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Iterated Function System in $\emptyset-$ Metric Spaces

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ABSTRACT: Fractals have gained great attention from researchers due to their wide applications in engineering and applied sciences. Especially, in several topics of applied sciences, the iterated function systems theory has important roles. As is well known, examples of fractals are derived from the fixed point theory for suitable operators in spaces with complete or compact structures. In this article, a new generalization of Hausdorff distance $h_\emptyset$ on $\mathcal{H}(\Omega)$, $\mathcal{H}(\Omega)$ is a class of all nonempty compact subsets of the generalized $b-$metric space $(\Omega, d_\emptyset)$. Completeness and compactness of $(\mathcal{H}(\Omega), h_\emptyset)$ are analogously obtained from its counterparts of $(\Omega, d_\emptyset)$. Furthermore, a fractal is presented under a finite set of generalized $\boldsymbol{\mu}$-contraction mappings. Also, other special cases are presented.

Key Words: Iterated function system, Generalized $\boldsymbol{\mu}$-contraction mappings, Fixed points, Generalized $b-$metric.

Contents

1. Introduction and Preliminaries

The important objects in the fractals field are the theory of the Iterated function system (IFS) and supplied powerful tools for the search of fractals. They are applied for both generating and modeling of very irregular forms. As it's known, a fractal is the attractor of the IFS. Interesting results in this field are found, for example, in [1,2,3,4,5]. (IFS) is developed by Hutchison (1981) [6], and Barnsley and others (1985, 1986, 1988) [2,7,8]. Many papers worked on the fixed point theory in different types of general metric spaces such as, [9,10,11,12,13,14,15,16,17,18,19,20]. Here, we construct a fractal set of (IFS), in a generalized $b-$metric space as the invariant set of a Hutchinson operator. This operator induced by the generalized $\mu F-$contraction mappings. The obtained fractal is the successive iterations of a generalized $\boldsymbol{\mu F}-$Hutchinson operator. Now, let Ω be a nonempty set and $\emptyset: \ \Omega\times\Omega\to[1,\ \infty)$ be a function, we recall the following:

Definition 1.1. *[18] "If a function $d_\emptyset:\Omega\times\Omega\longrightarrow[0,\ \infty)$ satisfies the following: for all $p,\ q,\ r\ \in\ \Omega$*

1. $d_\emptyset\ (p,\ q)=0\Longleftrightarrow p=q$
2. $d_\emptyset\ (p,\ q)=\ d_\emptyset\ (q,\ p)$
3. $d_\emptyset\ (p,\ q)\ \leq\ \emptyset(p,\ q)\ [d_\emptyset\ (p,\ r)+\ d_\emptyset\ (r,q)]$

Then, the pair $(\Omega,\ d_\emptyset)$ is called an extended b-metric (shortly, $\emptyset$- metric space). The following diagram is true, let $b\geq1$"
$\emptyset$- metric spaces $\overset{\emptyset(p,\ q)=b}{\Longrightarrow}$ b-metric spaces $\overset{b=1}{\Longrightarrow}$ metric spaces.

Example 1.2: [12] "Let $\Omega=C([a,b],\mathbb{R})$ be the space of all continuous real-valued functions defined on $[a,b]$.
Note that Ω is complete extended b-metric space by $d_\emptyset\,(a,b)=sup_{t\in[a,b]}|p\,(t)-q(t)|^2$,
with $\emptyset\,(p,q)=|p\,(t)|+|q\,(t)|+2$ where $\emptyset$: $\Omega\times\Omega\times\longrightarrow[1,\infty)$."

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Remark 1.2. *[16]*
(i) The $d_\emptyset$ -metric function may be discontinuous.
(i) A convergent sequence has a unique limit,
(iii) Each convergent sequence is Cauchy"

Definition 1.3. *[16] A sequence $\{p_n\} \subset \Omega$ is said to be:*

1. *Converge to a point $p \in \Omega$ if $\forall \varepsilon > 0$, $\exists\ k \in N \ni d_\emptyset\ (p_n,\ p) < \varepsilon\ \forall\ n > k$.*
2. *Cauchy sequence if $\forall \varepsilon > 0$, $\exists N = N(\varepsilon) \ni d_\emptyset\ (p_m,\ p_n) < \varepsilon$, $\forall m, \forall\ n > N$.*
3. *If any Cauchy sequence $< p_n > \subset \Omega$ converge to $p \in \Omega$ then Ω is called complete $\emptyset$-metric space."*

We reform the Hausdorff distance [2] in the case of $\emptyset$- metric spaces.

Definition 1.4. *[16] Let $\mathcal{H}(\Omega)$ be the class of all compact subsets of Ω. The extended Hausdorff distance $h_\emptyset$ between A and B in $\mathcal{H}(\Omega)$ is,*

$$h_\emptyset\,(A,\ B) = max\{\ d_\emptyset\,(A,\ B)\,,\ \ d_\emptyset\,(B,\ A)\},$$

where $d_\emptyset\,(A,\ B) = sup_{p\in A} d_\emptyset\,(p,\ B) = sup_{p\in A} inf_{q\in B} d_\emptyset\,(p,\ q)$.

