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A New Class of Higher-order Hypergeometric Bernoulli Polynomials Associated with Hermite Polynomials

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ABSTRACT: In this paper, we introduce a new class of higher-order hypergeometric Hermite-Bernoulli numbers and polynomials. We shall provide several properties of higher-order hypergeometric Hermite-Bernoulli polynomials including summation formulae, sums of product identity, recurrence relations.

Key Words: Hermite polynomials, Higher-order hypergeometric Bernoulli polynomials, Higher-order hypergeometric Hermite-Bernoulli polynomials, Recurrence relations.

Contents

1. Introduction

The Bernoulli polynomials $B_n(x)$ are defined by the following generating function

$$\left(\frac{t}{e^t-1}\right)e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}, \tag{1.1}$$

and $B_n=B_n(0)$ are named Bernoulli numbers. These numbers and polynomials have a long history, which arise from Bernoulli's calculations of power sums in 1713, that is,

$$\sum_{j=1}^{m}j^n=\frac{B_{n+1}(m+1)-B_{n+1}}{n+1},$$

(see [[19], p.5, (2.2)]). They have many applications in modern number theory, such as modular forms [11] and Iwasawa theory [9]. A recent book by Arakawa, Ibukiyama and Kaneko [1] give a nice introduction of Bernoulli numbers and polynomials including their connections with zeta functions.

In 1924, Nörlund [14] introduced and studied the generalized higher order Bernoulli polynomials defined by means of the following generating function

$$\left(\frac{t}{e^t-1}\right)^{\alpha}e^{xt}=\frac{e^{xt}}{\left(\frac{e^t-1}{t}\right)^{\alpha}}=\sum_{n=0}^{\infty}B_n^{(\alpha)}(x)\frac{t^n}{n!}. \tag{1.2}$$

We also have a similar expression of multiple power sums

$$\sum_{l_1\cdots l_n=0}^{m-1}(t+l_1+\cdots+l_n)^k,$$

in terms of higher order Bernoulli polynomials, (see ([12], Lemma 2.1)).

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Howard ([5], [6]) gave a generalization of Bernoulli polynomials by considering the following generating function

$$\frac{t^2e^{xt}/2}{e^t-1-t}=\sum_{n=0}^{\infty}A_n^{(\alpha)}(x)\frac{t^n}{n!}, \tag{1.3}$$

and more generally, for all positive integer N

$$\frac{\frac{t^N}{N!}}{e^t-T_{N-1}(t)}e^{xt}=\sum_{n=0}^{\infty}B_{N,n}(x)\frac{t^n}{n!}, \tag{1.4}$$

where $T_{N-1}(t)$ is the Taylor polynomial of order $N-1$ for the exponential function. For the case $N=1$ and $N=2$, (1.4) reduces to (1.1) and (1.3), respectively. We see that the polynomials $B_{N,n}(x)$ have rational coefficients.

The polynomials $B_{N,n}(x)$ are named hypergeometric Bernoulli polynomials, while the numbers $B_{N,n}=B_{N,n}(0)$ are named hypergeometric Bernoulli numbers since the generating function $f(t)=\frac{e^t-T_{N-1}(t)}{\frac{t^N}{N!}}$ can be expressed as ${}_1F_1(1;N+1;t)$, where the confluent hypergeometric function ${}_1F_1(a;b;t)$ is defined by

$${}_1F_1(a;b;t)=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{t^n}{n!}, \tag{1.5}$$

and $(a)_n$ is the Pochhammer symbol, (see [20])

$$(a)_0:=1,\ (a)_n=a(a+1)\cdots(a+n-1),(n\in\mathbb{N}:=\{1,2,3,\cdots\}).$$

For $N,r\in\mathbb{N}$, the higher-order hypergeometric Bernoulli polynomials $B_{N,n}^{(r)}(x)$ are defined by means of the generating function, (see [2], [7], [10])

$$\left(\frac{\frac{t^N}{N!}}{e^t-T_{N-1}(t)}\right)^r e^{xt}=\frac{1}{{}_1F_1(1;N+1;t)^r}e^{xt}=\sum_{n=0}^{\infty}B_{N,n}^{(r)}(x)\frac{t^n}{n!}. \tag{1.6}$$

For $x=0$ in (1.6), $B_{N,n}^{(r)}=B_{N,n}^{(r)}(0)$ are called the higher order hypergeometric Bernoulli numbers, (see [10], [13]). Again, on taking $r=1$ in (1.6), $B_{N,n}^{(1)}(x)=B_{N,n}(x)$ are called the the hypergeometric Bernoulli polynomials and if we put $x=0$ in (1.6), $B_{N,n}^{(1)}(0)=B_{N,n}$ are called the hypergeometric Bernoulli numbers.

The 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) $H_n(x,y)$ ([3], [4]]) are defined as

$$H_n(x,y)=n!\sum_{r=0}^{[\frac{n}{2}]}\frac{y^rx^{n-2r}}{r!(n-2r)!}. \tag{1.7}$$

It is easily seen that

$$H_n(2x,-1)=H_n(x),H_n(x,-\frac{1}{2})=He_n(x),$$

where $H_n(x)$ and $He_n(x)$ are called the ordinary Hermite polynomials. Also

$$H_n(x,0)=x^n.$$

The generating function for Hermite polynomial H_n(x,y) ([16]-[18]) are given by

$$e^{xt+yt^2}=\sum_{n=0}^{\infty}H_n(x,y)\frac{t^n}{n!}. \tag{1.8}$$

The object of this paper is to present a systematic account of these families in a unified and generalized form. We develop some elementary properties and derive the implicit summation formulae for the higher-order hypergeometric Hermite-Bernoulli polynomials by using different analytical means on their respective generating functions. The approach given in recent papers of Pathan and Khan ([16]-[18]) has indeed allowed the derivation of implicit summation formulae in the two-variable higher-order hypergeometric Hermite-Bernoulli polynomials. In addition to this, some relevant connections between Hermite and higher-order hypergeometric Bernoulli polynomials and recurrence relations are given.

