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Some Modified Types of Pitchfork Domination and Its Inverse

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ABSTRACT: Let G be a finite, simple graph, without isolated vertices. For any non-negative integers x and y, a set $D \subseteq V$ is a "pitchfork dominating set pds", when every vertex in D, dominates at most y and at least x vertices of $V - D$. A subset D^{-1} of $V - D$ is an inverse pds if it is a pitchfork set. The pitchfork domination number of G, $\gamma_{pf}(G)$, is the number of elements of a smallest pds. The "inverse pitchfork domination number" of G, $\gamma^{-1}_{pf}(G)$, is the number of elements of a smallest inverse pds. In this paper, some modified pitchfork dominations and its inverse dominations are introduced when $x = 1$ and $y = 2$. Several bounds and properties are given and proved. Then, these modified dominations are applied on some standard graphs such as: path, cycle, wheel, complete, complete bipartite graph and their complements.

Key Words: Dominating set, Pitchfork domination, Inverse pitchfork domination.

Contents

1. Introduction

Let G be a graph with V vertex set of order n and E edge set of size m. The degree of any vertex v in G is denoted by $deg(v)$ and defined as the number of edges incident on v. An isolated vertex is a vertex of degree 0, a leaf is a vertex of degree 1. The vertex that is adjacent to the leaf is said a support vertex. $\Delta(G)$ and $\delta(G)$ are respectively the maximum and minimum degrees in G. $N(t) = \{r \in V \,|\, t\,r \in E\}$ is the open neighborhood of t, while closed neighborhood of it, is $N[t] = N(t) \cup \{t\}$. The induced subgraph of a subset vertex M of V and the edges between them is $G[M]$. $\overline{G}$ is the complement of a simple graph G ,it is a graph with the same vertices of G, such that there is an edge between any two vertices in $\overline{G}$ if and only if there is no edge in G between them. See [10] for theoretic terminology and basic conceptus of graph. In graph theory, one of the fastest growing areas is the study of related subset problems of dominating sets, see [11,12,13]. In G , a set D of V is said a dominating set if every vertex out it, is adjacent to one vertex or more of it, such that $N[D]$ equals V. Furthermore, D is said to be a minimal dominating set, if it has no proper dominating subset. $\gamma(G)$ is the domination number(the cardinality of the minimum dominating set D of G.) Ore [18] is the one who introduced the concepts of domination number and dominating sets. According to the purpose used for and the importance of the concept in many applications, carry to the evolution of variant kinds of domination, see [2,3,5,6,7,8,9,14,15,16,17]. A new type of domination said "pitchfork domination" and its inverse are introduced by Al-Harere and Abdlhusein [1,4]. In this paper, these new types of domination are modified by adding new conditions on the graph. The independent pitchfork domination, inverse independent pitchfork domination, an annihilator pitchfork domination and an inverse annihilator pitchfork domination are defined and applied here.

Theorem 1.1. *[4] For any $G = (n, m)$ with pds, we have:*

$$\gamma_{pf}(G) \leq m \leq \binom{n}{2} + \gamma^2_{pf}(G) + (2 - n)\,\gamma_{pf}(G)$$

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Theorem 1.2. *[4] Let D be a pitchfork dominating set of a graph G, then D is a minimal if one of the following conditions holds:*
a. $|\,N(w) \cap V - D\,| = 2$,$\forall w \in D$.
b. $|\,N(x) \cap D\,| = 1$, $\forall x \in V - D$.
c. $G[\,D\,]$ *has no edges.*
d. D *has only support vertices.*
e. D *has only end vertices.*

Theorem 1.3. *[1] Let G be graph with $\gamma_{pf}(G)$ and $\gamma_{pf}^{-1}(G)$. Then, $\gamma_{pf}(G) + \gamma_{pf}^{-1}(G) = n$ if G satisfy one condition of:*
1- $G[\,V - D\,]$ is a null graph.
2- For any two vertices v_1, $v_2 \in V - D$, $N_D[v_1] \cap N_D[v_2] = \phi$.
3- For every $v \in V - D$, if the vertices that dominate v are dominate another vertices, then v is isolated in $G[V - D]$ or adjacent to a vertex in $V - D$ that is dominated by exactly two vertices.

