Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 ON LINE
SPM: www.spm.uem.br/bspm

(3s.) **v. 2022 (40)** : 1–10.
ISSN-0037-8712 IN PRESS
doi:10.5269/bspm.52003

Generalized Bézier Curves Based on Bernstein-Stancu-Chlodowsky Type Operators

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ABSTRACT: In this paper, we use the blending functions of Bernstein-Stancu-Chlodowsky type operators with shifted knots for construction of modified Chlodowsky Bézier curves. We study the nature of degree elevation and degree reduction for Bézier Bernstein-Stancu-Chlodowsky functions with shifted knots for $t \in [\frac{\gamma}{n+\delta}, \frac{n+\gamma}{n+\delta}]$. We also present a de Casteljau algorithm to compute Bernstein Bézier curves with shifted knots. The new curves have some properties similar to Bézier curves. Furthermore, some fundamental properties for Bernstein Bézier curves are discussed. Our generalizations show more flexibility in taking the value of γ and δ and advantage in shape control of curves. The shape parameters give more convenience for the curve modelling.

Key Words: Bernstein-Stancu-Chlodowsky type operators, Bézier curves, Degree elevation, de Casteljau algorithm, Shape parameters.

Contents

1. Introduction

Bézier curves were developed by Casteljau [4] and Bézier [3], and have been applied to many computer-aided design (CAD) applications. While their origin can be traced back to the design of car body shapes. A Bézier curve is defined in terms of a set of control points, though it only considers global information i.e. it does not consider local information and calculates the curve points in a linear recursive approach starting with the edges of the control polygon. Frequently, there is a large gap between the Bézier curve and its control polygon, which restricts the maximum length of a curve segment. While strategies such as degree elevation, composite Bézier curves, refinement and subdivision reduce this gap, they also increase the number of control points. A higher-degree Bézier curve obviously provides a better shape representation.
In this problem, we generalize some of the very well-known Bézier curve techniques by using a generalization of the Bernstein basis, called the Bernstein-Stancu-Chlodowsky basis.
S.N. Bernstein [2] in 1912, who first introduced his famous operators $B_n : C[0,1] \longrightarrow C[0,1]$ defined for any $n \in \mathbb{N}$ and for any function $f \in C[0,1]$

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right),\ 0 \le x \le 1. \tag{1.1}$$

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2010 *Mathematics Subject Classification:* Primary: 65D17; Secondary: 41A10, 41A25, 41A36.
Submitted January 29, 2020. Published April 13, 2020

and named it Bernstein polynomials to prove the Weierstrass theorem [16] and Bernstein polynomials possess many remarkable properties and has various applications in many areas such as approximation theory, numerical analysis, computer-aided geometric design, and solutions of differential equations due to its fine properties of approximation.
In computer aided geometric design (CAGD), Bernstein polynomials and its variants are used in order to preserve the shape of the curves or surfaces. One of the most important curve in CAGD [29] is the classical Bézier curve [3] constructed with the help of Bernstein basis functions. Other works related to different generalization of Bernstein polynomials and Bézier curves and surfaces can be found in [6-7, 8, 13-14, 17-18, 22-28].
Gadjiev and Gorhanalizadeh [9] introduced the following construction of Bernstein- Stancu type polynomials with shifted knots:

$$S_{n,\gamma,\delta}(f;x)=\left(\frac{n+\delta_2}{n}\right)\sum_{k=0}^{n}\binom{n}{k}\left(x-\frac{\gamma_2}{n+\delta_2}\right)^{k}\left(\frac{n+\gamma_2}{n+\delta_2}-x\right)^{n-k}f\left(\frac{k+\gamma_1}{n+\delta_1}\right) \tag{1.2}$$

where $\frac{\gamma_2}{n+\delta_2}\leq x\leq\frac{n+\gamma_2}{n+\delta_2}$ and γ_k,δ_k $(k=1,2)$ are positive real numbers provided $0\leq\gamma_1\leq\gamma_2\leq\delta_1\leq\delta_2$. It is clear that for $\gamma_1=\gamma_2=\delta_1=\delta_2=0$, then these operators reduces to the classical Bernstein operators. The classical Bernstein-Chlodowsky polynomials have the following form