2. Multiple hypergeometric Hermite-Bernoulli numbers and polynomials

For every positive integer N and r, the higher-order hypergeometric Hermite-Bernoulli numbers and polynomials ${}_HB^{(r)}_{N,n}(x,y)$ are defined by means of the following generating function defined in a suitable neighborhood of $t = 0$:

$$F_{r,N}(x,y,t)=\frac{1}{{}_1F_1(1;N+1;t)^r}e^{xt+yt^2}=\left(\frac{\frac{t^N}{N!}}{e^t-\sum\limits_{n=0}^{N-1}\frac{t^n}{n!!}}\right)^r e^{xt+yt^2}$$

$$=\sum_{n=0}^{\infty}{}_HB^{(r)}_{N,n}(x,y)\frac{t^n}{n!}. \tag{2.1}$$

For $x=y=0$, $B^{(r)}_{N,n}={}_HB^{(r)}_{N,n}(0,0)$ are called the higher-order hypergeometric Bernoulli numbers, (see [10, 13]). When $r=1$, we obtain the hypergeometric Hermite-Bernoulli polynomials ${}_HB_{N,n}(x,y)={}_HB^{(1)}_{N,n}(x,y)$ and $B_{N,n}={}_HB^{(1)}_{N,n}(0,0)$ is the hypergeometric Bernoulli numbers, (see [8, 15]). If we put $N=1$, the result reduces to the known result of Pathan and Khan, (see [16]).

Remark 2.1. *On setting $y=0$, (2.1) reduces to the known result of Aoki et al. [2] as follows:*

$$F_{r,N}(x,t)=\frac{1}{{}_1F_1(1;N+1;t)^r}e^{xt}=\left(\frac{\frac{t^N}{N!}}{e^t-\sum\limits_{n=0}^{N-1}\frac{t^n}{n!!}}\right)^r e^{xt}$$

$$=\sum_{n=0}^{\infty}B^{(r)}_{N,n}(x)\frac{t^n}{n!}. \tag{2.2}$$

In particular in terms of higher-order hypergeometric Bernoulli numbers $B^{(r)}_{N,n}$ and Hermite polynomials $H_s(x,y)$, the higher order Hermite-Bernoulli polynomials ${}_HB^{(r)}_{N,n}(x,y)$ are defined as

$${}_HB^{(r)}_{N,n}(x,y)=\sum_{s=0}^{n}\binom{n}{s}B^{(r)}_{N,n-s}H_s(x,y). \tag{2.3}$$

Taking $r=N=1$ and $x = 0$ in (2.1) gives the result

$$\sum_{m=0}^{[\frac{n}{2}]}\binom{n}{2m}B_{n-2m}y^m={}_HB^{(1)}_{1,n}(0,y). \tag{2.4}$$

Using $e^{it} = cost+i\sin t$ and $N=1$, the result reduces to

$$\sum_{n=0}^{\infty}f(n) = \sum_{n=0}^{\infty}f(2n)+\sum_{n=0}^{\infty}f(2n+1), \tag{2.5}$$

and

$$\left(\frac{it}{e^{it}-1}\right)^r=\left(\frac{it(\cos t-1-i\sin t)}{(\cos t-1+i\sin t)(\cos t-1-i\sin t)}\right)^r=\left(\frac{it(\cos t-1-i\sin t)}{(\cos t-1)^2+(\sin t)^2}\right)^r$$
$$=\left(\frac{(t\sin t)+it(\cos t-1)}{\Omega}\right)^r,$$

where $\Omega=(\cos t-1)^2+(\sin t)^2$, together with the definition (2.1) and the result (2.5), we get (see Pathan and Khan [16]):

$$e^{ixt-yt^2}\left(\frac{(t\sin t)+it(\cos t-1)}{\Omega}\right)^r$$
$$=\sum_{n=0}^{\infty}{}_HB_{2n}^{(r)}(x,y)\frac{(-1)^nt^{2n}}{(2n)!}+\sum_{n=0}^{\infty}{}_HB_{2n+1}^{(r)}(x,y)\frac{(-1)^nt^{2n+1}}{(2n+1)!}, \tag{2.6}$$

where $r\geq 1$, $\Omega=(\cos t-1)^2+(\sin t)^2$.

On setting $r=1$, $x=y=0$ in the above results , we get the following well known classical results involving Bernoulli numbers, (see [16]):

$$\frac{t}{2}\cot\left(\frac{t}{2}\right)=\sum_{n=0}^{\infty}B_{2n}\frac{(-1)^nt^{2n}}{(2n)!},\quad \frac{t}{2}\coth\left(\frac{t}{2}\right)=\sum_{n=0}^{\infty}B_{2n}\frac{t^{2n}}{(2n)!}.$$

Theorem 2.2. *For $n\geq 1$, we have*

$$H_n(x,y)=n!(N!)^r\sum_{m=0}^{n}\sum_{i_1+\cdots+i_r=n-m}\frac{{}_HB_{N,m}^{(r)}(x,y)}{m!(N+i_1)!\cdots(N+i_r)!}. \tag{2.7}$$

