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Spectral Properties of Non-Self-Adjoint Elliptic Differential Operators in Hilbert Space*

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ABSTRACT: Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$. In this article, we will investigate the spectral properties of a non-self adjoint elliptic differential operator $(Au)(x) = -\sum_{i,j=1}^{n} \omega^{2\alpha}(x) a_{ij}(x)\mu(x)u'_{x_i}(x))'_{x_j}$, acting in the Hilbert space $H = L^2(\Omega)$.with Dirichlet-type boundary conditions. Here $a_{ij}(x) = \overline{a_{ji}(x)} \quad (i,j = 1,\dots,n), \quad a_{ij}(x) \in C^2(\overline{\Omega})$, and the functions $a_{ij}(x)$ satisfies the uniformly elliptic condition, and let $0 \le \alpha < 1$. Furthermore, for $\forall x \in \overline{\Omega}$, the function $\mu(x)$ lie in the $\psi_{\theta_1\theta_2}$, where $\psi_{\theta_1\theta_2} = \{z \in \mathbf{C} : \ \pi/2 < \theta_1 \le |arg\ z| \le \theta_2 < \pi\}$.

Key Words: Resolvent, Asymptotic spectrum, Eigenvalue, Non-selfadjoint elliptic differential operator.

Contents

1. Introduction

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$ (i.e., $\partial\Omega \in C^\infty$). In this article, we investigate the spectral properties of a non-self adjoint elliptic differential operator

$$(Au)(x) = -\sum_{i,j=1}^{n} \left(\omega^{2\alpha}(x) a_{ij}(x)\mu(x)u'_{x_i}(x)\right)'_{x_j},$$

acting in the Hilbert space $H = L^2(\Omega)$ with Dirichlet-type boundary conditions, indeed this paper was written in continuing of earlier our papers, the functions $a_{ij}(x)$ satisfy the uniformly elliptic condition, i.e, there exist a fix positive number c such that: $c|s|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)s_i\overline{s_j} \quad (s = (s_1,\dots,s_n) \in \mathbf{C}^n, \ \ x \in \Omega)$, and we have $0 \le \alpha < 1$. Furthermore, let for $\forall x \in \overline{\Omega}$, let the function $\mu(x)$ lie in the sector $\psi_{\theta_1\theta_2}$, where $\psi_{\theta_1\theta_2} = \{z \in \mathbf{C} : \ \pi/2 < \theta_1 \le |arg\ z| \le \theta_2 < \pi\}$, for $\forall x \in \overline{\Omega}$.
More explanation on this research paper: This paper is in continuing of our earlier research papers. Here some important notes that is necessary to mention on this paper, to prove the existence of the resolvent of an operator is very important, in special if this operator as our operator A to be non self adjoint ones. Notice for self adjoint operator, it is easy to find its resolvent for example if our operator A to be self adjoint then we easily can find a complex number λ in $\mathbf{C}$ such that the following estimate: $\|(A-\lambda I)^{-1}\| \le M|\lambda|^{-1}$. but for a non self adjoint operator that our operator A is a non self adjoint ones, it is more difficult to find such λ in $\mathbf{C}$ that having the following estimate : $\|(A-\lambda I)^{-1}\| \le M|\lambda|^{-1}$. In this paper our operator is a non self adjoint ones and we will find its resolvent. The functions $a_{ij}(x)$ satisfies the uniformly elliptic condition, i.e., there exists $c > 0$ such that:

$$c|s|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)s_i\overline{s_j} \quad (s = (s_1,\dots,s_n) \in \mathbf{C}^n, \quad x \in \Omega).$$

Furthermore, for each $x \in \overline{\Omega}$, the function $\mu(x)$ lie in the $\psi_{\theta_1\theta_2}$, where $\psi_{\theta_1\theta_2} = \{z \in \mathbf{C} : \ \pi/2 < \theta_1 \le |arg\ z| \le \theta_2 < \pi\}$. With respect to define the operator;

$$(Au)(x) = -\sum_{i,j=1}^{n} \left(\omega^{2\alpha}(x) a_{ij}(x)\mu(x)u'_{x_i}(x)\right)'_{x_j},$$