Example1.6: Consider $\Omega = R$ with $d_\emptyset\,(p,\ q) = (p-q)^2$, $p, q \in \Omega$, then Ω is $\emptyset-$ metric space with $\emptyset\,(p,q) = 2$, let $A = [0,\ 20]\,,\ \ B = [32,\ 50]$.
We find that $sup_{q\in B}\ d_\emptyset\,(p,\ B) = sup_{q\in B} d_\emptyset\,(20, B) = d_\emptyset\,(20, 50) = 30^2 = 900$.
Similarly, $sup_{p\in A}\ d_\emptyset\,(q,\ A) = sup_{p\in A} d_\emptyset\,(32, A) = d_\emptyset\,(32, 0) = 32^2$ =1024.So, $h_\emptyset\,(A,\ B) = 1024$.
Throughout this article $\emptyset$ is symmetric, bounded above,

$$\emptyset\,(p, B) = inf_{q\in B}\emptyset(p, q)$$

and

$$\emptyset\,(A, B) = sup_{p\in A}\emptyset(p, A).$$

From Definitions 1.1 and 1.4, directly getting the following:

Lemma 1.5. *For all $A,\ B,\ C,\ D \in \mathcal{H}\,(\Omega)$, the following hold:*

1. $d_\emptyset\,(p,\ B) = 0$ iff $p \in B$.
2. $d_\emptyset\,(A,\ B) = 0\ iff\, A \subseteq B$.
3. $\exists a \in A \ni d\,(p, A) = d(p, a)$
4. *If $B \subseteq C$, then $sup_{a\in A}\ d_\emptyset\,(a,\ C) \le sup_{a\in A}\ d_\emptyset\,(a,\ B)$*
5. $sup_{p\in A\bigcup B}\ d_\emptyset\,(p,\ C) = max\{sup_{a\in A}\ d_\emptyset\,(a,\ C)\,,\ sup_{b\in B}\ d_\emptyset\,(b,\ C)$
6. $h_\emptyset\,(A\ \cup\ B, C\ \cup\ D) \le max\,\{h_\emptyset\,(A, C)\,, h_\emptyset\,(B, D)\}$.
7. *$d_\emptyset(p, B) \le h_\emptyset(A, B)$, for all $p \in A$.*
8. *$d_\emptyset\,(p, A) \le \emptyset(p, A)\ (d_\emptyset\,(p,\ q) +\ d_\emptyset\,(q,\ A))$, where $q \in \Omega$.*
9. *For $r > 1$ and $a \in A$, $\exists\ \ b \in B \ni d_\emptyset\,(a, b) \le r$ implies that $h_\emptyset(A.\ B) \le r$.*
10. *For $r > 1$ and $a \in A, \exists b \in B \ni d_\emptyset(a, b) \le r$ implies that $h_\emptyset(A.B) \le r$.*
11. *$d_\emptyset(p, A) = 0$ if and only if $p \in \overline{A}$, where $\overline{A}$ is the closure of A.*

2. The Fractals space $(\mathcal{H}(\Omega), h_\emptyset)$

We will benefit from the work that appeared in [15] and [16] in the following:

Proposition 2.1. *$h_\emptyset$ is $\emptyset$- metric on $\mathcal{H}(\Omega)$.*

Proof. We prove the conditions (i-iii) in Definition 1.1 are satisfied. Since $sup_{a\in A}\ d_\emptyset(a,\ B)\geq 0$, $sup_{b\in B}\ d_\emptyset(b,\ A)\geq 0$, then $h_\emptyset(A,\ B)=max\{sup_{a\in A}\ d_\emptyset(a,\ B),\ sup_{b\in B}\ d_\emptyset(b,\ A)\}\geq 0\ \forall\ A,\ B\in\ \mathcal{H}(\Omega)$.
For (i), suppose $h_\emptyset(A,\ B)=0$ this means
$sup_{a\in A}\ d_\emptyset(a,\ B)=sup_{b\in B}\ d_\emptyset(b,\ A)=0$ by Lemma 1.5, we see $A\subseteq B,\ B\subseteq A$, so $A=B$.
Now, suppose $A=B\implies A\subseteq B,\ B\subseteq A$ by Lemma 1.5 we find $sup_{a\in A}\ d_\emptyset(a,\ B)=0$ and $sup_{b\in B}\ d_\emptyset(b,\ A)=0\implies h_\emptyset(A,\ B)=0$.
The (ii) is proved from the symmetry of Definition 1.1

$$h_\emptyset(A,\ B)=max\{sup_{a\in A}\ d_\emptyset(a,\ B),\ sup_{b\in B}\ d_\emptyset(b,\ A)\}$$
$$=max\{sup_{b\in B}\ d_\emptyset(b,\ A),\ sup_{a\in A}\ d_\emptyset(a,\ B)\}$$
$$=h_\emptyset(A,\ B).$$

The final property (iii) is proved from Definition 1.4 and Lemma 1.5. Let $a'\in A, c'\in C\ni d_\emptyset(a',c')=d_\diamond(a',C)$
Now, $d_\emptyset(a',B)=inf_{b\in B}d_\emptyset(a',b')$

$$\leq inf_{b\in B}[\emptyset(a',b')[d_\emptyset(a',c')+d_\emptyset(c',b')]]$$
$$=inf_{b\in B}\emptyset(a',b')\,d_\emptyset(a',c')+inf_{b\in B}\emptyset(a',b')d_\emptyset(c',b')$$
$$=[\emptyset(a',B)[d_\emptyset(a',c')+d_\emptyset(c',B)]$$
$$\leq\emptyset(a',B)[d_\emptyset(a',C)+d_\emptyset(c',B)]$$