Proof. From definition (2.1), we have

$$\left(\frac{t^N}{N!}\right)^r e^{xt+yt^2}=\left(\frac{t^{i+N}}{(i+N)!}\right)^r\left(\sum_{n=0}^{\infty}{}_HB_{N,n}^{(r)}(x,y)\frac{t^n}{n!}\right)$$
$$=t^{rN}\left(\sum_{l=0}^{\infty}\sum_{i_1+\cdots+i_r=1}\frac{l!}{(N+i_1)!\cdots(N+i_r)!}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}{}_HB_{N,m}^{(r)}(x,y)\frac{t^m}{m!}\right)$$
$$\frac{t^{rN}}{(N!)^r}\sum_{n=0}^{\infty}H_n(x,y)\frac{t^n}{n!}$$
$$=t^{rN}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{i_1+\cdots+i_r=n-m}\frac{{}_HB_{N,m}^{(r)}(x,y)}{(N+i_1)!\cdots(N+i_r)!}\frac{t^n}{m!}.$$

Comparing the coefficients of t^n on both sides, we get (2.7). □

Corollary 2.3. *For $r=1$ in (2.7), we get*

$$H_n(x,y)=n!N!\sum_{m=0}^{n}\binom{n+N}{m}{}_HB_{N,m}(x,y). \tag{2.8}$$

Corollary 2.4. *For $x=y=0$ in (2.7), the result reduces to the known result of Aoki et al. [2] as follows*

$$\sum_{m=0}^{n}\sum_{i_1+\cdots+i_r=n-m}\frac{B_{N,m}^{(r)}}{m!(N+i_1)!\cdots(N+i_r)!}=0. \tag{2.9}$$

and $r=1$ in (2.8), the result reduces to (see [7]):

$$\sum_{m=0}^{n}\binom{n+N}{m}B_{N,m}(x,y)=0. \tag{2.10}$$

Theorem 2.5. *The following relationship holds true:*

$$H_n(x,y) = \sum_{m=0}^{n} \binom{n}{m} \frac{m!\Gamma(N+1)}{\Gamma(N+1+m)} {}_HB_{N,n-m}(x,y). \tag{2.11}$$

Proof. Using equations (2.1), (1.5) and (1.8), we have

$$\frac{1}{{}_1F_1(1;N+1;t)} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_{N,n}(x,y)\frac{t^n}{n!}$$

$$e^{xt+yt^2} = {}_1F_1(1;N+1;t) \sum_{n=0}^{\infty} {}_HB_{N,n}(x,y)\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} = \sum_{m=0}^{\infty} \frac{(1)_m}{(N+1)_m}\frac{t^m}{m!} \sum_{n=0}^{\infty} {}_HB_{N,n}(x,y)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \binom{n}{m} \frac{m!\Gamma(N+1)}{\Gamma(N+1+m)} {}_HB_{N,n-m}(x,y)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we arrive at the obtained result (2.11). □

Theorem 2.6. *The following relationship holds true:*

$$\int_0^1 (1-x)^{N-1} {}_HB^{(r)}_{N,n}(x,y)dx = (N-1)! \sum_{k=0}^{n} \binom{n}{k} \frac{(n-k)!}{(N+n-k)!} {}_HB^{(r)}_{N,k}(0,y). \tag{2.12}$$

Proof. From (2.1), we have

$$\frac{1}{{}_1F_1(1;N+1;t)^r} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB^{(r)}_{N,n}(x,y)\frac{t^n}{n!}$$

$$e^{xt} \sum_{n=0}^{\infty} {}_HB^{(r)}_{N,n}(0,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_HB^{(r)}_{N,n}(x,y)\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} {}_HB^{(r)}_{N,k}(0,y)x^{n-k}\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_HB^{(r)}_{N,n}(x,y)\frac{t^n}{n!}.$$

Thus, we have

$${}_HB^{(r)}_{N,n}(x,y) = \sum_{k=0}^{n} \binom{n}{k} {}_HB^{(r)}_{N,k}(0,y)x^{n-k}. \tag{2.13}$$

Therefore, by integrating (2.13) with weight $(1-x)^{N-1}$ and using the result ([20], p.26(48)), we obtain

$$\int_0^1 (1-x)^{N-1} {}_HB^{(r)}_{N,n}(x,y)dx = \sum_{k=0}^{n} \binom{n}{k} {}_HB^{(r)}_{N,k}(0,y) \int_0^1 (1-x)^{N-1}x^{n-k}dx$$

$$= (N-1)! \sum_{k=0}^{n} \binom{n}{k} \frac{(n-k)!}{(N+n-k)!} {}_HB^{(r)}_{N,k}(0,y),$$

which follows from (2.12). This completes the proof. □

Theorem 2.7. *The following representation for higher-order hypergeometric Hermite-Bernoulli polynomials ${}_HB_{N,n}^{(r)}(x,y)$ involving Hermite-Euler polynomials ${}_HE_n(x,y)$ holds true:*

$$
{}_HB_{N,n}^{(r)}(x,y)=\frac{1}{2}\left[\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\binom{m}{k}{}_HE_{n-m}(x,y)B_{N,m-k}^{(r)}\right.
$$

$$
\left.+\sum_{m=0}^{n}\binom{n}{m}{}_HE_{n-m}(x,y)B_{N,m}^{(r)}\right]. \tag{2.14}
$$

Proof. Using generating function for Hermite-Euler polynomials as follows

$$
e^{xt+yt^2}=\frac{e^t+1}{2}\sum_{n=0}^{\infty}{}_HE_n(x,y)\frac{t^n}{n!},\ \text{(see [18])}.
$$