Note 1. *[1] For any graph G of order n and pitchfork domination number γ_{pf}, if $\gamma_{pf}(G) > \frac{n}{2}$. Then, G without inverse pds.*

Remark 1.4. *[4]: For P_n and C_n, we have:*

1. $\gamma_{pf}(P_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$.
2. $\gamma_{pf}(C_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$.

Theorem 1.5. *[1] The cycle graph C_n; $(n \geq 3)$ has an inverse pitchfork domination such that: $\gamma_{pf}^{-1}(C_n) = \gamma_{pf}(C_n) = \lceil \frac{n}{3} \rceil$.*

Theorem 1.6. *[1] The path graph P_n; $(n \geq 2)$ has an inverse pitchfork domination such that:*

$$\gamma_{pf}^{-1}(P_n) = \begin{cases} \frac{n}{3} + 1 & if\ n \equiv 0\,(mod\,3) \\ \lceil \frac{n}{3} \rceil & \quad if\ n \equiv 1, 2\,(mod\,3) \end{cases}$$

where $\gamma_{pf}^{-1}(P_2) = 1$.

Proposition 1.7. *[4] For $n \geq 3$, $\gamma_{pf}(K_n) = n - 2$.*

Proposition 1.8. *[1] The complete graph K_n has an inverse pitchfork domination if and only if $n = 3, 4$ and $\gamma_{pf}^{-1}(K_n) = n - 2$.*

Theorem 1.9. *[4] Let G be a wheel graph W_n where $n \geq 3$, then:*

$$\gamma_{pf}(W_n) = \begin{cases} 2\lceil \frac{n}{4} \rceil - 1, & if\ n \equiv 1\ (mod\,4) \\ 2\lceil \frac{n}{4} \rceil, & otherwise \end{cases}$$

Theorem 1.10. *[1] The wheel graph W_n; $(n \geq 3)$ has an inverse pitchfork domination if and only if $n \equiv 0\ (mod\,4)$ or $n = 3$ where $\gamma_{pf}^{-1}(W_n) = 2\,\lceil \frac{n}{4} \rceil$.*

Theorem 1.11. *[1] The complete bipartite graph $K_{n,\,m}$ has an inverse pitchfork domination if and only if $K_{n,\,m} \equiv K_{1,\,2}, K_{2,\,2}, K_{2,\,3}, K_{2,\,4}, K_{3,\,3}, K_{3,\,4}$ or $K_{4,\,4}$ such that:*

$$\gamma_{pf}^{-1}(K_{n,\,m}) = \begin{cases} 2 & for\ K_{1,\,2} \\ n + m - 4 & if\, n,\ m\ =\ 2,\ 3,\ 4 \end{cases}$$

2. The Independent Pitchfork Domination

The independent pitchfork domination and the inverse independent pitchfork domination are defined here. Their bounds and properties are discussed and applied on some known graphs.

Definition 2.1. *Let G be a simple graph has no isolated vertices. A set D, is an independent pds if, D is a pds of G such that $G[D]$ has no edges. An independent pds is said minimal, if it has no independent pds as a subset.*

Definition 2.2. *The minimum independent pds D denoted by $\gamma^i_{pf}-$set, is the smallest minimal independent pds of G. The independent pitchfork domination number $\gamma^i_{pf}(G)$, is the order of the $\gamma^i_{pf}-$set.*

Definition 2.3. *Let $G = (V, E)$ be a graph with $\gamma^i_{pf}-$set D. A subset $D^{-1} \subseteq V - D$ is an inverse independent pds if it is a pds of G and $G[D^{-1}]$ has no edges. A set D^{-1} is said minimal inverse pds, if it has no independent pds as a subset.*

Definition 2.4. *The minimum inverse independent pds D^{-1} denoted by $\gamma^{-i}_{pf}-$set, is the smallest minimal inverse independent pds of G. The inverse independent pitchfork domination number $\gamma^{-i}_{pf}(G)$, is the order of the $\gamma^{-i}_{pf}-$set.*