Take a sup over a', we get

$$d_\emptyset(A,B)\leq\emptyset(A,B)[d_\emptyset(A,C)+d_\emptyset(C,B)]$$

$$\leq\emptyset(A,B)[\max(d_\emptyset(A,C),d_\emptyset(C,A))+\max(d_\emptyset(C,B),d_\emptyset(B,C))]$$

$$=\emptyset(A,B)[h_\emptyset(A,C)+h_\emptyset(C,B)]$$

Similarly, $d_\emptyset(B,A)=\emptyset(B,A)[h_\emptyset(C,B)+h_\emptyset(A,C)]$
Therefore, $h_\emptyset(A,B)\leq h_\emptyset(A,C)+h_\emptyset(C,B)$.
Now, let $A\in\mathcal{H}(\Omega)$ and $\epsilon>0$ define the set $A+\epsilon=\{p\in\Omega,\ d_\emptyset(p,\ A)<\epsilon\}$ □

Proposition 2.2. *Let Ω be a $\emptyset$- metric with continuous metric $d_\emptyset$ then:*

1. $A+\epsilon$ closed set if $A\in\mathcal{H}(\Omega)$,
2. $h_\emptyset(A,\ B)\ \leq\epsilon\iff A\subset B+\epsilon$ and $B\subset\ A+\epsilon$, for any $A,\ B$ in $\mathcal{H}(\Omega)$.

Proof. For (i), let $p\in\overline{A+\epsilon}$ (closure of $A+\epsilon$). Then, $\exists\ \{p_n\}\subset(A+\epsilon)\diagdown\{p\},\quad p_n\longrightarrow p$.
So $d_\emptyset(p_n,A)\leq\epsilon,\quad\forall n$
By Lemma (1.5-iii), $\forall n,\ \exists a_n\in A\ \ni d_\emptyset(a_n,A)=d_\emptyset(p_n,a_n)$. Thus, $d_\emptyset(p_n,a_n)\leq\epsilon,\ \forall n$. By compactness of A, there is a subsequence $\{a_{n_k}\}$ converges to $a\in A$. Since $p_n\longrightarrow p$ then the subsequence $\{p_{n_k}\}$ converge to p.
by continuity of $d_\emptyset$, we get $d_\emptyset(p_{n_k},a_{n_k})\longrightarrow d_\emptyset(p,a)$ and $d_\emptyset(p_{n_k},a_{n_k})\ \leq\epsilon,\ \forall k$.
This implies that $d_\emptyset(p,a)\leq\epsilon$. Then $d_\emptyset(p,A)\leq\epsilon$. So, $p\in A+\epsilon$ which means that $A+\epsilon$ is closed set.
For (ii), it is sufficient to prove that $d_\emptyset(B,\ A)<\epsilon\Leftrightarrow B\subseteq A+\epsilon$ and by symmetry $d_\emptyset(A,\ B)\leq\epsilon\Leftrightarrow A\subseteq B+\epsilon$ hold. Suppose $B\subseteq A+\epsilon\Leftrightarrow\forall q\in B, d_\emptyset(q,\ A)\leq\epsilon$ (by definition of $+\epsilon$), so, $supd_\emptyset(p,B)<\in\Leftrightarrow d_\emptyset(A,B)<\in$.

□

Proposition 2.3. *Let Ω be a $\emptyset$ - metric w.r.t. continuity of $d_{\emptyset}, \{A_{n_k}\}$ be a subsequence of a Cauchy sequence $\{A_n\} \subset \mathcal{H}(\Omega)$ and $\{p'_{n_k}\}$ be a sequence in $\Omega \ni p'_{n_k} \in A_{n_k}, \forall k$. Then, there is a Cauchy sequence $\{q'_n\} \subset \Omega \ni \in A_n \forall n$ and $q'_{n_k} = p'_{n_k}, \forall k$.*

Proof. Fix $n_0, q'_n = 0,\ \forall n,\ \ n_{k-1} < n < n_k$, we use Lemma (1.5-iii) to have $q'_n \in A_n \ni d_{\emptyset}\left(p'_{n_k},\ A_n\right) = d_{\emptyset}\left(p'_{n_k},\ q'_n\right)$.
Then, by definition of $\emptyset$-Hausdorff distance, we find $d_{\emptyset}\left(p'_{n_k},\ q'_n\right) \le h_{\emptyset}\left(A_{n_k},\ A_n\right)$. Since $p'_{n_k} \in A_{n_k}$ then $d_{\emptyset}\left(p'_{n_k},\ q'_{n_k}\right) = d_{\emptyset}\left(p'_{n_k},\ A_{n_k}\right) = 0$. It follows that $q'_{n_k} = p'_{n_k},\ \forall k$. Let $\epsilon_1 > 0$, since $\{p'_{n_k}\}$ is a Cauchy sequence $\Longrightarrow \exists J \in \mathbb{N} \ni d\left(p'_{n_k},\ p'_{n_j}\right) < \epsilon_1$, $\forall k, j > J$. Since $\{A_n\}$ is a Cauchy sequence $\Longrightarrow \exists L > n_k \ni h_{\emptyset}\left(\ A_n, A_m\right) \le \epsilon_1$, $\forall n, m > L$.
Suppose that $n, m > L \Longrightarrow \exists k, j > J\ \ni n_{k-1} < n < n_k$ and $n_{j-1} < m \le n_j$, so, by Definition 1.1, definition of $h_{\emptyset}$ and boundness of $\emptyset$ getting that