Substituting this value of e^{xt+yt^2} in (2.1) gives

$$
\sum_{n=0}^{\infty}{}_HB_{N,n}^{(r)}(x,y)\frac{t^n}{n!}=\frac{1}{{}_1F_1(1;N+1;t)^r}\frac{e^t+1}{2}\sum_{n=0}^{\infty}{}_HE_n(x,y)\frac{t^n}{n!}
$$

$$
=\frac{1}{2}\left[\sum_{n=0}^{\infty}{}_HE_n(x,y)\frac{t^n}{n!}\sum_{m=0}^{\infty}\sum_{k=0}^{m}B_{N,m-k}^{(r)}\frac{t^m}{(m-k)!k!}\right.
$$

$$
\left.+\sum_{n=0}^{\infty}\sum_{m=0}^{n}{}_HE_{n-m}(x,y)B_{N,m}^{(r)}\frac{t^n}{(n-m)!m!}\right]
$$

$$
=\frac{1}{2}\sum_{n=0}^{\infty}\left[\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\binom{m}{k}{}_HE_{n-m}(x,y)B_{N,m-k}^{(r)}\right.
$$

$$
\left.+\sum_{m=0}^{n}\binom{n}{m}{}_HE_{n-m}(x,y)B_{N,m}^{(r)}\frac{t^n}{(n-m)!m!}\right]\frac{t^n}{n!}.
$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we required at the result (2.14). □

Theorem 2.8. *For $n\geq 0$, $p,q\in\mathbb{R}$, the following formula for higher-order hypergeometric Hermite-Bernoulli polynomials ${}_HB_{N,n}^{(r)}(px,qy)$ holds true:*

$$
{}_HB_{N,n}^{(r)}(px,qy)
$$

$$
=n!\sum_{k=0}^{n}\sum_{j=0}^{[\frac{k}{2}]}{}_HB_{N,n-k}^{(r)}(x,y)((p-1)x)^{k-2j}((q-1)y)^j\frac{t^n}{(n-k-2j)!j!k!}. \tag{2.15}
$$

Proof. Rewrite the generating function (2.1), we have

$$
\sum_{n=0}^{\infty}{}_HB_{N,n}^{(r)}(px,qy)\frac{t^n}{n!}
$$

$$
=\frac{1}{{}_1F_1(1;N+1;t)^r}e^{xt+yt^2}e^{(p-1)xt}e^{(q-1)yt^2} \tag{2.16}
$$

.

$$
=\left(\sum_{n=0}^{\infty}{}_HB_{N,n}^{(r)}(x,y)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}((p-1)x)^k\frac{t^k}{k!}\right)\left(\sum_{j=0}^{\infty}((q-1)y)^j\frac{t^{2j}}{j!}\right)
$$

$$= \left(\sum_{n=0}^{\infty} {}_H B_{N,n}^{(r)}(x,y)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}((p-1)x)^k((q-1)y)^j\frac{t^{k+2j}}{n!k!j!}\right).$$

Replacing k by $k-2j$ in above equation, we have

$$L.H.S. = \left(\sum_{n=0}^{\infty} {}_H B_{N,n}^{(r)}(x,y)\frac{t^n}{n!}\right)\left(\sum_{k=2j}^{\infty}((p-1)x)^{k-2j}((q-1)y)^j\frac{t^k}{(k-2j)!j!}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{k=2j}^{\infty}\left(\sum_{n=0}^{\infty} {}_H B_{N,n}^{(r)}(x,y)\frac{t^n}{n!}\right)((p-1)x)^{k-2j}((q-1)y)^j\frac{t^{n+k}}{(k-2j)!j!n!}.$$

Again replacing n by $n-k$ in the above equation, we have

$$L.H.S. = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{[\frac{k}{2}]}\left(\sum_{n=0}^{\infty} {}_H B_{N,n-k}^{(r)}(x,y)\frac{t^n}{n!}\right)((p-1)x)^{k-2j}((q-1)y)^j\frac{t^n}{(n-k-2j)!j!k!}.$$

Finally, equating the coefficients of t^n on both sides, we acquire the result (2.15). □

Theorem 2.9. *For $n \geq 0$, $p, q \in \mathbb{R}$ and $x, y \in \mathbb{C}$, we have*

$$\begin{aligned} &{}_H B_{N,n}^{(r)}(px,qy) \\ &= \sum_{k=0}^{n}\binom{n}{k} {}_H B_{N,n-k}^{(r)}(x,y)H_k((p-1)x,(q-1)y). \end{aligned} \tag{2.17}$$

Proof. By using (2.16) and (1.8), we can easily derive (2.17). We omit the proof. □

3. Summation formulae for higher-order hypergeometric Hermite-Bernoulli polynomials

In this section, we derive the summation formula, the sum of the product of identity and recurrence relations. First, we prove the following results involving higher-order hypergeometric Hermite-Bernoulli polynomials ${}_H B_{N,n}^{(r)}(x,y)$.

