Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2022 (40)** : 1–12.
ISSN-0037-8712 in press
doi:10.5269/bspm.51464

Global Existence and Stability of Solution for a p-Kirchhoff type Hyperbolic Equation with Variable Exponents

Amar Ouaoua, Aya Khaldi and Messaoud Maouni

ABSTRACT: In this paper, we consider the following p-Kirchhoff type hyperbolic equation with variable exponents

$$u_{tt} - M\left(\int_\Omega |\nabla u|^p\, dx\right)\Delta_p u + |u_t|^{m(x)-2}\, u_t = |u|^{r(x)-2}\, u.$$

We prove the global existence of the solution with positive initial energy, the stability established based on Komornik's inequality.

Key Words: Kirchhoff type hyperbolic equation, Variable exponents, Global existence.

Contents

1. Introduction

We consider the following boundary value problem:

$$\begin{cases} u_{tt} - M\left(\int\limits_\Omega |\nabla u|^p\, dx\right)\Delta_p u + |u_t|^{m(x)-2}\, u_t = |u|^{r(x)-2}\, u, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\ T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n, n \geq 1$ with smooth boundary $\partial\Omega$ and $M(s) = a + bs$ with positive parameters a, b, $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, with $p \geq 2$. $r(.)$ and $m(.)$ are given measurable functions on Ω.

Equation (1.1) can be viewed as a generalization of a model introduced by Kirchhoff [6]. The following Kirchhoff type equation

$$u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + g(u_t) = f(u), \tag{1.2}$$

have been discussed by many authors. For $g(u_t) = u_t$, the global existence and blow up results can by found in [14, 18], for $g(u_t) = |u_t|^{p-2}\, u_t$, $p > 2$, the main results of existence and blow up are in [4, 13]. Many authors studied the existence and nonexistence of solutions for problem with variable exponents, can refer [2, 4, 9, 10, 15, 17, 19, 20]. Messaoudi et al. [13] considered the following equation:

$$u_{tt} - \Delta u + a\,|u_t|^{m(x)-2}\, u_t = |u|^{p(x)-2}\, u, \quad \text{in } \Omega \times (0,T),$$

and used the Faedo Galerkin method to establish the existence of a unique weak local solution. They also proved that the solutions with negative initial energy blow up in finite time. Messaoudi and Talahmeh [11, 12] , considered the following equation:

$$u_{tt} - div\left(|\nabla u|^{r(x)-2}\,\nabla u\right) + a\,|u_t|^{m(x)-2}\, u_t = b\,|u|^{p(x)-2}\, u, \quad \text{in } \Omega \times (0,T),$$

2010 *Mathematics Subject Classification:* 35B40, 35B40, 35L70, 35L10.

Submitted December 16, 2019. Published April 28, 2020

where a, b is a nonnegative constant. They proved a finite-time blow-up result for the solution with negative initial energy as well as for certain solutions with positive initial energy; in the cas where $m(x) = 2$ and under suitable conditions on the exponents, they established a blow-up result for solutions with arbitrary positive initial energy. Our objective in this paper is to study: In section 2, some notations, assumptions and preliminaries are introduced, section 3 the global existence of solution is proved and the main results of this article are shown in section 4.

2. Preliminaries

We begin this section with some notations and definitions. Denote by $\|.\|_p$, the $L^p(\Omega)$ norm of a Lebesgue function $u \in L^p(\Omega)$. We use $W_0^{1,p}(\Omega)$ to the well-known sobolev space such that u and $|\nabla u|$ are in $L^p(\Omega)$ equipped with the norm $\|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_p$.
Let $q : \Omega \to [1, +\infty]$ be a measurable function, where Ω is adomain of $\mathbb{R}^n$. We define the Lebesque space with a variale exponent $q(.)$ by:

$$L^{q(.)}(\Omega) := \left\{v : \Omega \to \mathbb{R} : \text{ measurable in } \Omega,\ \varrho_{q(.)}(\lambda v) < +\infty, \text{ for some } \lambda > 0\right\},$$

where $\varrho_{q(.)}(v) = \int_\Omega |v(x)|^{q(x)}\, dx$.
The set $L^{q(.)}(\Omega)$ equipped with the norm (Luxemburg's norm)

$$\|v\|_{q(.)} := \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{v(x)}{\lambda}\right|^{q(x)} dx \le 1\right\},$$

$L^{q(.)}(\Omega)$ is a Banach space [8].
Next,we define the variable-exponent Sobolev space $W^{1,q(.)}(\Omega)$ as follows:

$$W^{1,q(.)}(\Omega) := \left\{v \in L^{q(.)}(\Omega) \text{ such that } \nabla v \text{ exists and } |\nabla v| \in L^{q(.)}(\Omega)\right\}.$$