Example 2.5. *In the path graph P_4 of a vertex set $\{v_1, v_2, v_3, v_4\}$, if $D = \{v_2, v_4\}$, $\{v_1, v_4\}$ or $\{v_1, v_3\}$, then it is an independent pds. While if $D = \{v_2, v_3\}$, then D is a pds but not independent.*

Remark 2.6. *Let G be a graph has $\gamma^i_{pf}-$set D, then:*

1. $|V(G)| \geq 2$.
2. $\gamma^i_{pf}(G) \geq 1$.
3. $deg(v) \leq 2$ *for every* $v \in D$.
4. *If* $deg(v) \geq 3$, *then* v *is neither in* D *nor in* D^{-1}.
5. $\gamma^i_{pf}(C_n) = \gamma_{pf}(C_n) = \lceil \frac{n}{3} \rceil$.
6. $\gamma^i_{pf}(P_n) = \gamma_{pf}(P_n) = \lceil \frac{n}{3} \rceil$.

Theorem 2.7. *Let $G = (n, m)$ be a graph with independent pitchfork domination, then:*

$$\gamma^i_{pf}(G) \leq m \leq \binom{n}{2} + \frac{1}{2}(\gamma^i_{pf}(G))^2 + \frac{1}{2}(5 - 2n)\,\gamma^i_{pf}(G).$$

Proof. Let D be the $\gamma^i_{pf}-$ set in G, then:
Case 1: Let $G[V - D]$ be a null subgraph to be G has as few edges as possible since $G[D]$ has no edges. Between every vertex of D to $V - D$, there exist one edge at least. Then, there is $|D| = \gamma^i_{pf}(G)$ number of edges between D and $V - D$. Thus, $\gamma^i_{pf}(G) \leq m$.
Case 2: Suppose that $G[V - D]$ be a complete subgraph having a maximum number of edges. Let m_1 be the number of edges of $G[V - D]$, then:

$$m_1 = \frac{|V - D||V - D - 1|}{2} = \frac{(n - \gamma^i_{pf})(n - \gamma^i_{pf} - 1)}{2}$$

Between every vertex of D to $V - D$, there exist two edges at most. Then, there is $m_2 = 2|D| = 2\gamma^i_{pf}(G)$. Thus, the sum of edges in G is $m_1 + m_2 \leq \binom{n}{2} + \frac{1}{2}(\gamma^i_{pf}(G))^2 + \frac{1}{2}(5 - 2n)\,\gamma^i_{pf}(G)$. □

Theorem 2.8. *Let $G = (n, m)$ be a graph with an inverse independent pitchfork domination, if $D^{-1} = V - D$, then:*

$$\gamma^{-i}_{pf}(G) \leq m \leq 2\,\gamma^{-i}_{pf}(G).$$

Proof. Let D be a $\gamma^i_{pf}-$set of G and D^{-1} be a $\gamma^{-i}_{pf}-$set of G. Since D^{-1} is a $\gamma^{-i}_{pf}-$set of G, then $G[D]$ is a null graph. On other hand, since D^{-1} is $\gamma^{-i}_{pf}-$set of G, then $G[D^{-1}]$ is a null graph where $D^{-1} = V - D$. Hence, the sizes of $G[D]$ and $G[V - D]$ are zeros. Therefore, the upper and lower bounds are affected by only the number of edges between D^{-1} and D as follows:
Case 1: Suppose that from every $v \in D^{-1}$ there is one edge (at least) to set D. Thus, $m \geq |D^{-1}| = \gamma^{-i}_{pf}(G)$.
Case 2: Suppose that there are two edges (at most) from every $v \in D^{-1}$ to D. Thus, $m \leq 2\,|D^{-1}| = 2\,\gamma^{-i}_{pf}(G)$. □

