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Derivatives with Respect to Horizontal and Vertical Lifts of the Deformed Complete Lift Metric $\tilde{G}_f$ on Tangent Bundle

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ABSTRACT: In this paper, we define the deformed complete lift metric $\tilde{G}_f$ on tangent bundle, which is completely determined by its action on vector fields of type X^H and ω^V. Later, we obtain the covarient and Lie derivatives applied to the deformed complete lift metric $\tilde{G}_f$ with respect to the horizontal and vertical lifts of vector fields, respectively.

Key Words: Covarient derivative, Lie derivative, Ddeformed complete lift metric $\tilde{G}_f$, Horizontal Lift, Vertical lift, Tangent bundle.

Contents

1. Introduction

Let M be an $n-$dimensional Riemannian manifold with a Riemannian metric g and denote by $\pi : TM \to M$ its tangent bundle with fiber the tangent spaces to M. TM is then a $2n-$dimensional smooth manifold and some local charts induced naturally from local charts on M may be used. Namely, a system of local coordinates (U, x^i) in M induces on TM a system of local coordinates $(\pi^{-1}(U), x^i, x^{\bar{i}} = y^i)$, where (x^i), $i = 1, ..., n$ is a local coordinate system defined in the neighborhood U and (y^i) is the Cartesian coordinates in each tangent space $T_P M$ at an arbitrary point P in U with respect to the natural basis $\left\{ \frac{\partial}{\partial x^i} \middle| P \right\}$. Summation over repeated indices in always implied. The Riemannian manifolds and the tangent bundles studyed a lot of authors [1,2,7,8] too.

Let $X = X^i \frac{\partial}{\partial x^i}$ be the local expressions in U of a vector field X on M. The vertical lift X^V, the horizontal lift X^H, and the complete lift X^C of X are then given respectively by [6]

$$X^V = X^i \partial_{\bar{i}} \tag{1.1}$$

$$X^H = X^i \partial_i - y^j \Gamma^i_{jk} X^k \partial_{\bar{i}} \tag{1.2}$$

and

$$X^C = X^i \partial_i + y^j \partial_j X^i \partial_{\bar{i}} \tag{1.3}$$

with respect to the induced coordinates, where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial y^i}$ and Γ^i_{jk} are the coefficients of the Levi-Civita connection ∇ of g.

Given a $(p, q)-$tensor field S on M, $q > 1$, we then consider a tensor field $\gamma S \in \Im^p_{q-1}(TM)$ on $\pi^{-1}(U)$ by

$$\gamma S = (y^s S^{j_1 \ldots j_p}_{s i_2 \ldots i_q}) \partial_{\overline{j_1}} \otimes ... \otimes \partial_{\overline{j_p}} \otimes dx^{i_2} \otimes ... \otimes dx^{iq}$$

with respect to the induced coordinates (x^i, y^i) ([11], p.12).

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1.1. The deformed complete lift metric $\tilde{G}_f$

In this section, we give the Levi-Civita connection $\tilde{\nabla}$ of the tangent bundle TM with the deformed complete lift metric $\tilde{G}_f$ and study fiber-preserving Killing vector fields on TM. The deformed complete lift metric $\tilde{G}_f$ is defined by [6]

$$\begin{aligned} &\tilde{G}(X^H, Y^H) = fg(X,Y) \\ &\tilde{G}(X^H, Y^V) = \tilde{G}(X^V, Y^H) = g(X,Y) \\ &\tilde{G}(X^V, Y^V) = 0 \end{aligned} \tag{1.4}$$

for all $X, Y \in \Im^1_0(M)$. We now give expressions of the deformed complete lift metric $\tilde{G}_f$ and its inverse $\tilde{G}_f{}^{-1}$ with respect to the adapted frame $\{E_\beta\}$:

$$\tilde{G}_f = \begin{pmatrix} fg_{ij} & g_{ij} \\ g_{ij} & 0 \end{pmatrix} \text{ and } \tilde{G}_f^{-1} = \begin{pmatrix} 0 & g^{jh} \\ g^{jh} & -fg^{jh} \end{pmatrix}$$

for all vector fields X and Y on M, and $f > 0$, $f \in C^\infty(M)$ [6]. For $f = 1$, it follows that $\tilde{g} \; =^S g$, i.e. the metric $\tilde{g}$ is generalization of Sasakian metric ${}^S g$ [10].