$$d_{\emptyset}\left(q'_n, q'_m\right) \le \emptyset\left(q'_n, q'_m\right)\left[d_{\emptyset}\left(q'_n, p'_{n_k}\right) + d_{\emptyset}\left(p'_{n_k}, q'_m\right)\right]$$

$$\le \emptyset\left(q'_n, q'_m\right)\left[d_{\emptyset}\left(q'_n, p'_{n_k}\right) + \emptyset\left(p'_{n_k}, q'_m\right)\left[d_{\emptyset}\left(p'_{n_k}, p'_{n_j}\right) + d_{\emptyset}\left(p'_{n_j}, q'_m\right)\right]\right]$$

$$= \emptyset\left(q'_n, q'_m\right) d_{\emptyset}\left(p'_{n_k}, A_n\right) + \emptyset\left(q'_n, q'_m\right)\emptyset\left(p'_{n_k}, q'_m\right)\left[d_{\emptyset}\left(p'_{n_k}, p'_{n_j}\right) + d_{\emptyset}\left(p'_{n_j}, A_m\right)\ \right]$$

$$\le \emptyset\left(q'_n, q'_m\right) h_{\emptyset}\left(A_{n_k},\ A_n\right) + \emptyset\left(q'_n, q'_m\right)\emptyset\left(p'_{n_k}, q'_m\right)\left[d_{\emptyset}\left(p'_{n_k}, p'_{n_j}\right) + h_{\emptyset}\left(A_{n_j}, A_m\right)\ \right]$$

$\le M\epsilon_1 + M^2[\epsilon_1 + \epsilon_1]$, as $n, m, k \longrightarrow \infty$, and $M$is positive bound of $\emptyset$

$$(2M^2 + M)\epsilon_1 = \epsilon$$

This completes the proof. □

Proposition 2.4. *Let Ω, $\emptyset$ and $d_{\emptyset}$ in Proposition 2.3 and $\{A_n\}$ be a sequence in $\mathcal{H}(\Omega)$ and $A = \{p \in \Omega : \exists\{p_n\}$ converges to p and $p_n \in A_n,\ \forall n\}$. If $\{A_n\}$is Cauchy sequence, then $\emptyset \neq A$ closed.*

Proof. to prove $A \neq \emptyset$. Since $\{A_n\}$ is a Cauchy sequence, $\exists n_k\ \forall n \in \mathbb{N} \ni h_a\left(A_m, A_n\right) < \frac{1}{2}\ \ \forall m, n$. Let $p_{n_1} \in A_{n_1}$.
since h_a is $\emptyset$- metric then, we can choose $p_{n_2} \in A_{n_2}\ \ni d_{\emptyset}\left(p_{n_1}, p_{n_2}\right) = d_{\emptyset}\left(p_{n_1}, A_{n_2}\right)$ then $d_{\emptyset}\left(p_{n_1}, p_{n_2}\right) \le d_{\emptyset}\left(A_{n_1}, A_{n_2}\right) \le h_{\emptyset}\left(A_{n_1}, A_{n_2}\right) < \frac{1}{2}$. Continuing in this way, we get $\{p_{n_k}\} \ni\ \ p_{n_k} \in A_{n_k}\ \forall k$ and $d_{\emptyset}\left(p_{n_k}, p_{n_{k+1}}\right) \le h_{\emptyset}\left(A_{n_k}, A_{n_{k+1}}\right) < \frac{1}{2^k}$. This implies that $\{p_{n_k}\}$ is Cauchy sequence and $p_{n_k} \in A_{n_k}, \forall k$ by Proposition 2.3 $\exists\{q_n\} \in \Omega,\ \{q_n\}$ Cauchy sequence $\ni q_n \in A_n$ and $q_{n_k} = p_{n_k}$, $\forall k$. By completeness of $\Omega,\ q_n \longrightarrow q \longrightarrow \Omega$.Since $q_n \in A_n,\ \forall n$, then by definition of A, $q \in A$, i.e. $A \neq \emptyset$. To prove A is closed, we can follow the first part of the Proposition 2.2 and then using Proposition 2.3. □

Now, to prove that $A \in \mathcal{H}(\Omega)$, it only remains to prove that A is totally bounded. To get those results, the following proposition is required .