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that acting in the Hilbert space $H = L^2(\Omega)$. Here we closed extension the operator A with respect to the weighted Sobolev space $\mathcal{H} = W^2_{2,\alpha}(\Omega)$, then here we extend the domain of operator A from the domain in the Hilbert space $H = L^2(\Omega)$ to the close domain $\overset{\circ}{\mathcal{H}} \subset \mathcal{H} = W^2_{2,\alpha}(\Omega)$. therefore;

$$D(A) = \{u \in \overset{\circ}{\mathcal{H}}\colon \sum_{i,j=1}^{n} (\omega^{2\alpha} a_{ij} \mu u'_{x_i})'_{x_j} \in H\},$$

(see [8])

2. On the Resolvent Estimate of the Differential Operator in H

Theorem 2.1. As in Section 1, let the differential operator

$$(Au)(x) = -\sum_{i,j=1}^{n} \left(\omega^{2\alpha}(x) a_{ij}(x) \mu(x) u'_{x_i}(x)\right)'_{x_j}$$

acting in the Hilbert space $H = L^2(\Omega)$ with Dirichlet-type boundary conditions. Then for sufficiently large in modulus $\lambda \in \psi_{\theta_1\theta_2}$, The inverse operator $(A - \lambda I)^{-1}$ exists and is continuous in H, and also the resolvent of the operator A exist, i.e, the following estimates is valid

$$\|(A - \lambda I)^{-1}\| \leq M|\lambda|^{-1} \ (\ |\lambda| > C_{\psi_{\theta_1\theta_2}}, \quad \lambda \in \ \psi_{\theta_1\theta_2}). \tag{2.1}$$

Proof of Theorem 2.1. Let

$$(Au)(x) = -\sum_{i,j=1}^{n} \left(\omega^{2\alpha}(x) a_{ij}(x) \mu(x) u'_{x_i}(x)\right)'_{x_j},$$

to having as above conditions then we will have the following estimate

$$\|(A - \lambda I)^{-1}\| \leq M|\lambda|^{-1} \ (\ |\lambda| > C_{\psi_{\theta_1\theta_2}}, \quad \lambda \in \ \psi_{\theta_1\theta_2}), \tag{2.2}$$

To prove the assertion of Theorem 2.1. Now, we start to prove the following estimate:

$$\|(A\lambda I)^{-1}\| \leq M|\lambda|^{-1} \ (\ |\lambda| > C_{\psi_{\theta_1\theta_2}}),$$

for sufficiently large in modulus $\lambda \in \psi_{\theta_1\theta_2}$, Let $\mu(x)$ lie in the sector $\psi_{\theta_1\theta_2}$, where $\psi_{\theta_1\theta_2} = \{z \in \mathbf{C} : \pi/2 < \theta_1 \leq |arg\ z| \leq \theta_2 < \pi\}$, Therefore to estimate the resolvent of the operators A, by the below observations the estimates of the resolvent of A are obtained as follows: since $\mu(x)$ lie in the closed sector $\in \psi_{\theta_1\theta_2}$ we assume that the angles between the oscillation of variation of the functions $\mu(x)$ is less than $\frac{\pi}{8}$, i.e., we will have

$$\arg\{\mu(x_1)\mu^{-1}(x_2)\}| < \frac{\pi}{8},\ x_1,\ x_2 \in \overline{\Omega}). \tag{2.3}$$

As above by applying the function $\mu(x)$, as above we have

$$(Au)(x) = -\sum_{i,j=1}^{n} \left(\omega^{2\alpha}(x) a_{ij}(x) \mu(x) u'_{x_i}(x)\right)'_{x_j},$$

now by using the weighted Sobolve space we must closed extend their domains $D(A)$ as follows:

$$D(A) = \{u \in \overset{\circ}{\mathcal{H}}; \sum_{i,j=1}^{n} \left(\omega^{2\alpha} a_{ij} \mu u'_{x_i}\right)'_{x_j} \in H\}.$$

base on (2.3), there exists a complex number $z \in C$ such that $z = e^{i\gamma}$, for a fix real $\gamma \in (-\pi, \pi])$, such that: $|z = e^{i\gamma}| = 1$, and so

$$c' \leq Re\{z\mu(x)\},\ \ c'|\lambda| \leq -Re\{z\lambda\}, \quad c' > 0\ (\forall\ x \in \overline{\Omega},\ \lambda \in \psi_{\theta_1\theta_2}). \tag{2.4}$$