This is a Banach space with respect to the norm $\|v\|_{W^{1,q(.)}(\Omega)} = \|v\|_{q(.)} + \|\nabla v\|_{q(.)}$.
Furthemore, we set $W_0^{1,q(.)}(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in the space $W^{1,q(.)}(\Omega)$. Let us note that the space $W^{1,q(.)}(\Omega)$ has a differenet definition in the case of variable exponents.
However, under the log-Hölder continuity condition, both definitions are equivalent [8]. The space $W^{-1,q'(.)}(\Omega)$, dual of $W_0^{1,q(.)}(\Omega)$, is defined in the same way as the classical Sobolev spaces, where $\frac{1}{q(.)} + \frac{1}{q'(.)} = 1$.

Lemma 2.1 *If*

$$1 \le q_1 := ess\inf_{x\in\Omega} q(x) \le q(x) \le q_2 := ess\sup_{x\in\Omega} q(x) < \infty,$$

then we have

$$\min\left\{\|u\|_{q(.)}^{q_1},\ \|u\|_{q(.)}^{q_2}\right\} \le \varrho_{q(.)}(u) \le \max\left\{\|u\|_{q(.)}^{q_1},\ \|u\|_{q(.)}^{q_2}\right\},$$

for any $u \in L^{q(.)}(\Omega)$.

Lemma 2.2 (Hölder's Inequality) *Suppose that* p, q, $s \ge 1$ *are measurable functions defined on* Ω *such that*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \quad \text{for } a.e. \quad y \in \Omega.$$

For the existence of the local solution of problem (1.1), we refer the reader to [13]. Their result is given in the following theorem:

Theorem 2.1 *Suppose that* $r,\ m \in C\left(\overline{\Omega}\right)$ *with*

$$\begin{aligned} 2 &\le r_1 \le r(x) \le r_2 < 2\frac{n-1}{n-2}, \quad \textit{if } n \ge 3, \\ r(x) &\ge 2, \qquad\qquad\qquad\qquad \textit{if } n = 1,\ 2, \end{aligned}$$

and

$$\begin{aligned} 2 &\le m_1 \le m(x) \le m_2 < \frac{2n}{n-2}, \quad \textit{if } n \ge 3, \\ m(x) &\ge 2, \qquad\qquad\qquad\qquad \textit{if } n = 1,\ 2, \end{aligned}$$

$$\begin{aligned} r_1 &: = ess\inf_{x\in\Omega} r(x), \qquad r_2 := ess\sup_{x\in\Omega} r(x), \\ m_1 &: = ess\inf_{x\in\Omega} m(x), \qquad m_2 := ess\sup_{x\in\Omega} m(x). \end{aligned}$$

We also assume that $m(.)$ *and* $r(.)$ *satisfy the log-Hölder continuity condition:*

$$|q(x) - q(y)| \le -\frac{A}{\log|x-y|}, \text{ for a.e. } x,\ y \in \Omega, \text{ with } |x-y| < \delta, \tag{2.1}$$

$A > 0,\ 0 < \delta < 1.$

Then, for any $(u_0,\ u_1) \in W_0^{1,p}(\Omega) \times L^2(\Omega)$, *problem* (1.1) *has a unique weak local solution*

$$\begin{aligned} u &\in L^\infty\left((0,T),\ W_0^{1,p}(\Omega)\right), \\ u_t &\in L^\infty\left((0,T),\ L^2(\Omega)\right) \cap L^{m(.)}(\Omega \times (0,\ T)), \\ u_{tt} &\in L^2\left((0,T),\ W^{-1,p'}(\Omega)\right). \end{aligned}$$

3. Exponential growth

In the order to state and prove our result, we define the potential energy functional and the Nehari's functional, respectively, by the following

$$E(t) = E(u(t)) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{a}{p}\|\nabla u(t)\|_p^p + \frac{b}{2p}\|\nabla u(t)\|_p^{2p} - \int_\Omega \frac{1}{r(x)}|u(t)|^{r(x)}\,dx. \tag{3.1}$$

$$I(t) = I(u(t)) = a\|\nabla u(t)\|_p^p + b\|\nabla u(t)\|_p^{2p} - \int_\Omega |u(t)|^{r(x)}\,dx. \tag{3.2}$$

We can considering $a = b = 1$, and this does not change the general result.