Theorem 2.9. *Let $G = (n, m)$ be a graph with an inverse independent pitchfork domination number $\gamma^{-i}_{pf}(G)$ andD^{-1} be a $\gamma^{-i}_{pf}-$set D^{-1}. If $D^{-1} \neq V - D$. Then:*

$$n - \gamma^{i}_{pf} \leq m \leq \binom{n - \gamma^{i}_{pf} - \gamma^{-i}_{pf}}{2} + 2\gamma^{-i}_{pf} + \gamma^{i}_{pf}$$

Proof. Since $D^{-1} \neq V - D$, let $V - D = D^{-1} \cup T$ where $D^{-1} \cap T = \phi$, then $|T| = |V| - |D| - |D^{-1}| = n - \gamma^{i}_{pf} - \gamma^{-i}_{pf}$ where $V(G) = D \cup D^{-1} \cup T$. Since D is $\gamma^{i}_{pf}-$set, then $G[D]$ is a null graph by definition of independent pitchfork domination. Also, since D^{-1} is a $\gamma^{-i}_{pf}-$set, then $G[D^{-1}]$ is a null graph. Hence, $G[D]$ and $G[D^{-1}]$ are nulls.
Case 1: Let $G[T]$ is a null graph. From every $v \in D^{-1}$ there is at least one edge to $D \cup T$ (say m_1). Then, $m_1 = \gamma^{-i}_{pf}(G)$. So that from every vertex in T there is one edge to set D (say m_2). Therefore, $m_2 = |T| = n - \gamma^{i}_{pf} - \gamma^{-i}_{pf}$. Hence, $m = m_1 + m_2 = \gamma^{-i}_{pf}(G) + n - \gamma^{i}_{pf}(G) - \gamma^{-i}_{pf}(G) = n - \gamma^{i}_{pf}(G)$. Then, in general $m \geq n - \gamma^{i}_{pf}(G)$.
Case 2: Let $G[T]$ is a complete subgraph. Let m_1 be the number of edges in $G[T]$ which equals to $\binom{n-\gamma^{i}_{pf}-\gamma^{-i}_{pf}}{2}$. Suppose that there are at most two edges from every vertex in D^{-1} to both D and T such that the number of edges between D^{-1} and $D \cup T$ is $m_2 = 2\,|D^{-1}| = 2\,\gamma^{-i}_{pf}(G)$. So that, there is at most $|D|$ edges from D to T such that there is exactly one edge from every vertex in D to T (if there exist $v \in D$ with two edges with T, then v is not dominated by D^{-1} since it has no neighbor in D^{-1}). Then, the number of edges between D and T say m_3 equals $|D| = \gamma^{i}_{pf}$. Hence, $m = m_3 + m_4 + m_5 \leq \binom{n-\gamma^{i}_{pf}-\gamma^{-i}_{pf}}{2} + 2\gamma^{-i}_{pf} + \gamma^{i}_{pf}$. □

Remark 2.10. *Let $G = (n, m)$ be a graph with an independent pitchfork domination. Then:*
$\gamma(G) \leq \gamma_{pf}(G) \leq \gamma^{i}_{pf}(G)$ *and* $\gamma^{i}(G) \leq \gamma^{i}_{pf}(G)$.

Theorem 2.11. *Let $G = (n, m)$ be a graph with an independent pitchfork domination. Then every independent pds D, is a minimal independent pds.*

Proof. By condition 3 of Theorem (1.2) , $G[D]$ is a null graph. □

Proposition 2.12. *For W_n, K_n and $K_{n,m}$ graphs, we have:*
1- The wheel graph W_n has no independent pitchfork domination.
2- The complete graph K_n has no independent pitchfork domination for n geq4.
3- The complete bipartite graph $K_{n,m}$ has an independent pitchfork domination if and only if $n \leq 2$ such that:

$$\gamma^{i}_{pf}(K_{n,m}) = \begin{cases} \gamma_{pf}(K_{n,m}), & \text{if } n, m \leq 2 \\ m, & \text{if } n = 2 \wedge m > 2 \end{cases}$$

Proof. Since the induced subgraph of any pitchfork dominating set in W_n and in K_n has an edges. So that in $K_{n,m}$ for $n, m \geq 3$. □