$$C_n(f;x)=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^{k}\left(1-\frac{x}{b_n}\right)^{n-k}f\left(\frac{k}{n}b_n\right), \tag{1.3}$$

where $0\leq x\leq b_n$ and $\{b_n\}_{(n\geq1)}$ is a positive increasing sequence with the properties

$$\lim_{n\to\infty}b_n=\infty,\ \lim_{n\to\infty}\frac{b_n}{n}=0.$$

These polynomials were introduced by Chlodowsky [5] as a generalization of Bernstein polynomials on an unbounded set. Aral et al. [1] defined Bernstein-Stancu- Chlodowsky polynomials which are generalization of $S_{n,\gamma,\delta}(f;x)$ as:

$$S_{n,\gamma,\delta}(f;x)=\left(\frac{n+\delta_2}{n}\right)\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{b_n}-\frac{\gamma_2}{n+\delta_2}\right)^{k}\left(\frac{n+\gamma_2}{n+\delta_2}-\frac{x}{b_n}\right)^{n-k}f\left(\frac{k+\gamma_1}{n+\delta_1}b_n\right), \tag{1.4}$$

where $\frac{\gamma_2}{n+\delta_2}b_n\leq x\leq\frac{n+\gamma_2}{n+\delta_2}b_n$, γ_k,δ_k $(k=1,2)$ are positive real numbers provided $0\leq\gamma_1\leq\gamma_2\leq\delta_1\leq\delta_2$ and $\{b_n\}_{(n\geq1)}$ is a positive increasing sequence such that

$$\lim_{n\to\infty}b_n=\infty,\ \lim_{n\to\infty}\frac{b_n}{n}=0.$$

1. **Case 1.** Take $b_n=1$, then (1.4) reduces to (1.2).
2. **Case 2.** Take $\gamma_1=\gamma_2=\delta_1=\delta_2=0$, then (1.4) gives (1.3).
3. **Case 3.** Combined Case 1 and Case 2, we get classical Bernstein operators (1.1).

In recent years, generalization of the Bézier curve with shape parameters has received continuous attention. Several authors were concerned with the problem of changing the shape of curves and surfaces, while keeping the control polygon unchanged and thus they generalized the Bézier curves in [12, 13- 14, 25].
Recently, Mishra, et al. [20] studied on inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators and various generalization of Szász - Mirakjan operators have been studied by Mishra et al. [21], Gandhi et al. [11] and Mishra and Gandhi [19] and Gairola et al. [10] have discussed approximation properties of linear positive operators.
In 2017, Khatri and Mishra [15] introduced Generalized Szász-Mirakyan operators involving Brenke type

polynomials.
The rest of the paper is organized as follows: Section 2 introduces Bernstein-Stancu-Chlodowsky functions $H^k_{n,\gamma,\delta}$ and their Properties. Section 3 introduces Bernstein-Stancu-Chlodowsky Bézier curves, its properties, degree elevation and de Casteljau algorithm for $H^k_{n,\gamma,\delta}$. The effects on the shape of the curves by the shape parameters are presented in Section 4.

2. Properties of the Bernstein-Stancu-Chlodowsky functions

The Bernstein-Stancu-Chlodowsky functions are introduced as

$$H^k_{n,\gamma,\delta}(s) = \binom{n}{k}\left(\frac{n+\delta}{n}\right)^n\left(\frac{s}{b_n}-\frac{\gamma}{n+\delta}\right)^k\left(\frac{n+\gamma}{n+\delta}-\frac{s}{b_n}\right)^{n-k}, \tag{2.1}$$

where $\frac{\gamma}{n+\delta}b_n \leq s \leq \frac{n+\gamma}{n+\delta}b_n$ and γ, δ are positive real numbers provided $0 \leq \gamma \leq \delta$.