Lemma 3.1 *Under the assumptions of theorem* 2.1, *we have*

$$E'(t) = -\int_\Omega |u_t(t)|^{m(x)}\,dx \le 0, \quad t \in [0,\ T]. \tag{3.3}$$

and

$$E(t) \le E(0).$$

Proof: We multiply the first equation of (1.1) by u_t and integrating over the domain Ω, we get

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|u_t\|_2^2 + \frac{1}{p}\int_\Omega |\nabla u(t)|^p\,dx + \frac{1}{2p}\left(\int_\Omega |\nabla u(t)|^p\,dx\right)^2 - \int_\Omega \frac{1}{r(x)}|u|^{r(x)}\,dx\right) \\ &= -\int_\Omega |u_t(t)|^{m(x)}\,dx, \end{aligned}$$

then

$$E^{'}(t) = -\int_{\Omega} |u_t(t)|^{m(x)}\, dx \leq 0.$$

Integratying (3.3) over $(0,\ t)$, we obtain

$$E(t) \leq E(0).$$

Lemma 3.2 *Assume that the assumptions of theorem* 2.1 *and* $r_1 > 2p$, *hold,*

$$I(0) > 0,$$

and

$$\beta_1 + \beta_2 < 1, \tag{3.4}$$

where

$$\beta_1 \ : = \max\left\{\alpha c_*^{r_1}\left(\frac{pr_1}{r_1 - p}E(0)\right)^{\frac{r_1 - p}{p}},\ \alpha c_*^{r_2}\left(\frac{pr_1}{r_1 - p}E(0)\right)^{\frac{r_2 - p}{p}}\right\},$$

$$\beta_2 \ : = \max\left\{(1-\alpha)\, c_*^{r_1}\left(\frac{2pr_1}{r_1 - 2p}E(0)\right)^{\frac{r_1 - 2p}{2p}},\ (1-\alpha)\, c_*^{r_2}\left(\frac{2pr_1}{r_1 - 2p}E(0)\right)^{\frac{r_2 - 2p}{2p}}\right\},$$

with $0 < \alpha < 1$, c_* *is the best embedding constant of* $W_0^{1,\ p}(\Omega) \hookrightarrow L^{r(.)}(\Omega)$, *then* $I(t) > 0$, *for all* $t \in [0,\ T]$.

Proof: By continuity, there exists T_*, such that

$$I(t) \geq 0, \quad \text{for all } t \in [0,\ T_*]. \tag{3.5}$$

Now, we have for all $t \in [0,\ T]$:

$$\begin{aligned} J(t) &= J(u(t)) = \frac{1}{p}\|\nabla u(t)\|_p^p + \frac{1}{2p}\|\nabla u(t)\|_p^{2p} - \int_{\Omega} \frac{1}{r(x)} |u(t)|^{r(x)}\, dx \\ &\geq \frac{1}{p}\|\nabla u(t)\|_p^p + \frac{1}{2p}\|\nabla u(t)\|_p^{2p} - \frac{1}{r_1}\left(\|\nabla u(t)\|_p^p + \|\nabla u(t)\|_p^{2p} - I(t)\right) \\ &\geq \frac{r_1 - p}{pr_1}\|\nabla u(t)\|_p^p + \frac{r_1 - 2p}{2pr_1}\|\nabla u(t)\|_p^{2p} + \frac{1}{r_1}I(t) \end{aligned}$$

using (3.5), we obtain

$$\frac{r_1 - p}{pr_1}\|\nabla u(t)\|_p^p + \frac{r_1 - 2p}{2pr_1}\|\nabla u(t)\|_p^{2p} \leq J(t), \qquad \text{for all } t \in [0,\ T_*]. \tag{3.6}$$

By Lemma 3.1, we get

$$\|\nabla u(t)\|_p^p \leq \frac{pr_1}{r_1 - p}E(t) \leq \frac{pr_1}{r_1 - p}E(0) \tag{3.7}$$

and

$$\|\nabla u(t)\|_p^{2p} \leq \frac{2pr_1}{r_1 - 2p}E(t) \leq \frac{2pr_1}{r_1 - 2p}E(0) \tag{3.8}$$

On the other hand, by Lemma 2.1, we have

$$\begin{aligned} \int_{\Omega} |u(t)|^{r(x)}\, dx &\leq Max\left\{\|u(t)\|_{r(.)}^{r_1},\ \|u(t)\|_{r(.)}^{r_2}\right\} \\ &= \alpha\, Max\left\{\|u(t)\|_{r(.)}^{r_1},\ \|u(t)\|_{r(.)}^{r_2}\right\} \\ &\quad + (1-\alpha)\, Max\left\{\|u(t)\|_{r(.)}^{r_1},\ \|u(t)\|_{r(.)}^{r_2}\right\}. \end{aligned}$$

