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Cofficient Estimates for a General Subclass of Bi-univalent Functions

Khosrow Hosseinzadeh

ABSTRACT: In this paper, we introduce and investigate an interesting subclass $\mathcal{S}_{\Sigma}^{h,p}(A,B,C,\lambda)$ of bi-univalent functions in the open unit disk $\mathbb{U}$. Furthermore, we find estimates on the $|a_2|$ and $|a_3|$ coefficients for functions in this subclass. The coefficient bounds presented here generalize some recent works of several earlier authors.

Key Words: Analytic functions, Univalent functions, Bi-univalent functions, Coefficient estimates.

Contents

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions in the unit disk

$$\mathbb{U}=\{z\in\mathbb{C}:|z|<1\},$$

that have the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n. \tag{1.1}$$

Further, we shall denote by $\mathcal{S}$ the class of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$(for details see [1, 3, 5]).

Since univalent functions are one-to-one, they are invertible and inverse functions need not be defined on the entire unit disk $\mathbb{U}$. The Koebe one-quarter theorem [5] ensures that the image of $\mathbb{U}$ under every univalent function $f\in\mathcal{S}$ contains a disk of radius $\frac{1}{4}$. So every function $f\in\mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z))=z\ (z\in\mathbb{U})$$

and

$$f(f^{-1}(w))=w\ \left(|w|<r_0(f),\ r_0(f)\geq\frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots. \tag{1.2}$$

A function $f\in\mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$, if both f and f^{-1} are univalent in $\mathbb{U}$ (see [10]).

Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). The class of bi-univalent functions was first introduced and studied by Lewin [6], where it was proved that $|a_2|\leq 1.51$.

Brannan and Taha [1](see also [2]), also investigated certain subclasses of bi-univalent functions and found non-sharp estimates on the first two Taylor-Maclurin coefficients $|a_2|$ and $|a_3|$. For a brief history and interesting examples of functions in the class Σ, see [10].

Netanyahu [8], showed that $\max\limits_{f\in\Sigma}|a_2|=\frac{4}{3}$. The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ for $n=3,4,...$ is presumably still an open problem.

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Two of the most famous subclasses of univalent functions are the class $\mathcal{S}^*(\beta)$ of starlike functions of order $\beta(0 \le \beta < 1)$ and the class $K(\beta)$ of convex functions of order $\beta(0 \le \beta < 1)$. By definition, we have

$$\mathcal{S}^*(\beta) = \left\{ f \in \mathcal{S} : Re(\frac{zf'(z)}{f(z)}) > \beta \ , \ z \in \mathbb{U} \right\}$$

and

$$K(\beta) = \left\{ f \in \mathcal{S} : Re(1 + \frac{zf''(z)}{f'(z)}) > \beta \ , \ z \in \mathbb{U} \right\}.$$

For $\beta(0 \le \beta < 1)$, a function $f \in \Sigma$ is in the class $\mathcal{S}^*_\Sigma(\beta)$ of strongly bi-starlike functions of order $\beta(0 \le \beta < 1)$, or $K_\Sigma(\beta)$ of strongly bi-convex functions of order $\beta(0 \le \beta < 1)$, if both f and its inverse map f^{-1} are, respectively, starlike or convex of order $\beta(0 \le \beta < 1)$.

The object of the present paper is to introduce a new subclass of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this new subclass of the functions class Σ employing the techniques used earlier by Srivastava et al. (see [9]).