Theorem 3.1. *The following implicit summation formulae for higher-order hypergeometric Hermite-Bernoulli polynomials ${}_H B_{N,n}^{(r)}(x,y)$ holds true:*

$${}_H B_{N,k+l}^{(r)}(z,y) = \sum_{n,p=0}^{k,l}\frac{k!l!(z-x)^{n+p}\,{}_H B_{N,k+l-p-n}^{(r)}(x,y)}{(k-n)!(l-p)!n!p!}. \tag{3.1}$$

Proof. We replace t by $t+u$ and rewrite the generating function (2.1) as

$$\frac{1}{{}_1F_1(1;N+1;(t+u))^r}e^{y(t+u)^2} = e^{-x(t+u)}\sum_{k,l=0}^{\infty} {}_H B_{N,k+l}^{(r)}(x,y)\frac{t^k}{k!}\frac{u^l}{l!}. \tag{3.2}$$

Replacing x by z in the above equation and equating the resulting equation to the above equation, we get

$$e^{(z-x)(t+u)}\sum_{k,l=0}^{\infty} {}_H B_{N,k+l}^{(r)}(x,y)\frac{t^k}{k!}\frac{u^l}{l!} = \sum_{k,l=0}^{\infty} {}_H B_{N,k+l}^{(r)}(z,y)\frac{t^k}{k!}\frac{u^l}{l!}. \tag{3.3}$$

On expanding exponential function (3.3) gives

$$\sum_{M=0}^{\infty}\frac{[(z-x)(t+u)]^M}{M!}\sum_{k,l=0}^{\infty}{}_HB_{N,k+l}^{(r)}(x,y)\frac{t^k}{k!}\frac{u^l}{l!}=\sum_{k,l=0}^{\infty}{}_HB_{N,k+l}^{(r)}(z,y)\frac{t^k}{k!}\frac{u^l}{l!}, \tag{3.4}$$

which on using formula ([20], p.52(2))

$$\sum_{M=0}^{\infty}f(M)\frac{(x+y)^M}{M!}=\sum_{n,m=0}^{\infty}f(n+m)\frac{x^n}{n!}\frac{y^m}{m!}, \tag{3.5}$$

in the left hand side becomes

$$\sum_{n,p=0}^{\infty}\frac{(z-x)^{n+p}}{n!p!}\sum_{k,l=0}^{\infty}{}_HB_{N,k+l}^{(r)}(x,y)\frac{t^k}{k!}\frac{u^l}{l!}=\sum_{k,l=0}^{\infty}{}_HB_{N,k+l}^{(r)}(z,y)\frac{t^k}{k!}\frac{u^l}{l!}. \tag{3.6}$$

Now replacing k by $k-n$, l by $l-p$ and using the lemma ([20], p.100(1)) in the left hand side of (3.6), we get

$$\sum_{n,p=0}^{\infty}\sum_{k,l=0}^{\infty}\frac{(z-x)^{n+p}}{n!p!}{}_HB_{N,k+l-n-p}^{(r)}(x,y)\frac{t^k}{(k-n)!}\frac{u^l}{(l-p)!}=\sum_{k,l=0}^{\infty}{}_HB_{N,k+l}^{(r)}(z,y)\frac{t^k}{k!}\frac{u^l}{l!}. \tag{3.7}$$

Finally on equating the coefficients of the like powers of t and u in the above equation, we get the required result. □

Corollary 3.2. *On taking $l = 0$ in Theorem 3.1, the result reduces to*

$$_HB_{N,k}^{(r)}(z,y)=\sum_{n=0}^{k}\binom{k}{n}(z-x)^n\,{}_HB_{N,k-n}^{(r)}(x,y). \tag{3.8}$$

Corollary 3.3. *On replacing z by $z+x$ and setting $y=0$ in Theorem (3.1), we get the following result involving higher-order hypergeometric Hermite-Bernoulli polynomials of one variable:*

$$_HB_{N,k+l}^{(r)}(z+x)=\sum_{n,m=0}^{k,l}\frac{k!l!z^{n+m}\,{}_HB_{N,k+l-m-n}^{(r)}(x)}{(k-n)!(l-m)!n!m!}, \tag{3.9}$$

whereas by setting $z=0$ in Theorem 3.1, we get another result involving hypergeometric Hermite-Bernoulli polynomials of one and two variables:

$$_HB_{N,k+l}^{(r)}(y)=\sum_{n,m=0}^{k,l}\frac{k!l!(-x)^{n+m}\,{}_HB_{N,k+l-m-n}^{(r)}(x,y)}{(k-n)!(l-m)!n!m!}. \tag{3.10}$$

Theorem 3.4. *The following implicit summation formulae for higher-order hypergeometric Hermite-Bernoulli polynomials $_HB_{N,n}^{(r)}(x,y)$ holds true:*

$$_HB_{N,n}^{(r)}(x,y)=\sum_{m=0}^{n}\binom{n}{m}B_{N,n-m}^{(r)}(x-z)H_m(z,y). \tag{3.11}$$

Proof. By exploiting the generating function (2.1) and using (1.8), we can write equation (2.1) as

$$\frac{1}{{}_1F_1(1;N+1;t)^r}e^{(x-z)t}e^{zt+yt^2}=\sum_{n=0}^{\infty}B_{N,n}^{(r)}(x-z)\frac{t^n}{n!}\sum_{m=0}^{\infty}H_m(z,y)\frac{t^m}{m!}.$$

Replacing n by $n-m$ in above equation and using lemma ([20], p.101(1)), we get

$$\sum_{n=0}^{\infty} {}_HB_{N,n}^{(r)}(x,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} B_{N,n-m}^{(r)}(x-z)H_m(z,y)\frac{t^n}{(n-m)!m!}.$$

On equating the coefficients of the like powers of t, we get (3.11). □

Corollary 3.5. *Letting $z = x$ in Theorem 3.2 gives*

$$ {}_HB_{N,n}^{(r)}(x,y) = \sum_{m=0}^{n}\binom{n}{m} B_{N,n-m}^{(r)}H_m(x,y). \tag{3.12}$$