Now by using the uniformly elliptic condition, we have

$$c|s|^2 = c\sum_{i=1}^{n}|s_i|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)s_i\overline{s_j},\ (c>0,\ s=(s_1,\ldots,s_n)\in \mathbf{C}^n,\ x\in\Omega)$$

if instead of s_i we take $s_i = u'_{x_i}$ in the latter relation we implies that

$$c\sum_{i=1}^{n}|u'_{x_i}(x)|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)u'_{x_i}(x)\overline{u'_{x_j}(x)}.$$

From this, and according to $c' \leq Re\{z\mu(x)\}$ in (2.4), then we multiply these two positive relations with each other implies that

$$c_1\sum_{i=1}^{n}|u'_{x_i}(x)|^2 \leq Rez\mu(x)\sum_{i,j=1}^{n} a_{ij}(x)u'_{x_i}(x)\overline{u'_{x_j}(x)}.\text{for } u\in D(A)$$

Multiply both sides of the latter relation by the positive term $\omega^{2\alpha}(x)$, and then integrate from both sides, we will have

$$c_1\sum_{i=1}^{n}\int_\Omega \omega^{2\alpha}(x)|u'_{x_i}(x)|^2dx \leq Rez\sum_{i,j=1}^{n}\int_\Omega \omega^{2\alpha}(x)a_{ij}(x)\mu(x)u'_{x_i}(x)\overline{u'_{x_j}(x)}dx.$$

Now by applying the integration by parts, and using Dirichlet-type condition, then the right sides of the latter relation without multiple Rez becomes:

$$\begin{aligned}\sum_{i,j=1}^{n}\int_\Omega \omega^{2\alpha}(x)a_{ij}(x)\mu(x)u'_{x_i}(x)\overline{u'_{x_j}(x)}dx &= -\sum_{i,j=1}^{n}\int_\Omega(\omega^{2\alpha}(x)a_{ij}(x)\mu(x)u'_{x_i}(x))'_{x_j}\overline{u}(x)dx\\ &= (-\sum_{i,j=1}^{n}(\omega^{2\alpha}(x)a_{ij}(x)\mu(x)u'_{x_i}(x))'_{x_j}\ ,\ u(x)) = (Au,u). \quad (2.5)\end{aligned}$$

Since $(Au)(x) = -\sum_{i,j=1}^{n}\left(\omega^{2\alpha}(x)a_{ij}(x)\mu(x)u'_{x_i}(x)\right)'_{x_j}$. Here, the the symbol (,) denotes the inner product in H. Notice that the above equality in (2.5) obtains by the well known theorem of the m-sectorial operators which are closed by extending its domain to the closed domain in $\mathcal{H}$. These operators are associated with the closed sectorial bilinear forms that are densely defined in $\mathcal{H}$ (for more explanation see the well known Theorem 2.1, chapter 6 of [8]). Therefore

$$c_1\sum_{i=1}^{n}\int_\Omega \omega^{2\alpha}(x)|u'_{x_i}(x)|^2dx \leq Rez(Au\ ,\ u)$$

from (2.5) we have: $c'|\lambda| \leq -Re\{z\lambda\},\ c'>0,\ \forall\lambda\in\psi_{\theta_1\theta_2}$. Multiply this inequality by $\int_\Omega|u(x)|^2\ dx = (u\ ,\ u)^2 = \|u\|^2 > 0$. It follows that