Definition 1.1 ([7]). A function $f(z)$ given by (1.1) is said to be in the $\mathcal{S}_\Sigma(\alpha,\lambda)$ $(0 < \alpha \le 1,\ 0 \le \lambda \le 1)$, if the following conditions are satisfied:

$$f \in \Sigma,\ \left| arg\left(\frac{zf'(z) + (2\lambda^2 - \lambda)z^2 f''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)zf'(z) + (2\lambda^2 - 3\lambda + 1)f(z)} \right) \right| < \frac{\alpha\pi}{2} (0 < \alpha \le 1,\ 0 \le \lambda \le 1,\ z \in \mathbb{U})$$

and

$$\left| arg\left(\frac{wg'(w) + (2\lambda^2 - \lambda)w^2 g''(w)}{4(\lambda - \lambda^2)w + (2\lambda^2 - \lambda)wg'(w) + (2\lambda^2 - 3\lambda + 1)g(w)} \right) \right| < \frac{\alpha\pi}{2} (0 < \alpha \le 1,\ 0 \le \lambda \le 1,\ w \in \mathbb{U}),$$

where g is the extension of f^{-1} to $\mathbb{U}$.

Theorem 1.1 ([7]). *Let the function $f(z)$ given by (1.1) be in the $\mathcal{S}_\Sigma(\alpha,\lambda)$ $(0 < \alpha \le 1,\ 0 \le \lambda \le 1)$. Then*

$$|a_2| \le \frac{2\alpha}{\sqrt{\alpha(1 - 2\lambda + 25\lambda^2 - 44\lambda^3 + 20\lambda^4) + (1 + 3\lambda - 2\lambda^2)^2}}$$

and

$$|a_3| \le \frac{\alpha}{1 + 2\lambda^2} + \frac{4\alpha^2}{(1 + 3\lambda - 2\lambda^2)^2}.$$

Definition 1.2 ([7]). A function $f(z)$ given by (1.1) is said to be in the $\mathcal{S}_\Sigma(\beta,\lambda)$ $(0 \le \beta < 1,\ 0 \le \lambda \le 1)$, if the following conditions are satisfied:

$$f \in \Sigma,\ Re\left(\frac{zf'(z) + (2\lambda^2 - \lambda)z^2 f''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)zf'(z) + (2\lambda^2 - 3\lambda + 1)f(z)} \right) > \beta (0 \le \beta < 1,\ 0 \le \lambda \le 1,\ z \in \mathbb{U})$$

and

$$Re\left(\frac{wg'(w) + (2\lambda^2 - \lambda)w^2 g''(w)}{4(\lambda - \lambda^2)w + (2\lambda^2 - \lambda)wg'(w) + (2\lambda^2 - 3\lambda + 1)g(w)} \right) > \beta (0 \le \beta < 1,\ 0 \le \lambda \le 1,\ w \in \mathbb{U}),$$

where g is the extension of f^{-1} to $\mathbb{U}$.

It is stated that in Theorem 3.1 in [7], the calculations done by Magesh for the bound $|a_3|$ are inaccurate. To remove this remarkable mistake, we've revised the calculations appropriately (see Theorem1.2).

Theorem 1.2 ([7]). *Let the function $f(z)$ given by (1.1) be in the $\mathcal{S}_\Sigma(\beta,\lambda)$ $(0 \le \beta < 1,\ 0 \le \lambda \le 1)$. Then*

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1}}$$

and

$$|a_3| \le \frac{2(1-\beta)}{12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1} + \frac{1-\beta}{2\lambda^2 + 1}.$$

2. Coefficient bounds for the function class $\mathcal{S}_{\Sigma}^{h,p}(A,B,C,\lambda)$

In this section, we introduce the subclass $\mathcal{S}_{\Sigma}^{h,p}(A,B,C,\lambda)$ $(0 \le \lambda \le 1)$ and find the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this subclass.