2.1. Theorem

The Bernstein-Stancu-Chlodowsky functions possess the following properties:

1. Non-negativity:

$$H^k_{n,\gamma,\delta}(s) \geq 0,\ k=0,1,...,n,\ \frac{\gamma}{n+\delta}b_n \leq s \leq \frac{n+\gamma}{n+\delta}b_n.$$

2. Partition of unity:

$$\sum_{k=0}^{n} H^k_{n,\gamma,\delta}(s) = 1,\ \frac{\gamma}{n+\delta}b_n \leq s \leq \frac{n+\gamma}{n+\delta}b_n.$$

3. End-point property:

$$H^k_{n,\gamma,\delta}\left(\frac{\gamma}{n+\delta}b_n\right) = \begin{cases} 1, & \text{if } k=0, \\ 0, & k \neq 0, \end{cases}$$

$$H^k_{n,\gamma,\delta}\left(\frac{n+\gamma}{n+\delta}b_n\right) = \begin{cases} 1, & \text{if } k=n, \\ 0, & k \neq n, \end{cases}$$

clearly both side end point property holds.

4. Reducibility: when $\gamma = \delta = 0$, $b_n = 1$ formula (2.1) reduces to the classical Bernstein bases on [0, 1].

Proof: All these property can be proved easily from equation (2.1). Fig. 1 represents the Bernstein-Stancu-Chlodowsky functions of degree 3 for $\gamma = 6$, $\delta = 8$ and $b_n = (n)^{1/3}$. Here, we can see that sum of blending fuctions is always unity and also satisfies end point interpolation property. If $\gamma = \delta = 0$, $b_n = 1$ it gives classical Bernstein basis on $[0,1]$ which is presented in Fig. 2.
Apart from the basic properties above, Bernstein-Stancu-Chlodowsky functions also satisfy the following recurrence relations, as for the classical Bernstein basis.

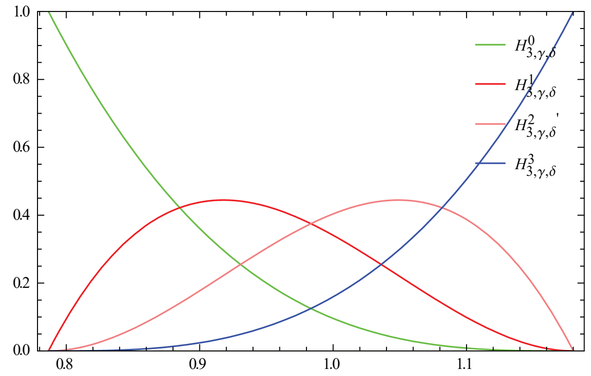

Figure 1: Cubic Bézier blending functions for $\gamma = 6$, $\delta = 8$, $b_n = (n)^{1/3}$, $n = 3$.

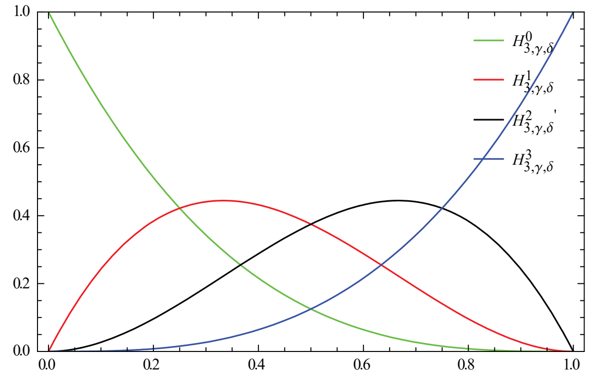

Figure 2: Cubic Bézier blending functions for $\gamma = 0$, $\delta = 0$, $b_n = 1$, $n = 3$.

2.2. Theorem

Each Bernstein-Stancu-Chlodowsky functions of degree n is a linear combination of two Bernstein-Stancu-Chlodowsky functions of degree $n+1$.

$$H^k_{n,\gamma,\delta}(s) = \left(\frac{n+1-k}{n+1}\right)H^k_{n+1,\gamma,\delta}(s) + \left(\frac{k+1}{n+1}\right)H^{k+1}_{n+1,\gamma,\delta}(s), \tag{2.2}$$

where $\frac{\gamma}{n+\delta}b_n \leq s \leq \frac{n+\gamma}{n+\delta}b_n$ and γ, δ are positive real numbers $0 \leq \gamma \leq \delta$.