Remark 2.13. *Let $G = C_n$, P_n, W_n, K_n and $K_{n,m}$, then we have:*

1. $\gamma^{-i}_{pf}(C_n) = \gamma^{-1}_{pf}(C_n)$.
2. $\gamma^{-i}_{pf}(P_n) = \gamma^{-1}_{pf}(P_n)$.
3. *W_n has no inverse independent pitchfork domination for all n.*
4. *K_n has no inverse independent pitchfork domination for $n \geq 4$.*
5. *$K_{n,m}$ has an inverse independent pitchfork domination if and only if n, $m = 1, 2$ where $\gamma^{-i}_{pf}(K_{n,m}) = \gamma^{-1}_{pf}(K_{n,m})$.*

Proposition 2.14. *Let C_n be a cycle graph, then:*
1. $\overline{C}_n$ has an independent pitchfork domination if and only if $n = 4, 5$ such that $\gamma^i_{pf}(\overline{C}_n) = 2$.
2. $\overline{C}_n$ has an inverse independent pitchfork domination if and only if $n = 4, 5$ such that $\gamma^{-i}_{pf}(\overline{C}_n) = 2$.
3. $\gamma^i_{pf}(\overline{C}_n) + \gamma^{-i}_{pf}(\overline{C}_n) = n$ if and only if $n = 4$.

Proof. It is clear, for $n = 3, 4, 5$. If $n \geq 6$, then $\overline{C}_n$ has no independent pitchfork dominating set. Since any independent dominating set has one or more vertices dominates more than two vertices. Any pitchfork dominating set has at least one edge between two of it's vertices. □

Proposition 2.15. *Let P_n be a path, then:*
1. $\overline{P}_n$ has an independent pitchfork domination if and only if $n = 4, 5$ where $\gamma^i_{pf}(\overline{P}_n) = 2$.
2. $\overline{P}_n$ has an inverse independent pitchfork domination if and only if $n = 4$ where $\gamma^{-i}_{pf}(\overline{P}_4) = 2$.
3. $\gamma^i_{pf}(\overline{P}_n) + \gamma^{-i}_{pf}(\overline{P}_n) = n$ if and only if $n = 4$.

Proof. Similar to proof of Proposition (2.14). □

Remark 2.16. *$\overline{K}_n$ and $\overline{W}_n$ has no independent pitchfork domination.*

3. An Annihilator Pitchfork Domination

In this section, an annihilator pitchfork domination and its inverse domination are introduced. Their bounds and properties are putted and discussed on some standard graphs.

Definition 3.1. *A subset $D \subseteq V(G)$ is an annihilator pds in $G = (V, E)$ if, D is a pds of G and $G[V - D]$ has no edges. A set D is a minimal annihilator pds if it has no annihilator pds as a subset.*

Definition 3.2. *The minimum annihilator pds D denoted by $\gamma^a_{pf}-$set, is the smallest minimal annihilator pds in G. The annihilator pitchfork domination number $\gamma^a_{pf}(G)$, is the order of the $\gamma^a_{pf}-$set.*

Definition 3.3. *Let $G = (V, E)$ be a graph with $\gamma^a_{pf}-$set D. A subset $D^{-1} \subseteq V - D$ is an inverse annihilator pds, if D^{-1} is pds of G and $G[V - D^{-1}]$ has no edges. D^{-1} is a minimal inverse annihilator pds, if it has no inverse annihilator pds as a subset.*

Definition 3.4. *The minimum inverse annihilator pds D^{-1}, denoted by $\gamma^{-a}_{pf}-$set is the smallest minimal inverse annihilator pds of G. The inverse annihilator pitchfork domination number $\gamma^{-a}_{pf}(G)$, is the order of the $\gamma^{-a}_{pf}-$set.*

Example 3.5. *In the cycle graph C_4 of a vertex set $\{v_1, v_2, v_3, v_4\}$, if $D = \{v_1, v_3\}$ or $\{v_2, v_4\}$, then it is an annihilator pitchfork dominating set. While if $D = \{v_1, v_2\}$, $\{v_1, v_4\}$, $\{v_2, v_3\}$ or $\{v_3, v_4\}$, then D is a pitchfork dominating set but not annihilator pitchfork dominating set.*