By the embedding of $W_0^{1,\,p}(\Omega) \hookrightarrow L^{r(.)}(\Omega)$, we obtain

$$\begin{aligned}\int_\Omega |u(t)|^{r(x)}\,dx &\leq \alpha\; Max\left\{c_*^{r_1}\|\nabla u(t)\|_p^{r_1},\; c_*^{r_2}\|\nabla u(t)\|_p^{r_2}\right\}\\ &\quad + (1-\alpha)\,Max\left\{c_*^{r_1}\|\nabla u(t)\|_p^{r_1},\; c_*^{r_2}\|\nabla u(t)\|_p^{r_2}\right\}\\ &\leq \alpha\; Max\left\{c_*^{r_1}\|\nabla u(t)\|_p^{r_1-p},\; c_*^{r_2}\|\nabla u(t)\|_p^{r_2-p}\right\}\times\|\nabla u(t)\|_p^p\\ &\quad + (1-\alpha)\,Max\left\{c_*^{r_1}\|\nabla u(t)\|_p^{r_1-2p},\; c_*^{r_2}\|\nabla u(t)\|_p^{r_2-2p}\right\}\times\|\nabla u(t)\|_p^{2p}\end{aligned}$$

By (3.7) and (3.8), we get

$$\int_\Omega |u(t)|^{r(x)}\,dx \leq \beta_1\|\nabla u(t)\|_p^p + \beta_2\|\nabla u(t)\|_p^{2p}, \qquad \text{for all } t\in[0,\;T_*]. \tag{3.9}$$

Since $\beta_1+\beta_2<1$, then

$$\int_\Omega |u(t)|^{r(x)}\,dx < \|\nabla u(t)\|_p^p + \|\nabla u(t)\|_p^{2p}, \qquad \text{for all } t\in[0,\;T_*].$$

This implies that

$$I(t)>0, \qquad \text{for all } t\in[0,\;T_*].$$

By repeating the above procedure, we can extend T_* to T.

Theorem 3.1 *Under the assumptions of lemma* 3.2, *the local solution of* (1.1) *is global.*

Proof: We have

$$\begin{aligned}E(u(t)) &= \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{p}\|\nabla u(t)\|_p^p + \frac{1}{2p}\|\nabla u(t)\|_p^{2p} - \int_\Omega \frac{1}{r(x)}|u(t)|^{r(x)}\,dx.\\ &\geq \frac{1}{2}\|u_t(t)\|_2^2 + \frac{r_1-p}{pr_1}\|\nabla u(t)\|_p^p + \frac{r_1-2p}{2pr_1}\|\nabla u(t)\|_p^{2p}.\end{aligned}$$

So that

$$\|u_t(t)\|_2^2 + \|\nabla u(t)\|_p^p \leq C\;E(t). \tag{3.10}$$

By Lemma 3.1, we obtain

$$\|u_t(t)\|_2^2 + \|\nabla u(t)\|_p^p \leq C\;E(0). \tag{3.11}$$

This implies that the local solution is global in time.

4. Stability solution

In this section our main result is based a Komornik's inequality [7], as in [5]. For this, we need the following Lemma:

Lemma 4.1 *Suppose that the assumptions of Lemma* 3.2 *and* $m_1>p$, *hold, then there exists a positive constant* c *such that*

$$\int_\Omega |u(t)|^{m(x)}\,dx \leq cE(t). \tag{4.1}$$

Proof: We have

$$\begin{aligned}\int_\Omega |u(t)|^{m(x)}\,dx &= \max\left\{\|u(t)\|_{m(.)}^{m_1},\;\|u(t)\|_{m(.)}^{m_2}\right\}\\ &\leq \max\left\{c_*^{m_1}\|\nabla u(t)\|_p^{m_1},\; c_*^{m_2}\|\nabla u(t)\|_p^{m_2}\right\}\\ &\leq \max\left\{c_*^{m_1}\|\nabla u(t)\|_p^{m_1-p},\; c_*^{m_2}\|\nabla u(t)\|_p^{m_2-p}\right\}\times\|\nabla u(t)\|_p^p\end{aligned}$$

By using (3.7), we obtain

$$\int_{\Omega} |u(t)|^{m(x)}\,dx \leq cE(t).$$

Now, we state our main result:

Theorem 4.1 *Let the assumptions of Lemma 3.2, then, there exists constants C, $\zeta > 0$, such that*

$$\begin{aligned} E(t) &\leq \frac{C}{(1+t)^{\frac{2}{m_2-2}}}, \quad \text{for all } t \geq 0 \ \text{ if } m_2 > 2. \\ E(t) &\leq Ce^{-\zeta t}, \qquad\qquad \text{for all } t \geq 0 \ \text{ if } m_2 = 2. \end{aligned}$$

Proof: Multiplying first equation of (1.1) by $u(t)\,E^q(t)$ $(q > 0)$ and integrating over $\Omega \times (S,\ T)$, we obtain

$$\begin{aligned} &\int_S^T\!\!\int_{\Omega} E^q(t)\left[u(t)\,u_{tt}(t) - u(t)\left(M\left(\int_{\Omega} |\nabla u|^p\,dx\right)\Delta_p u + |u_t|^{m(x)-2}\,u_t\right)\right]dxdt \\ =\ &\int_S^T E^q(t)\int_{\Omega} |u(t)|^{r(x)}\,dxdt \end{aligned}$$