Theorem 3.6. *The following implicit summation formulae for higher-order hypergeometric Hermite-Bernoulli polynomials ${}_HB_{N,n}^{(r)}(x,y)$ holds true:*

$$ {}_HB_{N,n}^{(r)}(x+1,y) = \sum_{m=0}^{n}\binom{n}{m} {}_HB_{N,n-m}^{(r)}(x,y). \tag{3.13}$$

Proof. Using the generating function (2.1), we have

$$\sum_{n=0}^{\infty} {}_HB_{N,n}^{(r)}(x+1,y)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_HB_{N,n}^{(r)}(x,y)\frac{t^n}{n!} = \frac{1}{{}_1F_1(1;N+1;t)^r}(e^t-1)e^{zt+yt^2}$$

$$= \sum_{n=0}^{\infty} {}_HB_{N,n}^{(r)}(x,y)\frac{t^n}{n!}\left(\sum_{m=0}^{\infty}\frac{t^m}{m!}-1\right)$$

$$= \sum_{n=0}^{\infty} {}_HB_{N,n}^{(r)}(x,y)\frac{t^n}{n!}\sum_{m=0}^{\infty}\frac{t^m}{m!} - \sum_{n=0}^{\infty} {}_HB_{N,n}^{(r)}(x,y)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m} {}_HB_{N,n-m}^{(r)}(x,y)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_HB_{N,n}^{(r)}(x,y)\frac{t^n}{n!}.$$

Finally equating the coefficients of the like powers of t, we get (3.13). □

Theorem 3.7. *The following implicit summation formula involving higher-order hypergeometric Hermite-Bernoulli polynomials ${}_HB_{N,n}^{(r)}(x,y)$ holds true:*

$$ {}_HB_{N,n}^{(r)}(z+x,u+y) = \sum_{m=0}^{n}\binom{n}{m} {}_HB_{N,n-m}^{(r)}(x,y)H_m(z,u). \tag{3.14}$$

Proof. We replace x by $x+z$ and y by $y+u$ in (2.1), use (1.2) and rewrite the generating function as

$$\frac{1}{{}_1F_1(1;N+1;t)^r}e^{(xt+yt^2}e^{zt+ut^2} = \sum_{n=0}^{\infty} {}_HB_{N,n}^{(r)}(x,y)\frac{t^n}{n!}\sum_{m=0}^{\infty}H_m(x,y)\frac{t^m}{m!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} {}_HB_{N,n}^{(r)}(x,y)H_m(x,y)\frac{t^{n+m}}{n!m!}.$$

Replacing n by $n-m$ in above equation, we have

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} {}_HB_{N,n-m}^{(r)}(x,y)H_m(x,y)\frac{t^n}{(n-m)!m!}.$$

Comparing the coefficients of t on both sides, we get the result (3.14). □

Theorem 3.8. *The following implicit summation formula involving higher-order hypergeometric Hermite-Bernoulli polynomials* ${}_HB^{(r)}_{N,n}(x,y)$ *holds true:*

$$ {}_HB^{(r)}_{N,n}(y,x)=\sum_{k=0}^{[\frac{n}{2}]}B^{(r)}_{N,n-2k}(y)\frac{x^k}{(n-2k)!k!}. \tag{3.15}$$

Proof. We replace x by y and y by x in equation (2.1) to get

$$\sum_{n=0}^{\infty}{}_HB^{(r)}_{N,n}(y,x)\frac{t^n}{n!}=\sum_{n=0}^{\infty}B^{(r)}_{N,n-2k}(y)\frac{t^n}{n!}\sum_{k=0}^{\infty}\frac{x^kt^{2k}}{k!}.$$

Now replacing n by $n-2k$ and comparing the coefficients of t, we get the result (3.15). □

Theorem 3.9. *The following implicit summation formula involving higher-order hypergeometric Hermite-Bernoulli polynomials* ${}_HB^{(r)}_{N,n}(x,y)$ *holds true:*

$$ {}_HB^{(r)}_{N,n}(z,u)=\sum_{m=0}^{n}\binom{n}{m}H_m(\alpha-x+z,\beta-y+u){}_HB^{(r)}_{N,n-m}(x-\alpha,y-\beta), \tag{3.16}$$

and

$$ {}_HB^{(r)}_{N,n}(z-\alpha-x,u-\beta+y)=\sum_{m=0}^{n}\binom{n}{m}H_m(z,u){}_HB^{(r)}_{N,n-m}(x-\alpha,y-\beta). \tag{3.17}$$

Proof. By exploiting the generating function (2.1), we can write

$$\sum_{n=0}^{\infty}{}_HB^{(r)}_{N,n}(z,u)\frac{t^n}{n!}=\frac{1}{{}_1F_1(1;N+1;t)^r}e^{zt+ut^2}$$

$$=e^{-(x-z-\alpha)t-(y-u-\beta)t^2}e^{(x-\alpha)t+(y-\beta)t^2}\frac{1}{{}_1F_1(1;N+1;t)^r}$$

$$=e^{-(x-z-\alpha)t-(y-u-\beta)t^2}\sum_{n=0}^{\infty}{}_HB^{(r)}_{N,n}(x-\alpha,y-\beta)\frac{t^n}{n!},$$

which yields

$$\sum_{n=0}^{\infty}{}_HB^{(r)}_{N,n}(x,y)\frac{t^n}{n!}=\sum_{m=0}^{\infty}H_m(\alpha-x+z,\beta-y+u)\frac{t^m}{m!}\sum_{n=0}^{\infty}{}_HB^{(r)}_{N,n}(x-\alpha,y-\beta)\frac{t^n}{n!}.$$