Definition 2.1. Let the functions $h, p : \mathbb{U} \to \mathbb{C}$ be analytic functions so that

$$\min\{\Re\left((h(z)\right), \Re\left(p(z)\right)\} > 0 \ (z \in \mathbb{U}) \ and \ h(0) = p(0) = 1.$$

Also, let the continuous functions $A, B, C : [0,1] \to \mathbb{R}$ be so constrained that

$$A(\lambda) + B(\lambda) + C(\lambda) = 1, \ C(\lambda) \neq 2 \ and \ 3 + 3A(\lambda) - C(\lambda) \neq 0; \ \lambda \in [0,1].$$

A function $f(z) \in \mathcal{A}$ given by (1.1) is said to be in the class $\mathcal{S}_{\Sigma}^{h,p}(A,B,C,\lambda)$ $(0 \le \lambda \le 1)$, if the following conditions are satisfied:

$$f \in \Sigma, \ \frac{zf'(z) + A(\lambda)z^2 f''(z)}{B(\lambda)z + A(\lambda)zf'(z) + C(\lambda)f(z)} \in h(\mathbb{U}) \ (z \in \mathbb{U}) \tag{2.1}$$

and

$$\frac{wg'(w) + A(\lambda)w^2 g''(w)}{B(\lambda)w + A(\lambda)wg'(w) + C(\lambda)g(w)} \in p(\mathbb{U}) \ (w \in \mathbb{U}), \tag{2.2}$$

where g is the extension of f^{-1} to $\mathbb{U}$.

Remark 2.1. There are many choices of the functions $h(z)$ and $p(z)$ which would provide interesting subclasses of the analytic function class $\mathcal{A}$. For example, if we get

$$h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \ (0 < \alpha \le 1, \ z \in \mathbb{U}),$$

it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1. If $f(z) \in \mathcal{S}_{\Sigma}^{h,p}(A,B,C,\lambda)$, $A(\lambda) = 2\lambda^2 - \lambda$, $B(\lambda) = 4(\lambda - \lambda^2)$ and $C(\lambda) = 2\lambda^2 - 3\lambda + 1$ then

$$f \in \Sigma, \ \left| arg\left(\frac{zf'(z) + (2\lambda^2 - \lambda)z^2 f''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)zf'(z) + (2\lambda^2 - 3\lambda + 1)f(z)}\right)\right| < \frac{\alpha\pi}{2} \ (0 < \alpha \le 1, \ 0 \le \lambda \le 1, \ z \in \mathbb{U})$$

and

$$\left| arg\left(\frac{wg'(w) + (2\lambda^2 - \lambda)w^2 g''(w)}{4(\lambda - \lambda^2)w + (2\lambda^2 - \lambda)wg'(w) + (2\lambda^2 - 3\lambda + 1)g(w)}\right)\right| < \frac{\alpha\pi}{2}(0 < \alpha \le 1, \ 0 \le \lambda \le 1, \ w \in \mathbb{U}).$$

In this case, the function f is said to be in the class $\mathcal{S}_{\Sigma}(\alpha,\lambda)$ introduced and studied by Magesh and Yamini [7].

By putting $\lambda = 0(A(\lambda) = B(\lambda) = 0$ and $C(\lambda) = 1)$, the class $\mathcal{S}_{\Sigma}(\alpha,\lambda)$ reduces to the class of strongly bi-starlike functions of order $\alpha(0 < \alpha \le 1)$ and denoted by $\mathcal{S}_{\Sigma}^{*}(\alpha)$.

By putting $\lambda = \frac{1}{2}(A(\lambda) = C(\lambda) = 0$ and $B(\lambda) = 1)$, the class $\mathcal{S}_{\Sigma}(\alpha,\lambda)$ reduces to the class $\mathcal{H}_{\Sigma}^{\alpha}$ introduced and studied by Srivastava et al. [10] and for $\lambda = 1(B(\lambda) = C(\lambda) = 0$ and $A(\lambda) = 1)$, the class $\mathcal{S}_{\Sigma}(\alpha,\lambda)$ reduces to the class of strongly bi-convex functions of order $\alpha(0 < \alpha \le 1)$ and denoted by $\mathcal{K}_{\Sigma}(\alpha)$.

