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Fekete-Szegö Problem for a Subclass of Analytic Functions Associated with Chebyshev Polynomials

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ABSTRACT: In this paper, we obtain initial coefficients $|a_2|$ and $|a_3|$ for a certain subclass by means of Chebyshev polynomials expansions of analytic functions in $\mathcal{D}$. Also, we solve Fekete-Szegö problem for functions in this subclass.

Key Words: Analytic and univalent functions, Subordination, Coefficient bounds, Chebyshev polynomial, Fekete-Szegö problem.

Contents

1. Introduction

Let $\mathcal{A}$ be the class of the functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{D} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and satisfying the conditions $f(0) = 0$ and $f'(0) = 1$. Also, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of the form (1.1) which are univalent in $\mathcal{D}$.

Let f and g be analytic functions in $\mathcal{D}$. We define that the function f is subordinate to g in $\mathcal{D}$ and denoted by

$$f(z) \prec g(z) \qquad (z \in \mathcal{D}),$$

if there exists a Schwarz function ω, which is analytic in $\mathcal{D}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathcal{D})$ such that

$$f(z) = g(\omega(z)) \qquad (z \in \mathcal{D}).$$

If g is a univalent function in $\mathcal{D}$, then

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathcal{D}) \subset g(\mathcal{D}).$$

Chebyshev polynomials play a considerable role in numerical analysis ([4], [8]). There are four kinds of Chebyshev polynomials. The first and second kinds of Chebyshev polynomials are defined by $T_n(t) = cosn\varphi$ and $U_n(t) = \frac{\sin(n+1)\varphi}{\sin\varphi}$ $(-1 < t < 1)$ where n denotes the polynomial degree and $t = \cos\varphi$. For a brief history of Chebyshev polynomials of the first kind $T_n(t)$, the second kind $U_n(t)$ and their applications one can refer [1]-[16].

Now, we define a subclass of analytic functions in $\mathcal{D}$ with the following subordination condition:

Definition 1.1. *A function $f \in \mathcal{A}$ given by (1.1) is said to be in the class $\mathcal{N}(\lambda, \beta, t)$ for $0 \leq \beta \leq \lambda \leq 1$ and $t \in \left(\frac{1}{2}, 1\right]$ if the following subordination hold:*

$$\frac{\lambda\beta z^3 f'''(z) + (2\lambda\beta + \lambda - \beta)z^2 f''(z) + zf'(z)}{\lambda\beta z^2 f''(z) + (\lambda - \beta)zf'(z) + (1 - \lambda + \beta)f(z)} \prec H(z, t) = \frac{1}{1 - 2tz + z^2} \qquad (z \in \mathcal{D}). \tag{1.2}$$

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We consider that if $t = cos\varphi$ $\left(\frac{-\pi}{3} < \varphi < \frac{\pi}{3}\right)$, then $H(z,t) = \frac{1}{1-2\cos\varphi z+z^2} = 1 + \sum_{n=1}^{\infty} \frac{\sin(n+1)\varphi}{\sin\varphi} z^n$ $(z \in \mathcal{D})$. Thus, $H(z,t) = 1 + 2\cos\varphi z + \left(3\cos^2\varphi - \sin^2\varphi\right) z^2 + \cdots$ $(z \in \mathcal{D})$.

So, according to [15], we write the Chebyshev polynomials of the second kind as following:

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \cdots \qquad (z \in \mathcal{D}, -1 < t < 1)$$

where $U_{n-1}(t) = \frac{\sin(n\arccos t)}{\sqrt{1-t^2}}$ $(n \in \mathbb{N})$ and we have $U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t)$,

$$U_1(t) = 2t, \quad U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t, \quad U_4(t) = 16t^4 - 12t^2 + 1, \cdots . \tag{1.3}$$

The Chebyshev polynomials $T_n(t)$, $t \in [-1,1]$ of the first kind have the generating function of the form $\sum_{n=0}^{\infty} T_n(t)z^n = \frac{1-tz}{1-2tz+z^2}$ $(z \in \mathcal{D})$.

There is the following connection by the Chebyshev polynomials of the first kind $T_n(t)$ and the second kind $U_n(t)$:

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t), \qquad T_n(t) = U_n(t) - tU_{n-1}(t), \qquad 2T_n(t) = U_n(t) - U_{n-2}(t).$$

In 1933, Fekete and Szegö [6] obtained a sharp bound of the functional $|a_3 - \mu a_2^2|$, with real μ $(0 \le \mu \le 1)$ for a univalent function f. Since then, the problem of finding the sharp bounds for this functional of any compact family of functions or $f \in \mathcal{A}$ with any complex μ is known as the classical Fekete-Szegö problem or inequality.

In this paper, we obtain initial coefficients $|a_2|$ and $|a_3|$ for subclass $\mathcal{N}(\lambda,\beta,t)$ by means of Chebyshev polynomials expansions of analytic functions in $\mathcal{D}$. Also, we solve Fekete-Szegö problem for functions in this subclass.