Proposition 2.5. *Let Ω, $\emptyset$ and $d_{\emptyset}$ in Proposition 2.3 and $\{C_n\}$ be a sequence of totally bounded sets in Ω and $C \subseteq \Omega$. If $\forall \epsilon > 0,\ \exists k \in \mathbb{N} \ni C \subseteq C_k + \epsilon$. Then C is totally bounded.*

Proof. Let $\epsilon > 0$. Choose $\in \mathbb{N} \ni C \subseteq C_k + \frac{\epsilon}{4}$. Since C_k is totally bounded $\Longrightarrow$ there is a finite set $\{p_i : 1 \le i \le L\} \ni p_i \in C_k$ and $C_k \subseteq \bigcup_{i=1}^{L} B(p_i, \frac{\epsilon}{4})$. By rearrangement of p_i's such that $B(p_i,\ \frac{\epsilon}{2}) \cap C \neq \emptyset$ for $1 \le i \le L$ and $B\left(p_i,\ \frac{\epsilon}{2}\right) \cap C = \emptyset$ for $L < i$. Then $\forall i,\ 1 \le i \le L$, let $q_i \in B(p_i,\ \frac{\epsilon}{2}) \cap C$. We claim that $C \subseteq \bigcup_{i=1}^{L} B\left(q_i, \epsilon\right)\ \ \ni$Let $a \in C$, then $a \in C_k + \frac{\epsilon}{4} \Longrightarrow d_{\emptyset}(a, C_k) \le \frac{\epsilon}{4}$ By Lemma (1.5-iii) $\exists p\ \in C_k\ \ni d_{\emptyset}\left(a, p\right) = d_{\emptyset}\left(a, C_k\right)$. Then finding that $d_{\emptyset}\left(a, p_i\right) \le \emptyset\left(a, p_i\right)\left[d_{\emptyset}\left(a, p\right) + d_{\emptyset}\left(p, p_i\right)\right] < M[\frac{\epsilon}{4} + \frac{\epsilon}{2} < \epsilon \Longrightarrow a \in B\left(p_i,\ \epsilon\right)$, for some $1 \le i \le L$. Since $q_i \in B(p_i,\ \frac{\epsilon}{2}) \cap C$ we have $d_{\emptyset}\left(a, q_i\right) \le$

$\emptyset(a,q_i)\left[d_{\emptyset}(a,p_i)+d_{\emptyset}(p_i,q_i)\right]\leq M\left[\epsilon+\frac{\epsilon}{2}\right]<\epsilon\Rightarrow a\in B(q_i,\epsilon)$ then it follows that $C\subseteq\bigcup_{i=1}^{L}B(q_i,\epsilon)$, so, C is totally bounded.

□

Now, we can give the main results:

Theorem 2.6. *Let Ω, $\emptyset$ and $d_{\emptyset}$ as in Proposition 2.3. If $(\Omega,d_{\emptyset})$ complete then $(\mathcal{H}(\Omega),h_{\emptyset})$ is complete.*

Proof. Let $\{A_n\}$ be a Cauchy sequence in $\mathcal{H}(\Omega)$ and $A=\{p\in\ \Omega:\exists\{p_n\}$ converges to p and $p_n\in A_n,\ \forall n\}$. We want to prove $A\in\mathcal{H}(\Omega)$ and $\{A_n\}$ converges to A. By proposition 2.4, $A\neq\emptyset$ and nonempty. Let $\epsilon>0$, since $\{A_n\}$ is Cauchy sequence then $\exists r>0\ \ni d_{\emptyset}(A_n,A_m)<\epsilon\ \forall\ n,m\geq r$. By Proposition 2.2, we get $A_m\subseteq A_n+\epsilon,\ \ \forall\ m>n\geq r$. Let $a\in A$, we want to prove $a\in A_n+\epsilon$, fix $n\geq r$, since A is the set of all points $p\in\ \Omega$ and $\{p_n\}\longrightarrow p$, $p_n\in A_n$ then $\exists\{p_i\}$ s.t. $p_i\in A_i\ \forall i\Longrightarrow\{p_i\}\longrightarrow a$. By Proposition2.4, $A_n+\epsilon$ is closed, since $p_i\in A_n+\epsilon\ \ \ \forall i\Longrightarrow\ \ a\in A_n+\epsilon$ this mean $A\subseteq A_n+\epsilon$. By Proposition 2.5, A is totally bounded, A is complete since it is closed subset of a complete space, $A\neq\emptyset\Longrightarrow A$ is compact and $A\in\mathcal{H}(\Omega)$. Let $\epsilon>0$, to show that $\{A_n\}$ converges to $A\in\mathcal{H}(\Omega)$, we must prove $\exists r>0\ni d_{\emptyset}(A_n,A)<\epsilon,\ \forall\ n\geq r$, $A\subseteq A_n+\epsilon$ and $A_n\subseteq A+\epsilon$ by Proposition 2.2. From the first part of our proof, $\exists r$ s.t. $A\subseteq A_n+\epsilon,\ \forall\ n\geq r$
To prove $A_n\subseteq A+\epsilon$, let $\epsilon>0$. Since $\{A_n\}$ Cauchy sequence, we can choose $r>0\ni d_{\emptyset}(A_m,A_n)<\frac{\epsilon}{2M}\ \ \forall m,n\geq r$ and $\exists\ \{n_i\}$ be a strictly increasing sequence of positive integers s.t. $n_1>r$, $d_{\emptyset}(A_m,A_n)<\epsilon 2^{-i-1}\ \ \forall m,n>n_i$. Now, we can use Lemma (1.5-iii) to get the following:
Since $A_n\subseteq A_{n_1}+\frac{\epsilon}{2M}$, $\exists p_{n_1}\in A_{n_1}\ni d_{\emptyset}(q,p_{n_1})\leq\frac{\epsilon}{2M}$.
Since $A_{n_1}\subseteq A_{n_2}+\frac{\epsilon}{4M^2}$, $\exists p_{n_2}\in A_{n_2}\ni d_{\emptyset}(p_{n_1},p_{n_2})\leq\frac{\epsilon}{4M^2}$.
Since $A_{n_2}\subseteq A_{n_3}+\frac{\epsilon}{8M^3}$, $\exists p_{n_3}\in A_{n_3}\ni d_{\emptyset}(p_{n_2},p_{n_3})\leq\frac{\epsilon}{8M^3}$.
By continuing this way, we have a sequence $\{p_{n_i}\}$, $\forall i>0$ then $p_{n_i}\in A_{n_i}$ and $d_{\emptyset}\left(p_{n_i},p_{n_{i+1}}\right)\leq\frac{\epsilon}{2^{i+1}M^{i+1}}$. Then $\{p_{n_i}\}$ is a Cauchy sequence, so by Proposition 2.3 the limit of the sequence a is in A. Also,