So that

$$\begin{aligned} &\int_S^T\!\!\int_{\Omega} E^q(t)\left[(u(t)\,u_t(t))_t - |u_t(t)|^2 + |\nabla u(t)|^p + \|\nabla u(t)\|_p^p\,|\nabla u(t)|^p\right. \\ &\qquad \left. +u(t)\,|u_t|^{m(x)-2}\,u_t\right]dxdt = \int_S^T E^q(t)\int_{\Omega} |u(t)|^{r(x)}\,dxdt \end{aligned}$$

We add and substract the term

$$\int_S^T E^q(t)\int_{\Omega}\left[\beta_1\,|\nabla u(t)|^p + \beta_2\,\|\nabla u(t)\|_p^p\,|\nabla u(t)|^p + (2+\beta_1+\beta_2)\,|u_t(t)|^2\right]dxdt$$

and use (3.9), to get

$$\begin{aligned} &(1-\beta_1)\int_S^T E^q(t)\int_{\Omega}\left[|\nabla u(t)|^p + |u_t(t)|^2\right]dxdt \\ &+(1-\beta_2)\int_S^T E^q(t)\int_{\Omega}\left[\|\nabla u(t)\|_p^p\,|\nabla u(t)|^p + |u_t(t)|^2\right]dxdt \\ &+\int_S^T E^q(t)\int_{\Omega}\left[(u(t)\,u_t(t))_t - (3-\beta_1-\beta_2)\,|u_t(t)|^2\right]dxdt \\ &+\int_S^T E^q(t)\int_{\Omega} u(t)\,u_t(t)\,|u_t(t)|^{m(x)-2}\,dxdt \\ =\ &-\int_S^T E^q(t)\int_{\Omega}\left[\beta_1\,|\nabla u(t)|^p + \beta_2\,\|\nabla u(t)\|_p^p\,|\nabla u(t)|^p - |u(t)|^{r(x)}\right]dxdt \leq 0 \end{aligned} \tag{4.2}$$

It is clear that

$$\begin{aligned}
&\gamma\int_S^T E^q(t)\int_\Omega\left[\frac{1}{p}|\nabla u(t)|^p+\frac{1}{2p}\|\nabla u(t)\|_p^p|\nabla u(t)|^p+\frac{|u_t(t)|^2}{2}-\frac{|u(t)|^{r(x)}}{r(x)}\right]dxdt\\
\leq\ &(1-\beta_1)\int_S^T E^q(t)\int_\Omega\left[\frac{1}{p}|\nabla u(t)|^p+\frac{|u_t(t)|^2}{2}\right]dxdt\\
&+(1-\beta_2)\int_S^T E^q(t)\int_\Omega\left[\frac{1}{2p}\|\nabla u(t)\|_p^p|\nabla u(t)|^p+\frac{|u_t(t)|^2}{2}\right]dxdt \qquad (4.3)
\end{aligned}$$

where $\gamma=Min\left((1-\beta_1),\ (1-\beta_2)\right)$. By (4.2), (4.3) and definition of $E(t)$, we get

$$\begin{aligned}
\gamma\int_S^T E^{q+1}(t)\,dt\ \leq\ &-\int_S^T E^q(t)\int_\Omega\left(u(t)\,u_t(t)\right)_t dxdt\\
&+(3-\beta_1-\beta_2)\int_S^T E^q(t)\int_\Omega|u_t(t)|^2\,dxdt\\
&-\int_S^T E^q(t)\int_\Omega u(t)\,u_t(t)\,|u_t(t)|^{m(x)-2}\,dxdt. \qquad (4.4)
\end{aligned}$$

Using the definition of $E(t)$ and the following expression

$$\begin{aligned}
\frac{d}{dt}\left(E^q(t)\int_\Omega u(t)\,u_t(t)\,dx\right)\ =\ &qE^{q-1}(t)\,\frac{d}{dt}E(t)\int_\Omega u(t)\,u_t(t)\,dx\\
&+E^q(t)\int_\Omega\left(u(t)\,u_t(t)\right)_t dx.
\end{aligned}$$

Inequality (4.4), becomes

$$\begin{aligned}
\gamma\int_S^T E^{q+1}(t)\,dt\ \leq\ &q\int_S^T E^{q-1}(t)\,\frac{d}{dt}E(t)\int_\Omega u(t)\,u_t(t)\,dx\\
&-\int_S^T\frac{d}{dt}\left(E^q(t)\int_\Omega u(t)\,u_t(t)\,dx\right)dt\\
&-\int_S^T E^q(t)\int_\Omega u(t)\,u_t(t)\,|u_t(t)|^{m(x)-2}\,dxdt\\
&+(3-\beta_1-\beta_2)\int_S^T E^q(t)\int_\Omega|u_t(t)|^2\,dxdt. \qquad (4.5)
\end{aligned}$$

We denote by c the various constants.