Remark 3.6. *Let D be a $\gamma^a_{pf}-$set, and D^{-1} be a $\gamma^{-a}_{pf}-$set in G, then:*

1. *$|V(G)| \geq 2$.*
2. *$\gamma^a_{pf}(G) \geq 1$.*
3. *$deg(v) \leq 2$ for every $v \in D^{-1}$.*

Proposition 3.7. *Let $G = (n, m)$ be a graph with $\gamma^a_{pf}-$set D, if G has $\gamma^{-a}_{pf}-$set D^{-1}, then $D^{-1} = V - D$ and $\gamma^a_{pf}(G) + \gamma^{-a}_{pf}(G) = n$.*

Proof. Since D is a $\gamma^a_{pf}-$set of G, then $G[V - D]$ is a null graph. Thus, $D^{-1} = V - D$ by Theorem (1.3). Hence, $|D^{-1}| = |V - D| = n - \gamma^a_{pf}(G)$. □

Theorem 3.8. *Let $G=(n,m)$ be a graph with an annihilator pds, then:*

$$\gamma^a_{pf}(G) \leq m \leq \frac{1}{2}[\,\gamma^a_{pf}(G)\,]^2 + \frac{3}{2}\,\gamma^a_{pf}(G).$$

Proof. If D be a $\gamma^a_{pf}-$ set in G, then:
Case 1: Let $G[D]$ be a a null graph, where $G[V-D]$ has no edges. Now, there is one edge from every $u \in D$ and $V-D$ (at least). Thus, the number of edges between D and $V-D$ is $|D| = \gamma^a_{pf}(G)$. Hence, $\gamma^a_{pf}(G) \leq m$.
Case 2: Let $G[D]$ be a complete subgraph to be having maximum number of edges. Let m_1 be the number of edges of $G[D]$, then:

$$m_1 = \frac{|D||D-1|}{2} = \frac{\gamma^a_{pf}\,(\gamma^a_{pf}-1)}{2}$$

Now, there is two edges between every $u \in D$ and $V-D$ (at most). Hence, the number of edges between D and $V-D$ equal $2|D| = 2\gamma^a_{pf}(G) = m_2$. Thus, the number of an edges in G is $m = m_1 + m_2 = \frac{1}{2}\,\gamma^a_{pf}(\gamma^a_{pf}-1) + 2\gamma^a_{pf}$. Hence, $m \leq \frac{1}{2}(\gamma^a_{pf}(G))^2 + \frac{3}{2}\,\gamma^a_{pf}(G)$. □

Theorem 3.9. *Let $G=(n,m)$ be a graph with an inverse annihilator pitchfork domination, then:*

$$\gamma^{-a}_{pf}(G) \leq m \leq 2\,\gamma^{-a}_{pf}(G).$$

Proof. Let D be a $\gamma^a_{pf}-$set in G and let D^{-1} be a $\gamma^{-a}_{pf}-$set in G. Since G has an inverse annihilator pitchfork domination, then $D^{-1} = V-D$ by Proposition (3.7). Also $G[V-D]$ is a null graph by definition of annihilator domination, then $G[D^{-1}]$ is a null graph. On other hand, since D^{-1} is a $\gamma^{-a}_{pf}-$set of G, then $G[D]$ is a null graph by definition of annihilator domination where $D = V - D^{-1}$. Hence, the sizes of $G[D]$ and $G[V-D]$ are zeros. Therefore, the upper and lower bounds are affected by only the number of edges between D^{-1} and D. □

Remark 3.10. *Let $G=(n,m)$ be a graph having an annihilator pds, then:*
$\gamma(G) \leq \gamma_{pf}(G) \leq \gamma^a_{pf}(G)$ *and* $\gamma^a(G) \leq \gamma^a_{pf}(G)$.