If we get

$$h(z) = p(z) = \frac{1 + (1-2\beta)z}{1-z} \ (0 \le \beta < 1, \ z \in \mathbb{U}),$$

it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1. If $f(z) \in \mathcal{S}_{\Sigma}^{h,p}(A,B,C,\lambda)$, $A(\lambda) = 2\lambda^2 - \lambda$, $B(\lambda) = 4(\lambda - \lambda^2)$ and $C(\lambda) = 2\lambda^2 - 3\lambda + 1$ then

$$f \in \Sigma,\ Re\left(\frac{zf'(z)+(2\lambda^2-\lambda)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)}\right) > \beta(0 \le \beta < 1,\ 0 \le \lambda \le 1,\ z \in \mathbb{U})$$

and

$$Re\left(\frac{wg'(w)+(2\lambda^2-\lambda)w^2g''(w)}{4(\lambda-\lambda^2)w+(2\lambda^2-\lambda)wg'(w)+(2\lambda^2-3\lambda+1)g(w)}\right) > \beta(0 \le \beta < 1,\ 0 \le \lambda \le 1,\ w \in \mathbb{U}).$$

In this case, the function f is said to be in the class $\mathcal{S}_\Sigma(\beta,\lambda)$ introduced and studied by Magesh and Yamini [7].

By putting $\lambda = 0(A(\lambda) = B(\lambda) = 0$ and $C(\lambda) = 1)$, the class $\mathcal{S}_\Sigma(\beta,\lambda)$ reduces to the class of strongly bi-starlike functions of order $\beta(0 \le \beta < 1)$ and denoted by $\mathcal{S}^*_\Sigma(\beta)$.

By putting $\lambda = \frac{1}{2}(A(\lambda) = C(\lambda) = 0$ and $B(\lambda) = 1)$, the class $\mathcal{S}_\Sigma(\beta,\lambda)$ reduces to the class $\mathcal{H}_\Sigma(\beta)$ introduced and studied by Srivastava et al. [10] and for $\lambda = 1(B(\lambda) = C(\lambda) = 0$ and $A(\lambda) = 1)$, the class $\mathcal{S}_\Sigma(\beta,\lambda)$ reduces to the class of strongly bi-convex functions of order $\beta(0 \le \beta < 1)$ and denoted by $\mathcal{K}_\Sigma(\beta)$.

Note: Let $A := A(\lambda), B := B(\lambda)$ *and* $C := C(\lambda)$.

Theorem 2.1. *A function $f(z)$ given by (1.1) is said to be in the $\mathcal{S}^{h,p}_\Sigma(A,B,C,\lambda)$ $(0 \le \lambda \le 1)$. Then*

$$|a_2| \le min \begin{cases} \sqrt{\frac{|h'(0)|^2+|p'(0)|^2}{2(C-2)^2}} \\ \frac{1}{2}\sqrt{\frac{|h''(0)|+|p''(0)|}{|(3+3A-C)+(C-2)(2A+C)|}};\ (3+3A-C)+(C-2)(2A+C) \neq 0 \end{cases}$$

and

$$|a_3| \le min \begin{cases} \frac{|h''(0)|+|p''(0)|}{4|3+3A-C|} + \frac{|h'(0)|^2+|p'(0)|^2}{2(C-2)^2} \\ \frac{|h''(0)|+|p''(0)|}{4|3+3A-C|} + \frac{|h''(0)|+|p''(0)|}{4|(3+3A-C)+(C-2)(2A+C)|}; \\ (3+3A-C)+(C-2)(2A+C) \neq 0. \end{cases}$$