Determining both the deformed complete metric $\tilde{G}_f$ and the almost complex structure J^H, and using the facts $X^V(fg(Y,Z)) = 0$ and $X^H(fg(Y,Z)) = X(f)g(Y,Z) + fX(g(Y,Z))$, we calculate

2. Main Results

Definition 2.1. *Let M be an $n-$dimensional diferentiable manifold. Differential transformation $D = L_X$ is called as Lie derivation with respect to vector field $X \in \Im^1_0(M)$ if*

$$\begin{aligned} L_X f &= Xf, \forall f \in \Im^0_0(M^n), \\ L_X Y &= [X,Y], \forall X, Y \in \Im^1_0(M^n). \end{aligned} \tag{2.1}$$

$[X,Y]$ is called by Lie bracked. The Lie derivative $L_X F$ of a tensor field F of type $(1,1)$ with respect to a vector field X is defined by [3,4,11]

$$(L_X F)Y = [X, FY] - F[X,Y]. \tag{2.2}$$

Definition 2.2. *The bracket operation of vertical and horizontal vector fields is given by the following formulas:*

$$\begin{aligned} [X^H, Y^H] &= [X,Y]^H - (R(X,Y)u)^V \\ [X^H, Y^V] &= (\nabla_X Y)^V \\ [X^V, Y^V] &= 0, \; X^H f^V = (Xf)^V \end{aligned} \tag{2.3}$$

for all $X, Y \in \Im^1_0(M)$, $f \in \Im^0_0(M)$ [5], where R is the Riemannian curvature of g defined by

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Theorem 2.3. *Let $\tilde{G}_f$ be the deformed complete lift metric, is defined by (1.4) and L_X the operator Lie derivation with respect to X. From (1.4), Definition (2.1) and Definition (2.2), we get the following results*

$$\begin{aligned}
i)\; (L_{X^V}\tilde{G}_f)(Y^V, Z^V) &= 0, \\
ii)\; (L_{X^V}\tilde{G}_f)(Y^V, Z^H) &= 0, \\
iii)\; (L_{X^V}\tilde{G}_f)(Y^H, Z^V) &= 0, \\
iv)\; (L_{X^H}\tilde{G}_f)(Y^V, Z^V) &= 0, \\
v)\; (L_{X^H}\tilde{G}_f)(Y^H, Z^V) &= (L_X g)(Y,Z) - g(Y, (\nabla_Z X)), \\
vi)\; (L_{X^V}\tilde{G}_f)(Y^H, Z^H) &= g((\hat{\nabla}_Y X), Z) + g(Y, (\hat{\nabla}_Z X)), \\
vii)\; (L_{X^H}\tilde{G}_f)(Y^V, Z^H) &= (L_X g)(Y,Z) - g((\nabla_Y X), Z), \\
viii)\; (L_{X^H}\tilde{G}_f)(Y^H, Z^H) &= (L_X fg)(Y,Z) + g((R(X,Y)U, Z) + g(Y, (R(X,Z)U)),
\end{aligned}$$

where the vertical, horizontal and complete lifts X^V,X^H, X^C, $\in \Im_0^1(TM)$ of $X \in \Im_0^1(M)$, defined by (1.1),(1.2),(1.3), respectively.

Proof. *i)*

$$\begin{aligned}(L_{X^V}\tilde{G}_f)(Y^V,Z^V) &= L_{X^V}\tilde{G}_f(Y^V,Z^V) - \tilde{G}_f(L_{X^V}Y^V,Z^V) - \tilde{G}_f(Y^V,L_{X^V}Z^V) \\ &= L_{X^V}\tilde{G}_f(Y^V,Z^V) \\ &= 0\end{aligned}$$

ii)

$$\begin{aligned}(L_{X^V}\tilde{G}_f)(Y^V,Z^H) &= L_{X^V}\tilde{G}_f(Y^V,Z^H) - \tilde{G}_f(L_{X^V}Y^V,Z^H) - \tilde{G}_f(Y^V,L_{X^V}Z^H) \\ &= X^Vg(Y,Z) - \tilde{G}_f(Y^V,[X,Z]^V - (\nabla_X Z)^V) \\ &= -\tilde{G}_f(Y^V,[X,Z]^V) + \tilde{G}_f(Y^V,(\nabla_X Z)^V) \\ &= 0\end{aligned}$$