$$
\begin{aligned}
d_{\emptyset}(q,p_{n_i}) &\leq \emptyset(q,p_{n_i})\left[d_{\emptyset}(q,p_{n_1})+\ d_{\emptyset}(p_{n_1},p_{n_i})\right]\\
&\leq M\left[\frac{\epsilon}{2M}+d_{\emptyset}(p_{n_1},p_{n_i})\right]\\
&\leq\frac{\epsilon}{2}+M\emptyset(q,p_{n_i})\left[d_{\emptyset}(p_{n_1},p_{n_2})+\ d_{\emptyset}(p_{n_2},p_{n_i})\right]\\
&\leq\frac{\epsilon}{2}+M^2\left[\frac{\epsilon}{4M^2}+\ d_{\emptyset}(p_{n_2},p_{n_i})\right]\\
&\ \ .\\
&\ \ .\\
&\leq\frac{\epsilon}{2}+\ \frac{\epsilon}{4}+\cdots+\frac{\epsilon}{2^i}=\epsilon
\end{aligned}
$$

Since, $d_{\emptyset}\left(q,p_{n_i}\right)\leq\epsilon\ \ \forall i$, it follows that $d_{\emptyset}(q,a)\leq\epsilon$ and therefore $q\in A+\epsilon$. Thus we know that there exists $r\ni A_n\subseteq A+\epsilon$, so it follows that $d_{\emptyset}(A_n,A)<\epsilon\ \forall n\geq r$ and thus $\{A_n\}\longrightarrow A\in\mathcal{H}(\Omega)$. Therefore, $(\Omega,d_{\emptyset})$ is complete, then $(\mathcal{H}(\Omega),h_{\emptyset})$ is complete. □

3. IFS for F-contraction mappings

Recall the following collection F, $\digamma:[0,\infty)\longrightarrow(\infty,-\infty)$, $\digamma\in$ F if
$\forall\ \gamma,\ \ \delta\ \in[0,\infty)\ni\ \gamma<\delta\Longrightarrow\digamma(\gamma)<\digamma(\delta)$. ...(1a)
$\forall\ \{\gamma_n\}\subset(0,\infty)$, $\lim_{n\to\infty}\digamma(\gamma_n)=0\iff-\infty$. ...(1b)
$\exists r\in(0,1)\ni\lim_{\gamma\to0^+}\digamma(\gamma)=0$ . ...(1c)
Let $\boldsymbol{\mu}$=$\{\mu:\mathbb{R}_+\longrightarrow\mathbb{R}_+;\ liminf_{t\to0}\ \ \mu(t)>0\ \forall\ t\ \geq\ 0\}$. Now we defined generalized $\digamma$-contractive (or $\boldsymbol{\mu}$-contractive):

Definition 3.1. *A self-mapping f on Ω is called $\boldsymbol{\mu}$-contractive, $\forall\ p,q\in\Omega,\ \exists F\in F,\ \exists\mu\in\boldsymbol{\mu}\ni$*
$\mu(d_\emptyset(p,q))+F(d_\emptyset(fp,fq))\le F\ (d_\emptyset(p,q))\ \ldots 2$
where $d_\emptyset(fp,fq)>0$.
Especially, if $\boldsymbol{\mu}$ is a single constant element μ then (1.1) will be
$\mu+F(d_\emptyset(fp,fq))\le F\ (d_\emptyset(p,q))\ \ldots 3$
and called $\boldsymbol{\mu}$-contractive.

Note: From (1a) and Definition (3.1), every $\boldsymbol{\mu}$-contractive is contractive and hence is continuous.