2. Coefficient bounds for the function class $\mathcal{N}(\lambda,\beta,t)$

We begin with the following result involving initial coefficient bounds for the function class $\mathcal{N}(\lambda,\beta,t)$.

Theorem 2.1. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{N}(\lambda,\beta,t)$. Then*

$$|a_2| \le \frac{2t}{2\lambda\beta + \lambda - \beta + 1} \tag{2.1}$$

and

$$|a_3| \le \frac{8t^2 - 1}{2(6\lambda\beta + 2\lambda - 2\beta + 1)}. \tag{2.2}$$

Proof. Let $f \in \mathcal{N}(\lambda,\beta,t)$. From (1.2), we have

$$\frac{\lambda\beta z^3 f'''(z) + (2\lambda\beta + \lambda - \beta)z^2 f''(z) + zf'(z)}{\lambda\beta z^2 f''(z) + (\lambda - \beta)zf'(z) + (1 - \lambda + \beta)f(z)} = 1 + U_1(t)\,p(z) + U_2(t)\,p^2(z) + \cdots \tag{2.3}$$

for some analytic functions

$$p(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots \qquad (z \in \mathcal{D}), \tag{2.4}$$

such that $p(0) = 0$, $|p(z)| < 1$ $(z \in \mathcal{D})$. Then, for all $j \in \mathbb{N}$,

$$|c_j| \le 1 \tag{2.5}$$

and for all $\mu \in \mathbb{R}$

$$\left|c_2 - \mu c_1^2\right| \le \max\{1, |\mu|\}. \tag{2.6}$$

It follows from (2.3) that

$$\frac{\lambda\beta z^3 f'''(z) + (2\lambda\beta + \lambda - \beta)z^2 f''(z) + zf'(z)}{\lambda\beta z^2 f''(z) + (\lambda - \beta)zf'(z) + (1 - \lambda + \beta)f(z)} = 1 + U_1(t)\,c_1 z + \left[U_1(t)\,c_2 + U_2(t)\,c_1^2\right] z^2 + \cdots . \tag{2.7}$$

It follows from (2.7) that

$$(2\lambda\beta+\lambda-\beta+1)\, a_2 = U_1(t)\, c_1, \tag{2.8}$$

and

$$2(6\lambda\beta+2\lambda-2\beta+1)\, a_3 - (2\lambda\beta+\lambda-\beta+1)^2 a_2^2 = U_1(t)\, c_2 + U_2(t)\, c_1^2. \tag{2.9}$$

From (1.3), (2.8) and (2.5), we have

$$|a_2| \leq \frac{2t}{2\lambda\beta+\lambda-\beta+1}. \tag{2.10}$$

By using (1.3) and (2.5) in (2.9), we obtain

$$|a_3| \leq \frac{8t^2-1}{2(6\lambda\beta+2\lambda-2\beta+1)}. \tag{2.11}$$

which completes the proof of Theorem 2.1. □

For $\lambda = 1$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.2. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{N}(1,\beta,t)$. Then*

$$|a_2| \leq \frac{2t}{\beta+2}$$

and

$$|a_3| \leq \frac{8t^2-1}{2(4\beta+3)}.$$

If we choose $\beta = 0$ in Theorem 2.1, we get the following corollary.

Corollary 2.3. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{N}(\lambda,0,t)$. Then*

$$|a_2| \leq \frac{2t}{\lambda+1}$$

and

$$|a_3| \leq \frac{8t^2-1}{2(2\lambda+1)}.$$

For $\beta = \lambda$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.4. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{N}(\beta,t)$. Then*

$$|a_2| \leq \frac{2t}{2\beta^2+1}$$

and

$$|a_3| \leq \frac{8t^2-1}{2(6\beta^2+1)}.$$

Remark 2.5. *For $\beta = 0$ and $\lambda = 1$ in Theorem 2.1, we obtain result of Dziok et al. [5, Theorem 6].*

3. Fekete-Szegö inequality for the function class $\mathcal{N}(\lambda, \beta, t)$

Now, we find the sharp bounds of Fekete-Szegö functional $\left|a_3 - \mu a_2^2\right|$ defined for $\mathcal{N}(\lambda, \beta, t)$.

Theorem 3.1. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{N}(\lambda, \beta, t)$. Then for some $\mu \in \mathbb{R}$,*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{t}{6\lambda\beta+2\lambda-2\beta+1}, & \mu \in [\mu_1, \mu_2], \\ \frac{t}{6\lambda\beta+2\lambda-2\beta+1}\left|\frac{8t^2-1}{2t} - \mu\frac{4t(6\lambda\beta+2\lambda-2\beta+1)}{(2\lambda\beta+\lambda-\beta+1)^2}\right|, & \mu \notin [\mu_1, \mu_2], \end{cases} \tag{3.1}$$

where $\mu_1 = \frac{(8t^2-2t-1)(2\lambda\beta+\lambda-\beta+1)^2}{8t^2(6\lambda\beta+2\lambda-2\beta+1)}$ and $\mu_2 = \frac{(8t^2+2t-1)(2\lambda\beta+\lambda-\beta+1)^2}{8t^2(6\lambda\beta+2\lambda-2\beta+1)}$.