Proof. First of all, we write the argument inequalities in (2.1) and (2.2) in their equivalent forms as follows:

$$\frac{zf'(z)+Az^2f''(z)}{Bz+Azf'(z)+Cf(z)} = h(z)\ (z \in \mathbb{U}) \tag{2.3}$$

and

$$\frac{wg'(w)+Aw^2g''(w)}{Bw+Awg'(w)+Cg(w)} = p(w)\ (w \in \mathbb{U}), \tag{2.4}$$

respectively, where functions $h(z)$ and $p(w)$ satisfy the conditions of Defintion 2.1. Furthermore, the functions $h(z)$ and $p(w)$ have the following Taylor-Maclaurin series expensions:

$$h(z) = 1 + h_1z + h_2z^2 + h_3z^3... \tag{2.5}$$

and

$$p(w) = 1 + p_1w + p_2w^2 + p_3w^3...\ , \tag{2.6}$$

respectively. Now, upon substituting from (2.5) and (2.6) into (2.3) and (2.4), respectively, and equating the coefficients, we get

$$(2-C)a_2 = h_1, \tag{2.7}$$

$$(3+3A-C)a_3 + (C-2)(2A+C)a_2^2 = h_2, \tag{2.8}$$

$$-(2-C)a_2 = p_1 \tag{2.9}$$

and

$$-(3+3A-C)a_3+\{2(3+3A-C)+(C-2)(2A+C)\}a_2^2=p_2. \tag{2.10}$$

From (2.7) and (2.9), we obtain

$$p_1=-h_1, \tag{2.11}$$

$$a_2^2=\frac{h_1^2+p_1^2}{2(2-C)^2}. \tag{2.12}$$

If $(3+3A-C)+(C-2)(2A+C)\neq 0$, then by adding (2.8) and (2.10), we get

$$a_2^2=\frac{h_2+p_2}{2[(3+3A-C)+(C-2)(2A+C)]}. \tag{2.13}$$

Therfore, we find from the equations (2.12) and (2.13) that

$$|a_2|^2\leq\frac{|h'(0)|^2+|p'(0)|^2}{2(C-2)^2}$$

and

$$|a_2|^2\leq\frac{|h''(0)|+|p''(0)|}{4|(3+3A-C)+(C-2)(2A+C)|},$$

respectively. So we get the desired estimate on the coefficient $|a_2|$ asserted. Next, in order to find the bound on the coefficient $|a_3|$, we subtract (2.10) from (2.8). We thus get

$$2(3+3A-C)a_3-2(3+3A-C)a_2^2=h_2-p_2. \tag{2.14}$$

Upon substituting the value of a_2^2 from (2.12) into (2.14), it follows that

$$a_3=\frac{h_2-p_2}{2(3+3A-C)}+\frac{h_1^2+p_1^2}{2(2-C)^2}. \tag{2.15}$$

We thus find that

$$|a_3|\leq\frac{|h''(0)|+|p''(0)|}{4|3+3A-C|}+\frac{|h'(0)|^2+|p'(0)|^2}{2(C-2)^2}.$$

If $(3+3A-C)+(C-2)(2A+C)\neq 0$, then by substituting the value of a_2^2 from (2.13) into (2.14), it follows that

$$a_3=\frac{h_2-p_2}{2(3+3A-C)}+\frac{h_2+p_2}{2[(3+3A-C)+(C-2)(2A+C)]}. \tag{2.16}$$

Consequently, we have

$$|a_3|\leq\frac{|h''(0)|+|p''(0)|}{4|3+3A-C|}+\frac{|h''(0)|+|p''(0)|}{4|(3+3A-C)+(C-2)(2A+C)|}.$$

□

3. Corollaries and Consequences

By putting

$$A(\lambda)=2\lambda^2-\lambda,\ B(\lambda)=4(\lambda-\lambda^2),\ C(\lambda)=2\lambda^2-3\lambda+1$$

and

$$h(z)=p(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}\quad(0<\alpha\leq 1,\ z\in\mathbb{U})$$

in Theorem 2.1, we obtain the following result.