Proof:

$$\left(\frac{n}{n+\delta}\right)H^k_{n,\gamma,\delta} = H^k_{n+1,\gamma,\delta}\left(\frac{n+\gamma}{n+\delta} - \frac{s}{b_n} + \{\frac{s}{b_n} - \frac{\gamma}{n+\delta}\}\right),$$

$$\begin{aligned}\left(\frac{n}{n+\delta}\right)H^k_{n,\gamma,\delta}(s) = \left(\frac{n+\gamma}{n+\delta} - \frac{s}{b_n}\right)H^k_{n+1,\gamma,\delta}(s) + \left(\frac{s}{b_n} - \frac{\gamma}{n+\delta}\right)H^k_{n+1,\gamma,\delta}(s) \\ = I_1 + I_2,\end{aligned} \tag{2.3}$$

$$\begin{aligned} I_1 &= \left(\frac{n+\gamma}{n+\delta} - \frac{s}{b_n}\right)\left(\binom{n}{k}\left(\frac{n+\delta}{n}\right)^n\left(\frac{s}{b_n} - \frac{\gamma}{n+\delta}\right)^k\left(\frac{n+\gamma}{n+\delta} - \frac{s}{b_n}\right)^{n-k}\right) \\ &= \left(\frac{n}{n+\delta}\right)\left(\frac{n+1-k}{n+1}\right)H^k_{n+1,\gamma,\delta}(s). \end{aligned}$$

Similarly,

$$\begin{aligned} I_2 &= \left(\frac{s}{b_n}-\frac{\gamma}{n+\delta}\right)\left(\binom{n}{k}\left(\frac{n+\delta}{n}\right)^n\left(\frac{s}{b_n}-\frac{\gamma}{n+\delta}\right)^k\left(\frac{n+\gamma}{n+\delta}-\frac{s}{b_n}\right)^{n-k}\right) \\ &= \left(\frac{n}{n+\delta}\right)\left(\frac{k+1}{n+1}\right)H^{k+1}_{n+1,\gamma,\delta}(s). \end{aligned}$$

By putting the values of I_1 and I_2 in (2.3), we get required result (2.2).

2.3. Theorem

Each Bernstein-Stancu-Chlodowsky functions of degree n is a linear combination of two Bernstein-Stancu-Chlodowsky functions of degree $n-1$.

$$H^k_{n,\gamma,\delta}(s)=\left(\frac{s}{b_n}-\frac{\gamma}{n+\delta}\right)\left(\frac{n+\delta}{n}\right)H^{k-1}_{n-1,\gamma,\delta}(s)+\left(\frac{n+\gamma}{n+\delta}-\frac{s}{b_n}\right)\left(\frac{n+\delta}{n}\right)H^{k}_{n-1,\gamma,\delta}(s), \tag{2.4}$$

where $\frac{\gamma}{n+\delta}b_n \leq s \leq \frac{n+\gamma}{n+\delta}b_n$ and γ, δ are positive real numbers $0 \leq \gamma \leq \delta$.
Proof: We use the Pascal-type relations, we have

$$\begin{aligned} H^k_{n,\gamma,\delta}(s) &= \binom{n}{k}\left(\frac{n+\delta}{n}\right)^n\left(\frac{s}{b_n}-\frac{\gamma}{n+\delta}\right)^k\left(\frac{n+\gamma}{n+\delta}-\frac{s}{b_n}\right)^{n-k} \\ &= \left\{\binom{n-1}{k-1}+\binom{n-1}{k}\right\}\left(\frac{n+\delta}{n}\right)^n\left(\frac{s}{b_n}-\frac{\gamma}{n+\delta}\right)^k\left(\frac{n+\gamma}{n+\delta}-\frac{s}{b_n}\right)^{n-k} \\ &= \binom{n-1}{k-1}\left(\frac{n+\delta}{n}\right)^n\left(\frac{s}{b_n}-\frac{\gamma}{n+\delta}\right)^k\left(\frac{n+\gamma}{n+\delta}-\frac{s}{b_n}\right)^{n-k} \\ &+ \binom{n-1}{k}\left(\frac{n+\delta}{n}\right)^n\left(\frac{s}{b_n}-\frac{\gamma}{n+\delta}\right)^k\left(\frac{n+\gamma}{n+\delta}-\frac{s}{b_n}\right)^{n-k} \\ &= \left(\frac{s}{b_n}-\frac{\gamma}{n+\delta}\right)\left(\frac{n+\delta}{n}\right)H^{k-1}_{n-1,\gamma,\delta}(s)+\left(\frac{n+\gamma}{n+\delta}-\frac{s}{b_n}\right)\left(\frac{n+\delta}{n}\right)H^{k}_{n-1,\gamma,\delta}(s). \end{aligned}$$