We estimate the terms in the right-hand side of (4.5) as follow:

By (3.3) and Young's inequality, we obtain

$$\begin{aligned} & q\int_S^T E^{q-1}(t)\frac{d}{dt}E(t)\int_\Omega u(t)\,u_t(t)\,dx \\ \leq\ & q\int_S^T E^{q-1}(t)\left(-E^{'}(t)\right)\int_\Omega\left[\frac{1}{p}\left|u(t)\right|^p+\frac{p-1}{p}\left|u_t(t)\right|^{\frac{p}{p-1}}\right]dxdt \end{aligned} \tag{4.6}$$

Since, $1\leq\frac{p}{p-1}<2$, by the embedding of $L^2(\Omega)\hookrightarrow L^{\frac{p}{p-1}}(\Omega)$, we have

$$\begin{aligned} & q\int_S^T E^{q-1}(t)\frac{d}{dt}E(t)\int_\Omega u(t)\,u_t(t)\,dx \\ \leq\ & q\int_S^T E^{q-1}(t)\left(-E^{'}(t)\right)\int_\Omega\left[\frac{1}{p}\left|u(t)\right|^p+c\frac{p-1}{p}\left|u_t(t)\right|^2\right]dxdt \end{aligned}$$

Thus, by (3.10), we find

$$\begin{aligned} & q\int_S^T E^{q-1}(t)\frac{d}{dt}E(t)\int_\Omega u(t)\,u_t(t)\,dx \\ \leq\ & c\int_S^T E^q(t)\left(-E^{'}(t)\right)dt \\ \leq\ & cE^{q+1}(S)-cE^{q+1}(T) \\ \leq\ & cE^q(0)\,E(S)\leq cE(S). \end{aligned} \tag{4.7}$$

For the second term, we have

$$\begin{aligned} & -\int_S^T\frac{d}{dt}\left(E^q(t)\int_\Omega u(t)\,u_t(t)\,dx\right)dxdt \\ \leq\ & \left|E^q(t)\int_\Omega u(S)\,u_t(S)\,dx-E^q(t)\int_\Omega u(T)\,u_t(T)\,dx\right| \\ \leq\ & E^q(t)\left|\int_\Omega u(x,S)\,u_t(x,S)\,dx\right|+E^q(t)\left|\int_\Omega u(x,T)\,u_t(x,T)\,dx\right| \\ \leq\ & cE^{q+1}(S)+cE^{q+1}(T) \\ \leq\ & cE^q(0)\,E(S)\leq cE(S). \end{aligned} \tag{4.8}$$

For the third term, we use the following Young inequality:

$$XY\leq\frac{\varepsilon}{\lambda_1}X^{\lambda_1}+\frac{1}{\lambda_2\varepsilon^{\frac{\lambda_2}{\lambda_1}}}Y^{\lambda_2},\ X,\ Y\geq0,\ \varepsilon>0\text{ and }\frac{1}{\lambda_1}+\frac{1}{\lambda_2}=1,$$

with $\lambda_1(x)=m(x)\,,\ \lambda_2(x)=\frac{m(x)}{m(x)-1}$.

By (3.3) and Lemma 4.1, we have

$$
\begin{aligned}
& -\int_S^T E^q(t) \int_\Omega u(t) u_t(t) |u_t(t)|^{m(x)-2} dxdt \\
\leq & \int_S^T E^q(t) \left(\varepsilon c \int_\Omega |u(t)|^{m(x)} dx + c_\varepsilon \int_\Omega |u_t(t)|^{m(x)} dx \right) dt \\
\leq & \varepsilon c \int_S^T E^q(t) \int_\Omega |u(t)|^{m(x)} dxdt + c_\varepsilon \int_S^T E^q(t) \left(-E'(t) \right) dt \\
\leq & \varepsilon c \int_S^T E^{q+1}(t) dt + c_\varepsilon E(S). \qquad (4.9)
\end{aligned}
$$