Theorem 3.11. *Let C_n be a cycle $(n \geq 3)$, then:*
1. C_n has an annihilator pitchfork domination for all n where $\gamma^a_{pf}(C_n) = \lceil \frac{n}{2} \rceil$.
2. C_n has an inverse annihilator pitchfork domination if and only if n is an even integer, where $\gamma^{-a}_{pf}(C_n) = \lceil \frac{n}{2} \rceil$.
3. $\gamma^a_{pf}(C_n) + \gamma^{-a}_{pf}(C_n) = n$ if and only if n is an even integer.

Proof. 1. Let us start from any vertex $v_1 \in C_n$ to choose it in D and leave the next vertex and so on. Then, D is a pds, where every u in it dominates exactly two vertices of $V-D$, unless when n is an odd integer, the vertices v_1 and v_n dominate one vertex. So that, $G[V-D]$ has no edges. Hence, D is a $\gamma^a_{pf}-$set and $\gamma^a_{pf}(C_n) = \lceil \frac{n}{2} \rceil$.
2. It is clear when n is an even integer, then, $D^{-1} = V-D$ and $\gamma^{-a}_{pf}(C_n) = \gamma^a_{pf}(C_n) = \lceil \frac{n}{2} \rceil$ from Proposition (3.7). But if n is an odd integer, then C_n has no $\gamma^{-a}_{pf}-$set since $\gamma^a_{pf}(C_n) > \frac{n}{2}$ by Note (1).
3. By Proposition (3.7). □

Theorem 3.12. *Let P_n be a path $(n \geq 2)$, then:*
1. P_n has an inverse annihilator pitchfork domination for all n where $\gamma^a_{pf}(P_n) = \lfloor \frac{n}{2} \rfloor$ and $\gamma^{-a}_{pf}(P_n) = \lceil \frac{n}{2} \rceil$.
2. $\gamma^a_{pf}(P_n) + \gamma^{-a}_{pf}(P_n) = n$ if and only if n is an even integer.

Proof. 1. Suppose that $D = \{v_i;\ i\, is\, an\, even\, no.\}$ and $D^{-1} = \{v_i;\ i\, is\, an\, odd\, no.\}$. Then, D is pds where every v in it dominates exactly two vertices of $V-D$, unless when n is an even integer, the last vertex v_n dominates only v_{n-1}. Also, $G[V-D]$ has no edges. Hence, D is a $\gamma^a_{pf}-$set and $\gamma^a_{pf}(P_n) = \lfloor \frac{n}{2} \rfloor$. Also

$D^{-1} = V - D$ in which every vertex dominates one or two vertices and all vertices of D are non-adjacent together. Hence, D^{-1} is a $\gamma_{pf}^{-a}-$set and $\gamma_{pf}^{-a}(P_n) = \lceil \frac{n}{2} \rceil$.
2. It is clear if n is an even integer, then P_n has an inverse annihilator pitchfork domination. So we get the result from Proposition (3.7) where $|D| = |V - D| = \frac{n}{2}$. □

Theorem 3.13. *Let W_n be the wheel graph of order $n+1$ $(n \geq 3)$, then:*
1. W_n has an annihilator pitchfork domination if and only if $n \leq 6$ where:

$$\gamma_{pf}^{a}(W_n) = \begin{cases} 3, & \text{if } n = 3 \\ n-1, & \text{if } n = 4, 5, 6 \end{cases}$$

2. W_n has no inverse annihilator pitchfork domination for all other values of n.

Proof. 1. Since v_{n+1} is adjacent with all other vertices. Therefore, if $v_{n+1} \in V - D$, then $G[V - D]$ has edges which is contradict our domination definition. Hence, $v_{n+1} \in D$. So D must be containing all the vertices unless one or two non-adjacent vertices. Therefore, if $n = 3, 4, 5$. Assume that D contains v_k and leave v_{k+1} and so on for any $1 \leq k \leq n$. If $n = 6$, D contains v_k, v_{k+1} and leave v_{k+2} and so on. Then, D is pds where every t in it dominates two or one vertices from $V - D$ with $G[V - D]$ has no edges. Hence, D is a $\gamma_{pf}^{a}-$set with order 3 for $n = 3, 4$ and order $n - 1$ for $n = 5, 6$.
2. Since $\gamma_{pf}^{a}(W_n) > \frac{n+1}{2}$, then W_n has no inverse annihilator pitchfork domination by Note (1). □