Replacing n by $n-m$ in above equation and comparing the coefficients of t, we obtain (3.16). On replacing z by $z-\alpha-x$ and u by $u-\beta+y$ in (3.16), we get (3.17). □

Corollary 3.10. *On setting $z=u=0$ in (3.16), we have the following result for higher-order hypergeometric Hermite-Bernoulli polynomials* ${}_HB^{(r)}_{N,n}(x,y)$ *holds true:*

$$B^{(r)}_{N,n}=\sum_{m=0}^{n}\binom{n}{m}H_m(\alpha-x,\beta-y){}_HB^{(r)}_{N,n-m}(x-\alpha,y-\beta).$$

Theorem 3.11. *The following implicit summation formula involving higher-order hypergeometric Hermite-Bernoulli polynomials* ${}_HB^{(r)}_{N,n}(x,y)$ *holds true:*

$$\sum_{m=0}^{[\frac{n}{2}]}\frac{B^{(r)}_{N,n-2m}(x)B^{(r)}_{N,m}(y)}{(n-2m)!m!}=\sum_{m=0}^{[\frac{n}{2}]}\frac{{}_HB^{(r)}_{N,n-2m}(x,y)B^{(r)}_{N,m}}{(n-2m)!m!},\tag{3.18}$$

and

$$\sum_{m=0}^{[\frac{n}{2}]}\frac{B^{(r)}_{N,n-2m}(x)B^{(r)}_{N,m}(y)}{(n-2m)!m!}=\sum_{k=0}^{n}\sum_{m=0}^{[\frac{n-k}{2}]}\frac{B^{(r)}_{N,n-k-2m}B^{(r)}_{N,m}H_k(x,y)}{(n-k-2m)!m!k!}.\tag{3.19}$$

Proof. Consider the definition of (2.1), we have

$$\sum_{n=0}^{\infty}B^{(r)}_{N,n}(y)\frac{t^{2n}}{n!}=\frac{1}{{}_1F_1(1;N+1;t^2)^r}e^{yt^2},\tag{3.20}$$

where x is replaced by y and t is replaced by t^2 in (2.1). On multiplying (2.1) and (3.20), we have

$$\sum_{n=0}^{\infty}B^{(r)}_{N,n}(x)\frac{t^n}{n!}\sum_{m=0}^{\infty}B^{(r)}_{N,m}(y)\frac{t^{2m}}{m!}=\frac{1}{{}_1F_1(1;N+1;t)^r}\frac{1}{{}_1F_1(1;N+1;t^2)^r}e^{xt+yt^2},\tag{3.21}$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]}B^{(r)}_{N,n-2m}(x)B^{(r)}_{N,m}(y)\frac{t^n}{(n-2m)!m!}=\sum_{n=0}^{\infty}{}_HB^{(r)}_{N,n}(x,y)\frac{t^n}{n!}\sum_{m=0}^{\infty}B^{(r)}_{N,m}\frac{t^{2m}}{m!}.$$

Using the Cauchy product and comparing the coefficients of t, we obtain (3.18). Another way of defining the right hand side of equation (3.21) is suggested by replacing e^{xt+yt^2} by its series representation

$$\sum_{n=0}^{\infty}B^{(r)}_{N,n}(x)\frac{t^n}{n!}\sum_{m=0}^{\infty}B^{(r)}_{N,m}(y)\frac{t^{2m}}{m!}=\frac{1}{{}_1F_1(1;N+1;t)^r}\frac{1}{{}_1F_1(1;N+1;t^2)^r}e^{xt+yt^2}$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]}B^{(r)}_{N,n-2m}(x)B^{(r)}_{N,m}(y)\frac{t^n}{(n-2m)!m!}=\sum_{k=0}^{\infty}H_k(x,y)\frac{t^k}{k!}\sum_{n=0}^{\infty}B^{(r)}_{N,n}\frac{t^n}{n!}\sum_{m=0}^{\infty}B^{(r)}_{N,m}\frac{t^{2m}}{m!}.$$

Using the Cauchy product and comparing the coefficients of t, we get (3.19).
□

Corollary 3.12. *For $y=0$ in Theorem 3.7, we have the following result for higher-order hypergeometric Bernoulli polynomials* ${}_HB^{(r)}_{N,n}(x,y)$ *holds true:*

$$\sum_{m=0}^{[\frac{n}{2}]}\frac{B^{(r)}_{N,n-2m}(x)B^{(r)}_{N,m}}{(n-2m)!m!}=\sum_{m=0}^{[\frac{n}{2}]}\frac{{}_HB^{(r)}_{N,n-2m}(x,0)B^{(r)}_{N,m}}{(n-2m)!m!},$$

and

$$\sum_{m=0}^{[\frac{n}{2}]}\frac{B^{(r)}_{N,n-2m}(x)B^{(r)}_{N,m}}{(n-2m)!m!}=\sum_{k=0}^{n}\sum_{m=0}^{[\frac{n-k}{2}]}\frac{B^{(r)}_{N,n-k-2m}B^{(r)}_{N,m}x^k}{(n-k-2m)!m!k!}.$$

Theorem 3.13. *The following implicit summation formula involving higher-order hypergeometric Hermite-Bernoulli polynomials* ${}_HB^{(r)}_{N,n}(x,y)$ *holds true:*

$$\sum_{m=0}^{n}\sum_{r=0}^{[\frac{n-m}{2}]}\frac{\left(\frac{x}{y^2}-\frac{y}{x^2}\right)^r B^{(k)}_{N,m}{}_HB^{(k)}_{N,n}(x,y)}{y^m m!r!(n-m-2r)!x^{n-m-2r}}=\sum_{m=0}^{n}\frac{B^{(k)}_{N,m}{}_HB^{(k)}_{N,n-m}(x,y)}{x^m m!y^{n-k}!(n-k)!}.\tag{3.22}$$