3. Bernstein-Stancu-Chlodowsky Bézier curves

We define the Bernstein-Stancu-Chlodowsky Bézier curves of degree n using the Bernstein-Stancu-Chlodowsky functions as the following:

$$R(s)=\sum_{k=0}^{n} R_k H^k_{n,\gamma,\delta}(s). \tag{3.1}$$

where R_k, $\mathbb{R}_3(j=0,1,...,n)$. R_k are control points. Joining up adjacent points $R_k, k=0,1,2,...,n$ to obtain a polygon which is called the control polygon of Bernstein-Stancu-Chlodowsky Bézier curves.

3.1. Theorem

The end-point property of derivative:

$$\mathbf{R}'\left(\frac{\gamma}{n+\delta}b_n\right)=\left(\frac{n+\delta}{b_n}\right)(\mathbf{R_1}-\mathbf{R_0})\left(\frac{n-1+\delta}{n-1}\right)^{n-1}\left(\frac{n-1+\gamma}{n-1+\delta}-\frac{\gamma}{n+\delta}\right)^{n-1-k} \tag{3.2}$$

$$\mathbf{R}'\left(\frac{n+\gamma}{n+\delta}b_n\right)=\left(\frac{n+\delta}{b_n}\right)(\mathbf{R_n}-\mathbf{R_{n-1}})\left(\frac{n-1+\delta}{n-1}\right)^{n-1}\left(\frac{n+\gamma}{n+\delta}-\frac{\gamma}{n-1+\delta}\right)^{n-1} \tag{3.3}$$

i.e. Bernstein-Stancu-Chlodowsky Bézier curves are tangent to fore-and-aft edges of its control polygon at end points.

Proof: Let

$$\begin{aligned}\mathbf{R}(s) &= \sum_{k=0}^{n}\mathbf{R_k}H^k_{n,\gamma,\delta}(s)\\ &= \sum_{k=0}^{n}\mathbf{R_k}\binom{n}{k}\left(\frac{n+\delta}{n}\right)^n\left(\frac{s}{b_n}-\frac{\gamma}{n+\delta}\right)^k\left(\frac{n+\gamma}{n+\delta}-\frac{s}{b_n}\right)^{n-k}\\ &= \mathbf{U}(s).\end{aligned}$$

or

$$\mathbf{R}(s)=\mathbf{U}(s).$$

Now, on differentiating both side with respect to s, we get

$$\mathbf{R}^{'}(s)=\mathbf{U}^{'}(s).$$

$$B^n_k(s)=\binom{n}{k}\left(\frac{n+\delta}{n}\right)^n\left(\frac{s}{b_n}-\frac{\gamma}{n+\delta}\right)^k\left(\frac{n+\gamma}{n+\delta}-\frac{s}{b_n}\right)^{n-k},$$