Corollary 3.1. *Let the function $f(z)$ given by (1.1) be in the bi-univalent function class $\mathcal{S}_\Sigma(\alpha,\lambda)$ $(0<\alpha\le 1,\ 0\le\lambda\le 1)$. Then*

$$|a_2| \le min\left\{\frac{2\alpha}{1+3\lambda-2\lambda^2}, \alpha\sqrt{\frac{2}{12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1}}\right\} = \alpha\sqrt{\frac{2}{12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1}}$$

and

$$\begin{aligned}|a_3| &\le min\left\{\frac{\alpha^2}{2\lambda^2+1}+\frac{4\alpha^2}{(1+3\lambda-2\lambda^2)^2}, \frac{\alpha^2}{2\lambda^2+1}+\frac{2\alpha^2}{12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1}\right\}\\ &= \frac{\alpha^2}{2\lambda^2+1}+\frac{2\alpha^2}{12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1}\end{aligned}$$

Remark 3.1. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.1 are better than those given in Theorem 1.1. Because

$$\alpha\sqrt{\frac{2}{12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1}} \le \frac{2\alpha}{\sqrt{\alpha(1-2\lambda+25\lambda^2-44\lambda^3+20\lambda^4)+(1+3\lambda-2\lambda^2)^2}}$$

$$(0\le\lambda\le 1,\ 0<\alpha\le 1)$$

and

$$\begin{aligned}\frac{\alpha^2}{2\lambda^2+1}+\frac{2\alpha^2}{12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1} &\le \frac{\alpha^2}{2\lambda^2+1}+\frac{4\alpha^2}{(1+3\lambda-2\lambda^2)^2}\\ &\le \frac{\alpha}{2\lambda^2+1}+\frac{4\alpha^2}{(1+3\lambda-2\lambda^2)^2}.\end{aligned}$$

By putting $\lambda=\frac{1}{2}$ in Corollary 3.1, we conclude the following corollary.

Corollary 3.2. *Let the function $f(z)$ given by (1.1) be in the bi-univalent function class H_Σ^α $(0<\alpha\le 1)$. Then*

$$|a_2| \le min\left\{\alpha, \sqrt{\frac{2}{3}}\alpha\right\} = \sqrt{\frac{2}{3}}\ \alpha$$

and

$$|a_3| \le min\left\{\frac{5}{3}\alpha^2, \frac{4}{3}\alpha^2\right\} = \frac{4}{3}\alpha^2.$$

Remark 3.2. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.2 are better than those given by Srivastava [10, Theorem 1].

By putting $\lambda=1$ in Corollary 3.1, we conclude the following corollary.

Corollary 3.3. *Let the function $f(z)$ given by (1.1) be in the bi-univalent function class $K_\Sigma(\alpha)$ $(0<\alpha\le 1)$. Then*

$$|a_2| \le \alpha \text{ and } |a_3| \le \frac{4}{3}\alpha^2.$$

Remark 3.3. The bound on $|a_3|$ given in Corollary 3.3 is better than that given by Xiao-Fei-li [11, Theorem 2.2], when $\lambda=1$.

By putting $\lambda=0$ in Corollary 3.1, we conclude the following corollary.

Corollary 3.4. *Let the function f given by (1.1) be in the bi-univalent function class $\mathcal{S}^*_\Sigma(\alpha)$ $(0<\alpha\le 1)$. Then*

$$|a_2| \le min\left\{2\alpha, \sqrt{2}\alpha\right\} = \sqrt{2}\alpha$$

and

$$|a_3| \le min\left\{5\alpha^2, 3\alpha^2\right\} = 3\alpha^2.$$

Remark 3.4. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.4 are better than those given by Çağlar [4, Corollary 2.5]. Because

$$\sqrt{2}\alpha \le \frac{2\alpha}{\sqrt{\alpha+1}}$$

and

$$3\alpha^2 \le 4\alpha^2+\alpha.$$

By putting

$$A(\lambda)=2\lambda^2-\lambda,\ B(\lambda)=4(\lambda-\lambda^2),\ C(\lambda)=2\lambda^2-3\lambda+1$$

and

$$h(z)=p(z)=\frac{1+(1-2\beta)z}{1-z}\ (0\le\beta<1,\ z\in\mathbb{U})$$

in Theorem 2.1, we obtain the following result.