For the last term of (4.5), we have

$$
\begin{aligned}
& (3-\beta_1-\beta_2) \int_S^T E^q(t) \int_\Omega |u_t(t)|^2 dxdt \\
\leq & (3-\beta_1-\beta_2) \int_S^T E^q(t) \left[\int_{\Omega^-} |u_t(t)|^2 dx + \int_{\Omega^+} |u_t(t)|^2 dx \right] dt \\
\leq & c \int_S^T E^q(t) \left[\left(\int_{\Omega^-} |u_t(t)|^{m_2} dx \right)^{\frac{2}{m_2}} + \left(\int_{\Omega^+} |u_t(t)|^{m_1} dx \right)^{\frac{2}{m_1}} \right] dt \\
\leq & c \int_S^T E^q(t) \left[\left(\int_\Omega |u_t(t)|^{m(x)} dx \right)^{\frac{2}{m_2}} + \left(\int_\Omega |u_t(t)|^{m(x)} dx \right)^{\frac{2}{m_1}} \right] dt.
\end{aligned}
$$

This implies

$$
\begin{aligned}
& (3-\beta_1-\beta_2) \int_S^T E^q(t) \int_\Omega |u_t(t)|^2 dxdt \\
\leq & c \int_S^T E^q(t) \left(-E'(t) \right)^{\frac{2}{m_2}} dt + c \int_S^T E^q(t) \left(-E'(t) \right)^{\frac{2}{m_1}} dt. \qquad (4.10)
\end{aligned}
$$

First, if we use Young's inequality with $\lambda_1 = (q+1)/q$ and $\lambda_2 = q+1$, we have

$$
\int_S^T E^q(t) \left(-E'(t) \right)^{\frac{2}{m_2}} dt \leq \varepsilon c \int_S^T E^{q+1}(t) dt + c_\varepsilon \int_S^T \left(-E'(t) \right)^{\frac{2(q+1)}{m_2}} dt.
$$

We take $q = \frac{m_2}{2} - 1$ to find

$$
\int_S^T E^q(t) \left(-E'(t) \right)^{\frac{2}{m_2}} dt \leq \varepsilon c \int_S^T E^{q+1}(t) dt + c_\varepsilon \int_S^T \left(-E'(t) \right) dt.
$$

This implies

$$\int_S^T E^q(t)\left(-E'(t)\right)^{\frac{2}{m_2}}dt \le \varepsilon c\int_S^T E^{q+1}(t)\,dt + c_\varepsilon E(S). \tag{4.11}$$

On the other hand, we have

$$\int_S^T E^q(t)\left(-E'(t)\right)^{\frac{2}{m_1}}dt \le \varepsilon c\int_S^T E^{q+1}(t)\,dt + c_\varepsilon E(S). \tag{4.12}$$

Indeed,

• if $m_1 = 2$ then

$$\int_S^T E^q(t)\left(-E'(t)\right)^{\frac{2}{m_1}}dt \le cE(S) \le \varepsilon c\int_S^T E^{q+1}(t)\,dt + c_\varepsilon E(S).$$

• if $m_1 > 2$, we use the Young's inequality $\lambda_1 = \frac{m_1}{m_1-2}$ and $\lambda_2 = \frac{m_1}{2}$, to obtain

$$\begin{aligned}\int_S^T E^q(t)\left(-E'(t)\right)^{\frac{2}{m_1}}dt &\le \varepsilon c\int_S^T E^{q\frac{m_1}{m_1-2}}(t)\,dt + c_\varepsilon\int_S^T\left(-E'(t)\right)dt\\ &\le \varepsilon c\int_S^T E^{q\frac{m_1}{m_1-2}}(t)\,dt + c_\varepsilon E(S).\end{aligned}$$

We notice that $q\frac{m_1}{m_1-2} = q+1+\frac{m_1-m_2}{m_1-2}$, then

$$\begin{aligned}\int_S^T E^q(t)\left(-E'(t)\right)^{\frac{2}{m_1}}dt &\le \varepsilon c\left(E(S)\right)^{\frac{m_1-m_2}{m_1-2}}\int_S^T E^{q+1}(t)\,dt + c_\varepsilon E(S)\\ &\le \varepsilon c\left(E(0)\right)^{\frac{m_1-m_2}{m_1-2}}\int_S^T E^{q+1}(t)\,dt + c_\varepsilon E(S)\\ &\le \varepsilon c\int_S^T E^{q+1}(t)\,dt + c_\varepsilon E(S).\end{aligned}$$

We substituting (4.11) and (4.12) in (4.10), we obtain

$$(3-\beta_1-\beta_2)\int_S^T E^q(t)\int_\Omega |u_t(t)|^2\,dxdt \le \varepsilon c\int_S^T E^{q+1}(t)\,dt + c_\varepsilon E(S). \tag{4.13}$$

By insert (4.7), (4.8), (4.9) and (4.13) in (4.5), we arrive at

$$\gamma\int_S^T E^{\frac{m_2}{2}}(t)\,dt \le \varepsilon c\int_S^T E^{\frac{m_2}{2}}(t)\,dt + c_\varepsilon E(S).$$

Choosing ε small enough for that

$$\int_S^T E^{\frac{m_2}{2}}(t)\,dt \le cE(S).$$

By taking T goes to ∞, we get

$$\int_S^{\infty} E^{\frac{m_2}{2}}(t)\,dt \leq cE(S).$$

By Komornik's integral inequality yields the result.