then

$$\mathbf{U}(s)=\sum_{k=0}^{n}\mathbf{R_k}B^n_k(s).$$

$$\begin{aligned}(B^n_k(s))^{'} &= \binom{n}{k}\left(\frac{n+\delta}{n}\right)^n\frac{k}{b_n}\left(\frac{s}{b_n}-\frac{\gamma}{n+\delta}\right)^{k-1}\left(\frac{n+\gamma}{n+\delta}-\frac{s}{b_n}\right)^{n-k}\\ &- \binom{n}{k}\left(\frac{n+\delta}{n}\right)^n\left(\frac{s}{b_n}-\frac{\gamma}{n+\delta}\right)^k\frac{n-k}{b_n}\left(\frac{n+\gamma}{n+\delta}-\frac{s}{b_n}\right)^{n-k-1}\\ &= \left(\frac{n+\delta}{b_n}\right)[B^{n-1}_{k-1}(s)+B^{n-1}_k(s)],\end{aligned}$$

then

$$\mathbf{U}^{'}(s)=\sum_{k=0}^{n}\mathbf{R_k}(B^n_k(s))^{'}.$$

Now

$$\mathbf{U}^{'}\left(\frac{\gamma}{n+\delta}b_n\right)=\mathbf{R}^{'}\left(\frac{\gamma}{n+\delta}b_n\right)=\left(\frac{n+\delta}{b_n}\right)(\mathbf{R_1}-\mathbf{R_0})B^{n-1}_0\left(\frac{\gamma}{n+\delta}b_n)\right)$$

and

$$\mathbf{R}^{'}\left(\frac{\gamma}{n+\delta}b_n\right)=\left(\frac{n+\delta}{b_n}\right)(\mathbf{R_1}-\mathbf{R_0})\left(\frac{n-1+\delta}{n-1}\right)^{n-1}\left(\frac{n-1+\gamma}{n-1+\delta}-\frac{\gamma}{n+\delta}\right)^{n-k-1}.$$

Similarly, we get

$$\mathbf{U}^{'}\left(\frac{n+\gamma}{n+\delta}b_n)\right)=\mathbf{R}^{'}\left(\frac{n+\gamma}{n+\delta}b_n\right)=\left(\frac{n+\delta}{b_n}\right)(\mathbf{R_n}-\mathbf{R_{n-1}})B^{n-1}_0\left(\frac{n+\gamma}{n+\delta}b_n)\right)$$

and

$$\mathbf{R}^{'}\left(\frac{n+\gamma}{n+\delta}b_n\right)=\left(\frac{n+\delta}{b_n}\right)(\mathbf{R_n}-\mathbf{R_{n-1}})\left(\frac{n-1+\delta}{n-1}\right)^{n-1}\left(\frac{n+\gamma}{n+\delta}-\frac{\gamma}{n-1+\delta}\right)^{n-1}.$$

Degree elevation and de Casteljau algorithm

Degree elevation

Bernstein-Stancu-Chlodowsky Bézier curves have a degree elevation algorithm that is similar to that possessed by the classical Bézier curves. To increase the flexibility of a given curve, we use the technique of degree elevation. A degree elevation algorithm calculates a new set of control points by taking a convex combination of the old set of control points which retains the old end points.

$$\mathbf{R}(s)=\sum_{k=0}^{n}\mathbf{R_k}H^{k}_{n,\gamma,\delta}(s)$$

and

$$\mathbf{R}(s)=\sum_{k=0}^{n+1}\mathbf{R}^*_{\mathbf{k}}H^{k}_{n+1,\gamma,\delta}(s).$$

where

$$\mathbf{R}^*_{\mathbf{k}}=\left(\frac{k}{n+1}\right)\mathbf{R_{k-1}}+\left(1-\frac{k}{n+1}\right)\mathbf{R_k},\ \mathrm{k}=0,1,\ \dots\ ,\mathrm{n}+1,\ \mathbf{R_{-1}}=\mathbf{R_{n+1}}=0. \tag{3.4}$$