$$c'|\lambda|\int_\Omega |u(x)|^2\ dt \leq -Re\{z\lambda\}(u\ ,\ u).$$

Then we will have

$$\begin{aligned}c_1\sum_{i=1}^{n}\int_\Omega \omega^{2\alpha}(x)|u'_{x_i}(x)|^2dx + c'|\lambda|\int_\Omega|u(x)|^2dx &\leq Re\{z(Au,u) - z\lambda(u,u)\}\\ &= Re\{z((A-\lambda I)u,u)\}\\ &\leq \|Z\|\|u\|\|(A-\lambda I)u\|\\ &= \|u\|\|(A-\lambda I)u\|; \quad (2.6)\end{aligned}$$

i.e.,

$$c_1 \sum_{i=1}^{n} \int_{\Omega} \omega^{2\alpha}(x)|u'_{x_i}(x)|^2 dx + c'|\lambda| \int_{\Omega} |u(x)|^2 \ dx \leq \|u\| \ \|(A \ - \ \lambda \ I)u\|.$$

Since $c_1 \sum_{i=1}^{n} \int_{\Omega} \omega^{2\alpha}(x)|u'_{x_i}(x)|^2 dx$ is positive. if we drop this positive term from the latter inequality this inequality again satisfy, therefore this imply we will have either $c'|\lambda|\|u(x)\|^2 = |\lambda| \int_{\Omega} |u(x)|^2 \ dx \leq \|u\|\|(A \ \ - \lambda \ I)u\|$. Or

$$|\lambda|\|u(x)\| \leq M_{\psi_{\theta_1 \theta_2}} \|(A \ \ - \lambda \ I)u\|. \tag{2.7}$$

This inequality ensures that the operator $(A \ - \ \lambda \ I)$ is one to one, which implies that $ker(A \ - \ \lambda \ I) = 0$. Therefore the inverse operator $(A - \ \lambda I)^{-1}$ exists, and its continuity follows from the proof of the estimate (2.2) of Theorem 2.1. To prove we set $u = (A - \lambda I)^{-1} f, \ f \in H$ in (2.6) implies that

$$|\lambda| \int_{\Omega} |(A - \lambda I)^{-1} f|^2 \ dx \leq M_{\psi_{\theta_1 \theta_2}} \|(A - \ \ \lambda I)^{-1} f\| \|(A \ \ - \lambda I)(A - \lambda I)^{-1} f\|.$$

Since $(A \ - \ \lambda \ I)(A - \lambda I)^{-1} f = I(f) = f$. Then

$$|\lambda| \int_{\Omega} |(A \ \ - \lambda I)^{-1} f|^2 \ dx \leq M_{\Phi} \|(A \ \ - \lambda I)^{-1} f\| |f|.$$

So

$$|\lambda| \|(A \ \ - \lambda I)^{-1}(f)\|^2 \leq M_{\psi_{\theta_1 \theta_2}} \|(A \ \ - \lambda I)^{-1}(f)\| |f|.$$

Which this implies that $|\lambda|\|(A \ \ \ - \lambda I)^{-1}(f)\| \ \leq \ M_{\psi_{\theta_1 \theta_2}} |f|$. Since $\lambda \neq 0$. Then $\|(A - \lambda I)^{-1}(f)\| \leq M_{\psi_{\theta_1 \theta_2}} |\lambda|^{-1} |f|$; $\|(A - \lambda I)^{-1}(f)\| \leq M_{\psi_{\theta_1 \theta_2}} |\lambda|^{-1}$; from this we will have the following estimates

$$\|(A - \lambda I)^{-1}(f)\| \leq M_{\psi_{\theta_1 \theta_2}} |\lambda|^{-1};$$

then from this consequently we proved the following estimates:

$$\|(A - \lambda I)^{-1}(f)\| \leq M_{\Phi} |\lambda|^{-1}. \tag{2.8}$$

This estimate completes the proof of the assertion of Theorem 2.1.
i.e, it follows that

$$\|(A - \lambda I)^{-1}\| \leq M_{\Phi} |\lambda|^{-1}, \ \ (|\lambda| \geq C, \quad \ \lambda \in_{\psi_{\theta_1 \theta_2}}).$$

This complete the proof of Theorem 2.1.

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