Theorem 3.14. *Let $K_{n,m}$ be the complete bipartite graph, then:*
1. $K_{n,m}$ has an annihilator pitchfork domination if and only if $n \leq 2$ such that:

$$\gamma_{pf}^{a}(K_{n,m}) = \begin{cases} 1, & \text{if } n = 1 \wedge m = 2 \\ 2, & \text{if } n = m = 2, \\ m, & \text{if } n = 1, 2 \wedge m \geq 3 \end{cases}$$

2. $K_{n,m}$ has an inverse annihilator pitchfork domination if and only if $K_{n,m} = K_{1,2} \vee K_{2,2}$ where $\gamma_{pf}^{-a}(K_{n,m}) = 2$.
3. $\gamma_{pf}^{a}(K_{n,m}) + \gamma_{pf}^{-a}(K_{n,m}) = n + m$ if and only if $K_{n,m} = K_{1,2} \vee K_{2,2}$.

Proof. 1. If $n \leq 2$ the proof is clear. Now since the induced subgraph $G[V - D]$ of any pitchfork dominating set in $K_{n,m}$ has an edges for $n \geq 3$ because it has vertices from the two sets of the graph. Hence, $K_{n,m}$ has no annihilator pitchfork domination.
2. Since $\gamma_{pf}^{a}(K_{n,m}) > \frac{n+m}{2}$ for $m > 2$, then $K_{n,m}$ has no inverse annihilator pitchfork domination by Note (1).
3. By Proposition (3.7). □

Remark 3.15. *The complete graph K_n has an annihilator pitchfork domination for $n \geq 2$ such that $\gamma_{pf}^{a}(K_n) = n - 1$.*

Theorem 3.16. *Let C_n be a cycle $(n \geq 4)$. Then:*
1. $\gamma_{pf}^{a}(\overline{C}_n) = n - 2$.
2. $\overline{C}_n$ has an inverse annihilator pitchfork domination if and only if $n = 4$ such that $\gamma_{pf}^{-a}(\overline{C}_4) = 2$.
3. $\gamma_{pf}^{a}(\overline{C}_n) + \gamma_{pf}^{-a}(\overline{C}_n) = n$ if and only if $n = 4$.

Proof. 1. Since $deg(v) = n - 3$ for all $v \in V(\overline{C}_n)$, then D must be containing all the vertices of $\overline{C}_n$ unless two respective vertices. Therefore, D is a $\gamma_{pf}^{a}-$set. Since all vertices in D dominate two or one vertices with $G[V - D]$ has no edges.
2. It is clear for $n = 4$. If $n > 4$, then $\overline{C}_n$ has no annihilator pitchfork domination since $\gamma_{pf}^{a}(\overline{C}_n) > \frac{n}{2}$ by Note (1).
3. By Proposition 3.7. □

Proposition 3.17. *Let P_n be a path $(n \geq 4)$, then:*
1. $\gamma_{pf}^{a}(\overline{P}_n) = n - 2$.
2. $\overline{P}_n$ *has an inverse annihilator pitchfork domination if and only if* $n = 4$ *such that* $\gamma_{pf}^{-a}(\overline{P}_4) = 2$.
3. $\gamma_{pf}^{a}(\overline{P}_n) + \gamma_{pf}^{-a}(\overline{P}_n) = n$ *if and only if* $n = 4$.

Proof. Similar to proof of Theorem (3.16) where $deg(v) = n - 2$ for the two end vertices while $deg(v) = n - 3$ for the others. □

Remark 3.18. $\overline{K}_n$ *and* $\overline{W}_n$ *has no annihilator pitchfork domination.*

Theorem 3.19. *Let* $G = \overline{C_4}$ *or* $K_{2,m}$ $(m \geq 3)$, *then every annihilator pds in G is a minimal annihilator pds.*

Proof. Since $G[D]$ is a null graph, then by condition 3 of Theorem (1.2), we get the result. □

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