Corollary 3.5. *Let the function $f(z)$ given by (1.1) be in the bi-univalent function class $\mathcal{S}_\Sigma(\beta,\lambda)$ $(0\le\beta<1,\ 0\le\lambda\le1)$. Then*

$$|a_2|\le min\left\{\frac{2(1-\beta)}{1+3\lambda-2\lambda^2},\sqrt{\frac{2(1-\beta)}{12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1}}\right\}$$

and

$$|a_3|\le min\left\{\frac{1-\beta}{2\lambda^2+1}+\frac{4(1-\beta)^2}{(1+3\lambda-2\lambda^2)^2},\frac{1-\beta}{2\lambda^2+1}+\frac{2(1-\beta)}{12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1}\right\}.$$

By setting $\lambda=\frac{1}{2}$ in Corollary 3.5, we conclude the following corollary.

Corollary 3.6. *Let the function $f(z)$ given by (1.1) be in the bi-univalent function class $H_\Sigma(\beta)$ $(0\le\beta<1)$. Then*

$$|a_2|\le\begin{cases}\sqrt{\frac{2}{3}(1-\beta)}\ ;\ 0\le\beta\le\frac{1}{3}\\[2ex](1-\beta)\ ;\ \frac{1}{3}\le\beta<1\end{cases}$$

and

$$|a_3|\le\begin{cases}\frac{4}{3}(1-\beta)\ ;\ 0\le\beta\le\frac{1}{3}\\[2ex]\frac{(1-\beta)(5-3\beta)}{3}\ ;\ \frac{1}{3}\le\beta<1.\end{cases}$$

Remark 3.5. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.6 are better than those given by Srivastava [10, Theorem 2].

By putting $\lambda=1$ in Corollary 3.5, we conclude the following corollary.

Corollary 3.7. *Let the function $f(z)$ given by (1.1) be in the bi-univalent function class $K_\Sigma(\beta)$ $(0\le\beta<1)$. Then*

$$|a_2|\le min\left\{(1-\beta),\sqrt{1-\beta}\right\}=(1-\beta)$$

and

$$|a_3|\le min\left\{\frac{4}{3}(1-\beta),\frac{1}{3}(1-\beta)+(1-\beta)^2\right\}=\frac{1}{3}(1-\beta)+(1-\beta)^2.$$

Remark 3.6. The bound on $|a_2|$ given in Corollary 3.7 is better than that given by Xiao-Fei-li [11, Theorem 3.2], when $\lambda=1$.

By putting $\lambda=0$ in Corollary 3.5, we conclude the following corollary.

Corollary 3.8. *Let the function $f(z)$ given by (1.1) be in the bi-univalent function class $\mathcal{S}^*_\Sigma(\beta)$ $(0 \leq \beta < 1)$. Then*

$$|a_2| \leq \begin{cases} \sqrt{2(1-\beta)} \ ; \ 0 \leq \beta \leq \frac{1}{2} \\ \\ 2(1-\beta) \ ; \ \frac{1}{2} \leq \beta < 1 \end{cases}$$

and

$$|a_3| \leq \begin{cases} 3(1-\beta) \ ; \ 0 \leq \beta \leq \frac{1}{2} \\ \\ (1-\beta)(5-4\beta) \ ; \ \frac{1}{2} \leq \beta < 1. \end{cases}$$

Remark 3.7. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.8 are better than those given by Çağlar [4, Corollary 3.5].

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Khosrow Hosseinzadeh,
Faculty of Mathematical Sciences,
Shahrood University of Technology,
Shahrood, Iran.
E-mail address: zadeh.khosrow@gmail.com, kh.hosseinzadeh@shahroodut.ac.ir