iii)

$$\begin{aligned}(L_{X^V}\tilde{G}_f)(Y^H,Z^V) &= L_{X^V}\tilde{G}_f(Y^H,Z^V) - \tilde{G}_f(L_{X^V}Y^H,Z^V) - \tilde{G}_f(Y^H,L_{X^V}Z^V) \\ &= X^Vg(Y,Z) - \tilde{G}_f([X,Y]^V - (\nabla_X Y)^V,Z^V) \\ &= -\tilde{G}_f([X,Y]^V,Z^V) + \tilde{G}_f((\nabla_X Y)^V,Z^V) \\ &= 0\end{aligned}$$

iv)

$$\begin{aligned}(L_{X^H}\tilde{G}_f)(Y^V,Z^V) &= L_{X^H}\tilde{G}_f(Y^V,Z^V) - \tilde{G}_f(L_{X^H}Y^V,Z^V) - \tilde{G}_f(Y^V,L_{X^H}Z^V) \\ &= -\tilde{G}_f((\hat{\nabla}_X Y)^V,Z^V) - \tilde{G}_f(Y^V,(\hat{\nabla}_X Z)^V) \\ &= 0\end{aligned}$$

v)

$$\begin{aligned}(L_{X^H}\tilde{G}_f)(Y^H,Z^V) &= L_{X^H}\tilde{G}_f(Y^H,Z^V) - \tilde{G}_f(L_{X^H}Y^H,Z^V) - \tilde{G}_f(Y^H,L_{X^H}Z^V) \\ &= Xg(Y,Z) - \tilde{G}_f([X,Y]^H - (R(X,Y)U)^V,Z^V) \\ &\quad -\tilde{G}_f(Y^H,[X,Z]^V + (\nabla_Z X)^V) \\ &= Xg(Y,Z) - g([X,Y],Z) - g(Y,[X,Z]) - g(Y,(\nabla_Z X)) \\ &= (L_X g)(Y,Z) - g(Y,(\nabla_Z X))\end{aligned}$$

vi)

$$\begin{aligned}(L_{X^V}\tilde{G}_f)(Y^H,Z^H) &= L_{X^V}\tilde{G}_f(Y^H,Z^H) - \tilde{G}_f(L_{X^V}Y^H,Z^H) - \tilde{G}_f(Y^H,L_{X^V}Z^H) \\ &= X^V(fg(Y,Z)) - \tilde{G}_f([X,Y]^V - (\nabla_X Y)^V,Z^H) \\ &\quad -\tilde{G}_f(Y^H,[X,Z]^V - (\nabla_X Z)^V) \\ &= -\tilde{G}_f([X,Y]^V,Z^H) + \tilde{G}_f((\nabla_X Y)^V,Z^H) - \tilde{G}_f(Y^H,[X,Z]^V) \\ &\quad +\tilde{G}_f(Y^H,(\nabla_X Z)^V) \\ &= g(-[X,Y] + (\nabla_X Y),Z) + g(Y,-[X,Z] + (\nabla_X Z)) \\ &= g([Y,X] + (\nabla_X Y),Z) + g(Y,[Z,X] + (\nabla_X Z)) \\ &= g((\hat{\nabla}_Y X),Z) + g(Y,(\hat{\nabla}_Z X))\end{aligned}$$

vii)

$$\begin{aligned}(L_{X^H}\tilde{G}_f)(Y^V,Z^H) &= L_{X^H}\tilde{G}_f(Y^V,Z^H)-\tilde{G}_f(L_{X^H}Y^V,Z^H)-\tilde{G}_f(Y^V,L_{X^H}Z^H)\\ &= X^Hg(Y,Z)-\tilde{G}_f([X,Y]^V+(\nabla_XY)^V,Z^H)\\ &\quad -\tilde{G}_f(Y^V,[X,Z]^H-(R(X,Z)U)^V)\\ &= Xg(Y,Z)-\tilde{G}_f([X,Y]^V,Z^H)-\tilde{G}_f((\nabla_YX)^V,Z^H)\\ &\quad -\tilde{G}_f(Y^V,[X,Z]^H)+\tilde{G}_f(Y^V(R(X,Z)U)^V)\\ &= Xg(Y,Z)-g([X,Y],Z)-g((\nabla_YX),Z)-g(Y,[X,Z])\\ &= (L_Xg)(Y,Z)-g((\nabla_YX),Z)\end{aligned}$$

