1}f(x)dx.$ $[u=1]]$
4	$\theta(x)=x, \psi(s)=\frac{1}{s}, \lambda(x)=x,$ $p=1, q=-1=k$	$F(s,u)=\int_0^\infty se^{-\frac{x}{s}}x^{s-1}f(xu)dx; [F(s)=\int_0^\infty se^{-\frac{x}{s}}x^{s-1}f(x)dx.$ $[u=1]]$
5	$\theta(x)=x, \psi(s)=s, \lambda(x)=x,$ $p=-1, q=1=k$	$F(s,u)=\int_0^\infty \frac{1}{s}e^{-sx}x^{s-1}f(xu)dx; [F(s)=\int_0^\infty \frac{1}{s}e^{-sx}x^{s-1}f(x)dx.$ $[u=1]]$
6	$\theta(x)=x, \psi(s)=s, \lambda(x)=x,$ $p=0, q=1=k$	$F(s,u)=\int_0^\infty e^{-sx}x^{s-1}f(xu)dx; [F(s)=\int_0^\infty e^{-sx}x^{s-1}f(x)dx.$ $[u=1]]$
7	$\theta(x)=x, \psi(s)=\frac{1}{s}, \lambda(x)=x,$ $p=0, q=1=k$	$F(s,u)=\int_0^\infty e^{-\frac{x}{s}}x^{s-1}f(xu)dx; [F(s)=\int_0^\infty e^{-\frac{x}{s}}x^{s-1}f(x)dx.$ $[u=1]]$

Note: 4

In (2.1) if we assume $c \to -\infty$ and $d \to \infty$, f is periodic on every finite interval having some finite period, say T, and $\psi(s)$ is purely imaginary with $Im\psi(s) \equiv mod(s)$, i.e. $Im\psi(s) = rs$ for some $r \in \mathbb{R}^+$, then expression in Sl. (2) gives Fourier transform of $f(xu)$ with parameter u, with the choices, $\theta(x) > 0, \forall x \in \mathbb{R}, \theta(x) = x, \lambda(x) = 1, p = 0, q = 1, k = \frac{1}{2\pi}$. If $u = 1$, then expression in (2) becomes Fourier transform of $f(x)$ with the above choices.

4. Conclusion

Theorem-A together with Note-4 gives a strong convergence condition for the existence of many well-known integral transforms mentioned above in Table-I including Dirichlet's condition for Fourier Transform.

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