Proof. Let $f \in \mathcal{N}(\lambda, \beta, t)$. By using (2.8) and (2.9) for some $\mu \in \mathbb{R}$, we have

$$\left|a_3 - \mu a_2^2\right| = \frac{U_1(t)}{2(6\lambda\beta+2\lambda-2\beta+1)}\left|c_2 + \left\{\frac{U_2(t)}{U_1(t)} + U_1(t) - 2\mu\frac{(6\lambda\beta+2\lambda-2\beta+1)U_1(t)}{(2\lambda\beta+\lambda-\beta+1)^2}\right\}c_1^2\right|. \tag{3.2}$$

Then, in view of (2.6), we conclude that

$$\left|a_3 - \mu a_2^2\right| \leq \frac{U_1(t)}{2(6\lambda\beta+2\lambda-2\beta+1)}\max\left\{1, \left|\frac{U_2(t)}{U_1(t)} + U_1(t) - 2\mu\frac{(6\lambda\beta+2\lambda-2\beta+1)U_1(t)}{(2\lambda\beta+\lambda-\beta+1)^2}\right|\right\}. \tag{3.3}$$

Finally, by using (1.3) in (3.3), we get

$$\left|a_3 - \mu a_2^2\right| \leq \frac{t}{6\lambda\beta+2\lambda-2\beta+1}\max\left\{1, \left|\frac{8t^2-1}{2t} - 4\mu\frac{(6\lambda\beta+2\lambda-2\beta+1)t}{(2\lambda\beta+\lambda-\beta+1)^2}\right|\right\}.$$

Because $t > 0$, we obtain

$$\left|\frac{8t^2-1}{2t} - 4\mu\frac{(6\lambda\beta+2\lambda-2\beta+1)t}{(2\lambda\beta+\lambda-\beta+1)^2}\right| \leq 1$$

$$\Leftrightarrow \left\{\frac{(8t^2-2t-1)(2\lambda\beta+\lambda-\beta+1)^2}{8t^2(6\lambda\beta+2\lambda-2\beta+1)} \leq \mu \leq \frac{(8t^2+2t-1)(2\lambda\beta+\lambda-\beta+1)^2}{8t^2(6\lambda\beta+2\lambda-2\beta+1)}\right\}$$
$$\Leftrightarrow \mu_1 \leq \mu \leq \mu_2.$$

This proves Theorem 3.1. □

For $\lambda = 1$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{N}(1, \beta, t)$. Then for some $\mu \in \mathbb{R}$,*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{t}{4\beta+3}, & \mu \in [\mu_1, \mu_2], \\ \frac{t}{4\beta+3}\left|\frac{8t^2-1}{2t} - \mu\frac{4t(4\beta+3)}{(\beta+2)^2}\right|, & \mu \notin [\mu_1, \mu_2], \end{cases}$$

where $\mu_1 = \frac{(8t^2-2t-1)(\beta+2)^2}{8t^2(4\beta+3)}$ and $\mu_2 = \frac{(8t^2+2t-1)(\beta+2)^2}{8t^2(4\beta+3)}$.

If we choose $\beta = 0$ in Theorem 3.1, we get the following corollary.

Corollary 3.3. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{N}(\lambda, 0, t)$. Then for some $\mu \in \mathbb{R}$,*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{t}{2\lambda+1}, & \mu \in [\mu_1, \mu_2], \\ \frac{t}{2\lambda+1}\left|\frac{8t^2-1}{2t} - \mu\frac{4t(2\lambda+1)}{(\lambda+1)^2}\right|, & \mu \notin [\mu_1, \mu_2], \end{cases}$$

where $\mu_1 = \frac{(8t^2-2t-1)(\lambda+1)^2}{8t^2(2\lambda+1)}$ and $\mu_2 = \frac{(8t^2+2t-1)(\lambda+1)^2}{8t^2(2\lambda+1)}$.

For $\beta = \lambda$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.4. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{N}(\beta, t)$. Then for some $\mu \in \mathbb{R}$,*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{t}{6\beta^2+1}, & \mu \in [\mu_1, \mu_2], \\ \frac{t}{6\beta^2+1}\left|\frac{8t^2-1}{2t} - \mu\frac{4t\left(6\beta^2+1\right)}{\left(2\beta^2+1\right)^2}\right|, & \mu \notin [\mu_1, \mu_2], \end{cases}$$

where $\mu_1 = \frac{\left(8t^2-2t-1\right)\left(2\beta^2+1\right)^2}{8t^2\left(6\beta^2+1\right)}$ *and* $\mu_2 = \frac{\left(8t^2+2t-1\right)\left(2\beta^2+1\right)^2}{8t^2\left(6\beta^2+1\right)}$

Remark 3.5. *For $\beta = 0$ in Theorem 3.1, we obtain result of Mustafa and Akbulut [10].*

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