Proof. On replacing t by $\frac{t}{x}$ and r by k, we can write equation (2.1) as

$$\sum_{n=0}^{\infty} {}_HB_{N,n}^{(k)}(x,y)\frac{t^n}{x^n n!} = \frac{1}{{}_1F_1(1;N+1;\frac{t}{x})^k}e^{t+y\frac{t^2}{x^2}}. \tag{3.23}$$

Now interchanging x and y, we have

$$\sum_{n=0}^{\infty} {}_HB_{N,n}^{(k)}(y,x)\frac{t^n}{y^n n!} = \frac{1}{{}_1F_1(1;N+1;\frac{t}{y})^k}e^{t+x\frac{t^2}{y^2}}. \tag{3.24}$$

Comparison of (3.23) and (3.24) yields

$$e^{x\frac{t^2}{y^2}-y\frac{t^2}{x^2}}\frac{1}{{}_1F_1(1;N+1;\frac{t}{y})^k}\sum_{n=0}^{\infty} {}_HB_{N,n}^{(k)}(x,y)\frac{t^n}{x^n n!}$$

$$= \frac{1}{{}_1F_1(1;N+1;\frac{t}{x})^k}\sum_{n=0}^{\infty} {}_HB_{N,n}^{(k)}(y,x)\frac{t^n}{y^n n!}$$

$$\sum_{r=0}^{\infty}\frac{\left(\frac{x}{y^2}-\frac{y}{x^2}\right)^r t^{2r}}{r!}\sum_{m=0}^{\infty} B_{N,m}^{(k)}\frac{t^m}{y^m m!}\sum_{n=0}^{\infty} {}_HB_{N,n}^{(k)}(x,y)\frac{t^n}{x^n n!}$$

$$= \sum_{m=0}^{\infty} B_{N,m}^{(k)}\frac{t^m}{x^m m!}\sum_{n=0}^{\infty} {}_HB_{N,n}^{(k)}(x,y)\frac{t^n}{y^n n!}.$$

Using the Cauchy product and comparing the coefficients of t, we get (3.22). □

Theorem 3.14. *The following implicit summation formula involving higher-order hypergeometric Hermite-Bernoulli polynomials* ${}_HB_{N,n}^{(r)}(x,y)$ *holds true:*

$$ {}_HB_{N,n}^{(r)}(w,u)\,{}_HB_{N,m}^{(r)}(W,U) = \sum_{s,k=0}^{m,n}\binom{n}{s}\binom{m}{k}H_s(w-x,u-y)\,{}_HB_{N,n-s}^{(r)}(x,y)$$

$$\times H_k(W-X,U-Y)\,{}_HB_{N,m-k}^{(r)}(X,Y). \tag{3.25}$$

Proof. Consider the product of higher-order hypergeometric Hermite-Bernoulli polynomials, equation (2.1) in the following form

$$\frac{1}{{}_1F_1(1;N+1;t)^r}e^{xt+yt^2}\frac{1}{{}_1F_1(1;N+1;T)^r}e^{XT+YT^2}$$

$$= \sum_{n=0}^{\infty} {}_HB_{N,n}^{(r)}(x,y)\frac{t^n}{n!}\sum_{m=0}^{\infty} {}_HB_{N,m}^{(r)}(X,Y)\frac{T^m}{m!}. \tag{3.26}$$

Replacing x by w, y by u, X by W and Y by U in (3.26) and equating the resultant equation to itself, we find

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} {}_HB_{N,n}^{(r)}(w,u)\,{}_HB_{N,m}^{(r)}(W,U)\frac{t^n}{n!}\frac{T^m}{m!}$$

$$= \exp\left((w-x)t+(u-y)t^2\right)\exp\left((W-X)T+(U-Y)T^2\right)$$

$$\times\sum_{n=0}^{\infty} {}_HB_{N,n}^{(r)}(x,y)\frac{t^n}{n!}\sum_{m=0}^{\infty} {}_HB_{N,m}^{(r)}(X,Y)\frac{T^m}{m!}$$

$$=\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}H_s(w-x,u-y)\,{}_HB_{N,n}^{(r)}(x,y)\frac{t^{n+s}}{n!s!}.$$

$$\times H_k(W-X,U-Y)\,{}_HB_{N,m}^{(r)}(X,Y)\frac{T^{m+k}}{m!k!}.$$

Finally, replacing n by $n-s$ and m by $m-k$ in the r.h.s. of the above equation and then equating the coefficients of like powers of t and T, we get assertion (3.25) of Theorem (3.8). □

Remark 3.15. *Replacing u by y and U by Y in assertion (3.25) of Theorem (3.9), we deduce the the following consequence of Theorem (3.9).*

Corollary 3.16. *The following implicit summation formula involving higher-order hypergeometric Hermite-Bernoulli polynomials ${}_HB_{N,n}^{(r)}(x,y)$ holds true:*

$${}_HB_{N,n}^{(r)}(w,y)\,{}_HB_{N,m}^{(r)}(W,Y)=\sum_{s,k=0}^{m,n}\binom{n}{s}\binom{m}{k}(w-x)^s\,{}_HB_{N,n-s}^{(r)}(x,y)$$

$$\times(W-X)^k\,{}_HB_{N,m-k}^{(r)}(X,Y).$$

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