viii)

$$\begin{aligned}(L_{X^H}\tilde{G}_f)(Y^H,Z^H) &= L_{X^H}\tilde{G}_f(Y^H,Z^H)-\tilde{G}_f(L_{X^H}Y^H,Z^H)-\tilde{G}_f(Y^H,L_{X^H}Z^H)\\ &= X^H(fg(Y,Z))-\tilde{G}_f([X,Y]^H-(R(X,Y)U)^V,Z^H)\\ &\quad -\tilde{G}_f(Y^H,[X,Z]^H-(R(X,Z)U)^V)\\ &= X(f)g(Y,Z)+fXg(Y,Z)-fg([X,Y],Z)+g((R(X,Y)U),Z)\\ &\quad -fg(Y,[X,Z])+g(Y,(R(X,Z)U))\\ &= (L_Xfg)(Y,Z)+g((R(X,Y)U,Z)+g(Y,(R(X,Z)U))\end{aligned}$$

□

Definition 2.4. *Differantial transformation of algebra $T(M)$, deffined by*

$$D=\nabla_X:T(M)\to T(M),X\in\Im_0^1(M),$$

is called as covarient derivation with respect to vector field X if

$$\begin{aligned}\nabla_{fX+gY}t &= f\nabla_Xt+g\nabla_Yt,\\ \nabla_Xf &= Xf,\end{aligned}\tag{2.4}$$

where $\forall f,g\in\Im_0^0(M),\forall X,Y\in\Im_0^1(M),\forall t\in\Im(M)$.

On the other hand, a transformation deffined by

$$\nabla:\Im_0^1(M)\times\Im_0^1(M)\to\Im_0^1(M),$$

is called as affin connection [9,11]. In addition, the horizontal lift of an affine connection ∇ in M to $T(M)$, denoted by ∇^H, defined by

$$\begin{aligned}\nabla^H_{X^V}Y^V &= 0,\ \nabla^H_{X^V}Y^H=0,\\ \nabla^H_{X^H}Y^V &= (\nabla_XY)^V,\nabla^H_{X^H}Y^H=(\nabla_XY)^H\end{aligned}\tag{2.5}$$

for any $X,Y\in\Im_0^1(M)$.

Theorem 2.5. *Let $\tilde{G}_f$ be the deformed complete lift metric, is defined by (1.4) and the horizontal lift ∇^H of a symetric affine connection ∇ in M to $T(M)$. From (1.4) and Definition 2.4, we get the following results*

$$\begin{aligned}i)\ (\nabla^H_{X^V}\tilde{G}_f)(Y^V,Z^V) &= 0,\\ ii)\ (\nabla^H_{X^V}\tilde{G}_f)(Y^V,Z^H) &= 0,\\ iii)\ (\nabla^H_{X^V}\tilde{G}_f)(Y^H,Z^V) &= 0,\\ iv)\ (\nabla^H_{X^V}\tilde{G}_f)(Y^H,Z^H) &= 0,\\ v)\ (\nabla^H_{X^H}\tilde{G}_f)(Y^V,Z^V) &= 0,\\ vi)\ (\nabla^H_{X^H}\tilde{G}_f)(Y^V,Z^H) &= (\nabla_Xg)(Y,Z),\\ vii)\ (\nabla^H_{X^H}\tilde{G}_f)(Y^H,Z^V) &= (\nabla_Xg)(Y,Z),\\ viii)\ (\nabla^H_{X^H}\tilde{G}_f)(Y^H,Z^H) &= (\nabla_Xfg)(Y,Z),\end{aligned}$$

where the vertical, horizontal and complete lifts X^V,X^H, X^C, $\in \Im_0^1(TM)$ of $X \in \Im_0^1(M)$, defined by (1.1),(1.2),(1.3), respectively.