The statement above can be derived from Theorem (2.2). When $\gamma=\delta=0$ and $b_n=1$ formula (3.4) reduce to the degree evaluation formula of the classical Bézier curves. If we let $\mathbf{R}=(\mathbf{R_0},\mathbf{R_1},...,\mathbf{R_n})^T$ denote the vector of control points of the initial Bernstein-Stancu-Chlodowsky Bézier curves of degree n, and $\mathbf{R^{(1)}}=(\mathbf{R}^*_{\mathbf{0}},\mathbf{R}^*_{\mathbf{1}},...,\mathbf{R}^*_{\mathbf{n+1}})$ the vector of control points of the degree elevated Bernstein-Stancu-Chlodowsky Bézier curves of degree $n+1$, then we can represent the degree elevation procedure as:

$$\mathbf{R^{(1)}}=T_{n+1}\mathbf{R}, \tag{3.5}$$

where

$$T_{n+1}=\frac{1}{n+1}\begin{bmatrix} n+1 & 0 & \cdots & 0 & 0\\ n+1-n & n & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & n+1-2 & 2 & 0\\ 0 & 0 & \cdots & n+1-1 & 1\\ 0 & 0 & \cdots & 0 & n+1\end{bmatrix}_{(n+2)\times(n+1)}$$

For any $l\in\mathbb{N}$, the vector of control points of the degree elevated Bernstein-Stancu-Chlodowsky Bézier curves of degree $n+l$ is:

$$\mathbf{R^{(1)}}=T_{n+l}...T_{n+2}T_{n+1}\mathbf{R}. \tag{3.6}$$

As $l\to 0$, the control polygon $\mathbf{R^{(1)}}$ converges to a Bernstein-Stancu-Chlodowsky Bézier curves.

De Casteljau algorithm

Bernstein-Stancu-Chlodowsky Bézier curves of degree n can be written as two kinds of linear combination of two Bernstein-Stancu-Chlodowsky Bézier curves of degree $n-1$, and we can get the two selectable algorithms to evaluate Bernstein-Stancu-Chlodowsky Bézier curves. The algorithms can be expressed as:

Algorithm 1.

$$\begin{cases}\mathbf{R^0_k}(s)\equiv\mathbf{R^0_k}\equiv\mathbf{R_k}, k=0,1,2,...,n,\\ \mathbf{R^r_k}(s)=\frac{n+\delta}{n}\left(\frac{s}{b_n}-\frac{\gamma}{n+\delta}\right)\mathbf{R^{r-1}_{k+1}}(s)+\frac{n+\delta}{n}\left(\frac{n+\gamma}{n+\delta}-\frac{s}{b_n}\right)\mathbf{R^{r-1}_k}(s),\\ r=1,...,n, k=0,1,2,...,n-r,\ \frac{\gamma}{n+\delta}b_n\le s\le\frac{n+\gamma}{n+\delta}b_n,\ 0\le\gamma\le\delta.\end{cases} \tag{3.7}$$

$$\mathbf{R}(s)=\sum_{k=0}^{n-1}\mathbf{R^1_k}(s)=...\sum\mathbf{R^r_i}(s)H^{k}_{n-r,\gamma,\delta}(s)=...\mathbf{R^n_0}(s). \tag{3.8}$$

It is clear that the results can be obtained from Theorem (2.3). When $\gamma = \delta = 0$ and $b_n = 1$, formula (3.7) and (3.8) recover the de Casteljau algorithms of classical Bézier curves. Let $\mathbf{R^0} = (\mathbf{R_0}, \mathbf{R_1}, ..., \mathbf{R_n})^T$, $\mathbf{R^r} = (\mathbf{R^r_0}, \mathbf{R^r_1}, ..., \mathbf{R^r_{n-r}})^T$, then de Casteljau algorithm can be expressed as:
Algorithm 2.

$$\mathbf{R^r}(s) = M_r(s)...M_2(s)M_1(s)\mathbf{R^0}, \tag{3.9}$$

where $M_r(s)$ is a $(n-r+1) \times (n-r+2)$ matrix and

$$M_r(s) = \frac{n+\delta}{n}\begin{bmatrix} \left(\frac{n+\gamma}{n+\delta} - \frac{s}{b_n}\right) & \left(\frac{s}{b_n} - \frac{\gamma}{n+\delta}\right) & \cdots & 0 & 0 \\ 0 & \left(\frac{n+\gamma}{n+\delta} - \frac{s}{b_n}\right) & \left(\frac{s}{b_n} - \frac{\gamma}{n+\delta}\right) & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \left(\frac{n+\gamma}{n+\delta} - \frac{s}{b_n}\right) & \left(\frac{s}{b_n} - \frac{\gamma}{n+\delta}\right) & 0 \\ 0 & 0 & \cdots & \left(\frac{n+\gamma}{n+\delta} - \frac{s}{b_n}\right) & \left(\frac{s}{b_n} - \frac{\gamma}{n+\delta}\right) \end{bmatrix}$$