Acknowledgement *The authors wish to thank deeply the anonymous referee for useful remarks and careful reading of the proofs presented in this paper.*

References

1. A. Benaissa, SA. *Messaoudi, Blow-up of solutions for Kirchhoff equation of q-Laplacian type with nonlinear dissipation.* Colloq. Math. 94(1), 103-109 (2002).

2. H. Chen, GW. Liu, *Global existence, uniform decay and exponential growth for a class of semilinear wave equation with strong damping.* Acta Math. Sci. 33B(1), 41-58 (2013).

3. Q. Gao, Y. Wang, *Blow-up of the solution for higher-order Kirchhoff-type equations with nonlinear dissipation.* Cent. Eur. J. Math. 9(3), 686-698 (2011).

4. V. Georgiev, G. Todorova, *Existence of a solution of the wave equation with nonlinear damping and source term*, J Differ Equations 1994; 109: 295–308.

5. S. Ghegal, I. Hamchi and SA. Messaoudi, *Global existence and stability of a nonlinear wave equation with variable-exponent nonlinearities*, Applicable Analysis, DOI: 10.1080/00036811.2018.1530760.

6. Kirchhoff G, Mechanik, Teubner, 1883.

7. Komornik V, *Exact controllability and stabilization the multiplier method. Paris*: Masson-JohnWiley; 1994.

8. D. Lars, P. Harjulehto, P. Hasto, M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents*, in: Lecture Notes in Mathematics, Vol. 2017, 2017.

9. HA. Levine, *Instability and nonexistence of global solutions to nonlinear wave equations of the form*, Trans Amer Math Soc 1974; 192: 1–21.

10. HA. Levine, Some additional remarks on the nonexistence of global solutions to nonlinear wave equations, SIAM J Math Anal 1974; 5:138–146.

11. S. Messaoudi, A. Talahmeh, H. Jamal, *Nonlinear damped wave equation: existence and blow-up.* Comput Math Appl. 2017; 74:3024–3041.

12. S. Messaoudi, A. Talahmeh, *A blow-up result for a nonlinear wave equation with variable-exponent nonlinearities.* Appl Anal. 2017;96:1509–1515.

13. S. Messaoudi, A. Talahmeh, *Blowup in solutions of a quasilinear wave equation with variable-exponent nonlinearities.* Math Methods Appl Sci. 2017;40:1099–1476.

14. K. Ono, *Global existence, decay, and blow-up of solutions for some mildly degenerate nonlinear Kirchhoff strings*, J Differ Equations 1997; 137: 273-301.

15. A. Ouaoua, M. Maouni, *Blow-up, exponential growth of solution for a nonlinear parabolic equation with $p(x)$-Laplacian*, International Journal of Analysis and Applications, V 17, N 4, (2019), 620-629.

16. A. Stanislav, S. Sergey, *Evolution PDEs with nonstandard growth conditions: existence, uniqueness, localization, blow-up.* Atlantis Stud Differential Equations. 2015;4:1–417.

17. E. Vitillaro, *Global existence theorems for a class of evolution equations with dissipation*, Arch Rational Mech Anal 1999; 149: 155–182.

18. ST. Wu, LY. Tsai, *Blow-up solutions for some nonlinear wave equations of Kirchhoff type with some dissipation*, Nonlinear Anal 2006; 65: 243-264.

19. YZ. Xu, Y. Ding, *Global solutions and finite time blow-up for damped Klein-Gordon equation.* Acta Math. Sci. 33B(1), 643-652 (2013).

20. SQ. Yu, *On the strongly damped wave equation with nonlinear damping and source terms.* Electron. J. Qual. Theory Differ. Equ. 2009, 39 (2009).

Amar Ouaoua,
Department of Mathematics,
Laboratory of Applied Mathematics and History and Didactics of Mathematics (LAMAHIS)
University of 20 August 1955, Skikda,
Algeria.
E-mail address: ouaouaam21@gmail.com

and

Aya Khaldi
Department of Mathematics,
Laboratory of Applied Mathematics and History and Didactics of Mathematics (LAMAHIS)
University of 20 August 1955, Skikda,
Algeria.
E-mail address: ayakhaldi21@gmail.com

and

Messaoud Maouni,
Department of Mathematics,
Laboratory of Applied Mathematics and History and Didactics of Mathematics (LAMAHIS)
University of 20 August 1955, Skikda,
Algeria.
E-mail address: m.maouni@univ-skikda.dz