Proof. i)

$$\begin{aligned}(\nabla^H_{X^V}\tilde{G}_f)(Y^V,Z^V) &= \nabla^H_{X^V}\tilde{G}_f(Y^V,Z^V)-\tilde{G}_f(\nabla^H_{X^V}Y^V,Z^V)-\tilde{G}_f(Y^V,\nabla^H_{X^V}Z^V)\\ &= X^V\tilde{G}_f(Y^V,Z^V)\\ &= 0\end{aligned}$$

ii)

$$\begin{aligned}(\nabla^H_{X^V}\tilde{G}_f)(Y^V,Z^H) &= \nabla^H_{X^V}\tilde{G}_f(Y^V,Z^H)-\tilde{G}_f(\nabla^H_{X^V}Y^V,Z^H)-\tilde{G}_f(Y^V,\nabla^H_{X^V}Z^H)\\ &= X^Vg(Y,Z)\\ &= 0\end{aligned}$$

iii)

$$\begin{aligned}(\nabla^H_{X^V}\tilde{G}_f)(Y^H,Z^V) &= \nabla^H_{X^V}\tilde{G}_f(Y^H,Z^V)-\tilde{G}_f(\nabla^H_{X^V}Y^H,Z^V)-\tilde{G}_f(Y^H,\nabla^H_{X^V}Z^V)\\ &= X^Vg(Y,Z)\\ &= 0\end{aligned}$$

iv)

$$\begin{aligned}(\nabla^H_{X^V}\tilde{G}_f)(Y^H,Z^H) &= \nabla^H_{X^V}\tilde{G}_f(Y^H,Z^H)-\tilde{G}_f(\nabla^H_{X^V}Y^H,Z^H)-\tilde{G}_f(Y^H,\nabla^H_{X^V}Z^H)\\ &= X^Vfg(Y,Z)\\ &= 0\end{aligned}$$

v)

$$\begin{aligned}(\nabla^H_{X^H}\tilde{G}_f)(Y^V,Z^V) &= \nabla^H_{X^H}\tilde{G}_f(Y^V,Z^V)-\tilde{G}_f(\nabla^H_{X^H}Y^V,Z^V)-\tilde{G}_f(Y^V,\nabla^H_{X^H}Z^V)\\ &= -\tilde{G}_f((\nabla_XY)^V,Z^V)-\tilde{G}_f(Y^V,(\nabla_XZ)^V)\\ &= 0\end{aligned}$$

vi)

$$\begin{aligned}(\nabla^H_{X^H}\tilde{G}_f)(Y^V,Z^H) &= \nabla^H_{X^H}\tilde{G}_f(Y^V,Z^H)-\tilde{G}_f(\nabla^H_{X^H}Y^V,Z^H)-\tilde{G}_f(Y^V,\nabla^H_{X^H}Z^H)\\ &= Xg(Y,Z)-\tilde{G}_f((\nabla_XY)^V,Z^H)-\tilde{G}_f(Y^V,(\nabla_XZ)^H)\\ &= Xg(Y,Z)-g((\nabla_XY),Z)-g(Y,(\nabla_XZ))\\ &= (\nabla_Xg)(Y,Z)\end{aligned}$$

vii)

$$\begin{aligned}(\nabla^H_{X^H}\tilde{G}_f)(Y^H,Z^V) &= \nabla^H_{X^H}\tilde{G}_f(Y^H,Z^V)-\tilde{G}_f(\nabla^H_{X^H}Y^H,Z^V)-\tilde{G}_f(Y^H,\nabla^H_{X^H}Z^V)\\ &= Xg(Y,Z)-g((\nabla_XY),Z)-g(Y,(\nabla_XZ))\\ &= (\nabla_Xg)(Y,Z)\end{aligned}$$

viii)

$$\begin{aligned}(\nabla^H_{X^H}\tilde{G}_f)(Y^H,Z^H) &= \nabla^H_{X^H}\tilde{G}_f(Y^H,Z^H)-\tilde{G}_f(\nabla^H_{X^H}Y^H,Z^H)-\tilde{G}_f(Y^H,\nabla^H_{X^H}Z^H)\\ &= X(f)g(Y,Z)+fXg(Y,Z)-\tilde{G}_f((\nabla_XY)^H,Z^H)\\ &\quad -\tilde{G}_f(Y^H,(\nabla_XZ)^H)\\ &= X(f)g(Y,Z)+fXg(Y,Z)-fg((\nabla_XY),Z)-g(Y,(\nabla_XZ))\\ &= (\nabla_Xfg)(Y,Z)\end{aligned}$$

□

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