4. Shape control of Bernstein-Stancu-Chlodowsky Bézier curves

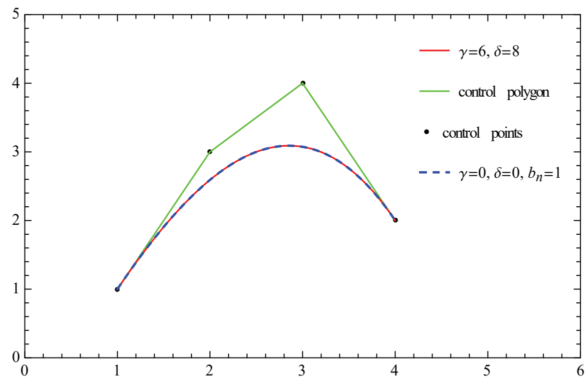

Figure 3: The effect of the shape of Cubic Bézier curve

The Bernstein-Stancu-Chlodowsky Bézier curves is generated by setting $\gamma = 6, \delta = 8$ (red lines), the classical Bézier curve generated by setting $\gamma = 0, \delta = 0, b_n = 1$ (dashed bule lines). From fig. 3, Bernstein-Stancu-Chlodowsky Bézier curves move close to the control polygon approximately same as classical Bézier curves. Similarly, in order to construct closed, we can set $R_n = R_0$. The Bernstein-Stancu-Chlodowsky Bézier curves is generated by setting $\gamma = 6, \delta = 8$ (red line), the classical Bézier curve is generated by setting $\gamma = 0, \delta = 0, b_n = 1$ (dashed blue line). From fig. 4, Bernstein-Stancu-Chlodowsky Bézier curves is closer to the control polygon than classical Bézier curves.

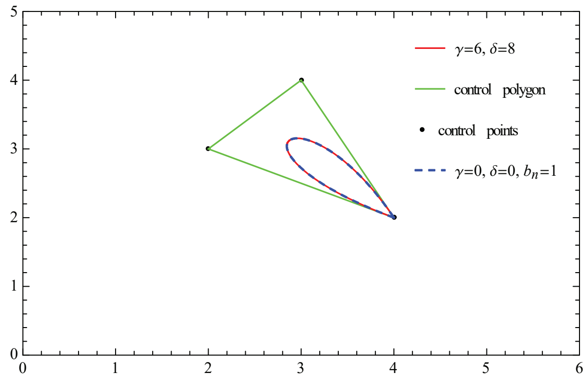

Figure 4: Closed cubic Bézier curve

5. Future work

Bernstein-Stancu-Chlodowsky Bézier curves share most properties of classical Bézier curves. Moreover, the shape of Bernstein-Stancu-Chlodowsky Bézier curves can be adjusted by altering the value of shape parameters. In the future, we will construct Bernstein-Stancu-Chlodowsky Bézier surfaces and will discuss some fundamental properties for Bernstein-Stancu-Chlodowsky Bézier surfaces, study de Casteljau algorithm and degree evaluation properties for surfaces. Similarly, we will determine q-analogue of Bernstein-Stancu-Chlodowsky Bézier curves and surfaces. We will also explain de Casteljau algorithm and degree evaluation properties for curves and surfaces. We also hope to construct generalizations of classical rational Bézier curves and surfaces based on these operator.

Competing Interests

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

Acknowledgments

The authors would like to express their deep gratitude to the anonymous learned referees of the journal for their valuable suggestions and constructive comments for better improvement of the manuscript. The first author Kejal Khatri acknowledges the Department of Atomic Energy, National Board Higher Mathematics, Mumbai, India for supporting this research article, DAE Ref. Number: 2/40(58)/2015/R&D-II/13262 on dated 29/09/2015.

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