Theorem 3.2. *Let $f:\Omega\longrightarrow\Omega$ be $\boldsymbol{\mu}$ -contractive. Then $f:\ \mathcal{H}(\Omega)\longrightarrow\mathcal{H}(\Omega)$ is $\boldsymbol{\mu}$ –contractive when $f(C')=\{f(p):p\in C'\}$ for any $C'\in\mathcal{H}(\Omega)$.*

Proof. : Firstly, by the continuity of f and compactness of C, we get $f(C)\in\mathcal{H}(\Omega)$.
Now, $C',\ D\in\mathcal{H}(\Omega)$ and $h_\emptyset(f(C'),f(D))\neq 0$. Then we have

$$d_\emptyset(fp,f(D))=inf_{q\in D}\ d_\emptyset(fp,fq)<inf_{q\in D}\ d_\emptyset(p,q)=d_\emptyset(p,D)$$

In addition,

$$d_\emptyset(fp,f(C'))=inf_{p\in C}\ d_\emptyset(fp,fq)<inf_{p\in C}\ d_\emptyset(q,p)=d_\emptyset(q,C')$$

Now,

$$h_\emptyset(f(C'),f(D))=\max\{sup_{p\in C}\ d_\emptyset(fp,f(D)),sup_{q\in D}\ d_\emptyset(fp,f(C'))\}$$
$$<\left\{sup_{p\in C}\ d_\emptyset(p,D),sup_{q\in\mathcal{B}}\ d_\emptyset(q,f(C'))\right\}\ =\ h_\emptyset(C',D)$$

By (1a), we get

$$F(h_\emptyset(f(C'),f(D)))<F(h_\emptyset(C',\ D))$$

So, $\exists\ \mu:\mathbb{R}_+\longrightarrow\mathbb{R}_+\ ,\ liminf_{t\to 0}\ \mu(t)>0\ \forall\ t\ \ge 0\ \ni\ \mu(h_\emptyset(C',D))+F(h_\emptyset(f(C'),f(D)))\le F(h_\emptyset(C',D))$
Therefore, $f:\ \mathcal{H}(\Omega)\longrightarrow\mathcal{H}(\Omega)$ is $\boldsymbol{\mu}$ -contractive.

□

Definition 3.3. *Let $\{f_n:n=1,2,\ldots,k\},\ k\in\mathbb{N}$ is MF -contractive Ω and $\mathcal{T}:\mathcal{H}(\Omega)\longrightarrow\mathcal{H}(\Omega)$ by $\mathcal{T}(E)=\bigcup_{n=1}^{k}f_n(\ E),\ \forall\ E\in\mathcal{H}(\Omega)$ then $\mathcal{T}$ is called $h_\emptyset$- Hutchison- Barnsley operator on $\mathcal{H}(\Omega)$.*

Theorem 3.4. *The $h_\emptyset-HB$operator (shortly, $h_\emptyset-HB$) is $\boldsymbol{\mu}$ –contractive.*

Proof. We claim that $k=2$. Let f_1,f_2 be two F- contraction defined as $f_1,f_2:\ \Omega\longrightarrow\Omega$, let $C',D\in\mathcal{H}(\Omega)$ with $h_\emptyset(\mathcal{T}(C'),\mathcal{T}(D))\neq 0$.From Lemma (1.5-vi), we get the following

$$\mu(h_\emptyset(C',\ D))+F(h_\emptyset(\mathcal{T}(C'),\mathcal{T}(D)))=\mu(h_\emptyset(C',\ D))+\ F(h_\emptyset(f_1(C')\cup f_1(C'),(f_2(C')\cup f_2(C')))\le$$
$$\mu(h_\emptyset(C',\ D))+\ F(\max\{h_\emptyset(f_1(C'),f_1(D)),h_\emptyset(f_2(C'),f_2(D))\})\ \le F(h_\emptyset(C',\ D)).$$

As a consequence, $\mathcal{T}$ is $h_\emptyset-HB$operator. □

In the following, the generalization of condition (2) is presented.

Definition 3.5. *Let $\mathcal{T}:\ \mathcal{H}(\Omega)\longrightarrow\mathcal{H}(\Omega)$ is $\boldsymbol{\mu}$ -contractive if $F\in F,\ \mu\in\boldsymbol{\mu},\ C',D\in\mathcal{H}(\Omega)$, $h_\emptyset(\mathcal{T}(C'),\mathcal{T}(D))\neq 0$,*
$\mu(M_{\mathcal{T}}(C',\ D))+\ F(h_\emptyset(\mathcal{T}(C'),\mathcal{T}(D)))\le F(M_{\mathcal{T}}(C',\ D))\ \ldots(4)$
where

$$M_{\mathcal{T}}(C',\ D)=\max\{h_\emptyset(C',\ D)\ ,h_\emptyset(C',\mathcal{T}(C')),h_\emptyset(D,\mathcal{T}(D)),\frac{h_\emptyset((C',\ \mathcal{T}(D)+h_\emptyset(D,\ \mathcal{T}(C'))}{2\emptyset(C',\ D)},$$
$$h_\emptyset\left(\mathcal{T}^2(C'),\mathcal{T}(C')\right),h_\emptyset\left(\mathcal{T}^2(C'),D\right),h_\emptyset(\mathcal{T}^2(C'),\mathcal{T}(D))\}.$$

Then $\mathcal{T}$ is called Ciric type $\boldsymbol{\mu}$ -contractive

Theorem 3.6. *Let* $\{f_n : n = 1, 2, \ldots, k\}$ *is* Ciric type $\boldsymbol{\mu}$ −contractive Ω *and* $\mathcal{T} : \mathcal{H}(\Omega) \longrightarrow \mathcal{H}(\Omega)$ *by*

$$\mathcal{T}(\mathcal{A}) = \bigcup_{n=1}^{k} f_n(\mathcal{A}), \qquad \forall\, \mathcal{A} \in \mathcal{H}(\Omega).$$

Then $\mathcal{T}$ *is a Ciric type* $\boldsymbol{\mu}$ *-contractive on* $\mathcal{H}(\Omega)$.

Proof. Using Definition (3.5) and (1a), we get $\mathcal{T}$ is a Ciric type $\boldsymbol{\mu}$ −contractive. □

An important result is the following:

Theorem 3.7. *Let* $\mathcal{T}$ *is* $h_\emptyset$*-* $H - B$ *operator on* $\mathcal{H}(\Omega)$ *w.r.t. family* $\{f_i\}_{i=1}^{n}$ *then*

1. $\mathcal{T} : \mathcal{H}(\Omega) \longrightarrow \mathcal{H}(\Omega)$ *is be* $\mathcal{T}(\mathcal{A}) = \bigcup_{n=1}^{k} f_n(\mathcal{A}),\ \forall\, \mathcal{A} \in \mathcal{H}(\Omega)$ *is Ciric* $\boldsymbol{\mu}$*-contractive.*
2. $\mathcal{T}$ *has a unique fixed point* $\mathcal{V} \in\ \mathcal{H}(\Omega) \ni \mathcal{V} =\ \mathcal{T}(\mathcal{V}) = \bigcup_{n=1}^{k} f_n(\mathcal{V})$
3. $\mathcal{A}_0$ *is initial set, the sequence* $\mathcal{A}_0 \in \mathcal{H}(\Omega)$, $\{\mathcal{A}_0, \mathcal{T}(\mathcal{A}_0), \mathcal{T}^2(\mathcal{A}_0), \ldots\}$ *of compact sets converges to a fixed point of* $\mathcal{T}$.

Proof. Part (i) follows from Theorem (3.6). For parts (ii) and (iii), let $\mathcal{A}_0 \in \mathcal{H}(\Omega)$, if $\mathcal{A}_0 = \mathcal{T}(\mathcal{A}_0)$ the proof is complete. Now, suppose that $\mathcal{A}_0 \neq \mathcal{T}(\mathcal{A}_0)$, let $\mathcal{A}_1 = \mathcal{T}(\mathcal{A}_0), \mathcal{A}_2 = \mathcal{T}(\mathcal{A}_1), \ldots\ \mathcal{A}_{m+1} = \mathcal{T}(\mathcal{A}_m)$ for $m \in \mathbb{N}$ and $\mathcal{A}_0 \neq \mathcal{T}(\mathcal{A}_0)$. Let $\mathcal{A}_1 = \mathcal{T}(\mathcal{A}_0),\ \mathcal{A}_2 = \mathcal{T}(\mathcal{A}_1),\ \ldots,\ \mathcal{A}_{m+1} = \mathcal{T}(\mathcal{A}_m)$ for $m \in \mathbb{N}$. If $\mathcal{A}_k = \mathcal{T}(\mathcal{A}_{k+1})$ for some k. So, the proof is also complete. Now, we take $\mathcal{A}_m \neq \mathcal{T}(\mathcal{A}_{m+1})$, $\forall\, m \in \mathbb{N}$. Form (2), we get

$$\mu(M_\mu(\mathcal{A}_m,\ \mathcal{A}_{m+1})) +\ F(h_\emptyset(\mathcal{A}_{m+1}, \mathcal{A}_{m+2})) = \mu(M_\mu(\mathcal{A}_m,\ \mathcal{A}_{m+1})) +\ F(h_\emptyset(\mathcal{A}_m, \mathcal{A}_{m+1}))$$
$$\leq F(M_\mu(\mathcal{A}_m,\ \mathcal{A}_{m+1}))$$

where

$$M_\mu(\mathcal{A}_m,\ \mathcal{A}_{m+1}) = \max\{h_\emptyset(\mathcal{A}_m,\ \mathcal{A}_{m+1})\ , h_\emptyset(\mathcal{A}_m,\ \mathcal{T}(\mathcal{A}_m)), h_\emptyset(\mathcal{A}_{m+1},\ \mathcal{T}(\mathcal{A}_{m+1})),$$
$$\frac{h_\emptyset((\mathcal{A}_m,\ \mathcal{T}(\mathcal{A}_{m+1}) + h_\emptyset(\mathcal{A}_{m+1},\ \mathcal{T}(\mathcal{A}_m))}{2\emptyset(\mathcal{A}_m,\ \mathcal{A}_{m+1})},$$
$$h_\emptyset\left(\mathcal{T}^2(\mathcal{A}_m), \mathcal{T}(\mathcal{A}_m)\right), h_\emptyset\left(\mathcal{T}^2(\mathcal{A}_m), \mathcal{A}_{m+1}\right), h_\emptyset(\mathcal{T}^2(\mathcal{A}_m), \mathcal{T}(\mathcal{A}_{m+1}))\}$$
$$= \max\{h_\emptyset(\mathcal{A}_m,\ \mathcal{A}_{m+1})\ , h_\emptyset(\mathcal{A}_m,\ \mathcal{A}_{m+1}), h_\emptyset(\mathcal{A}_{m+1},\ \mathcal{A}_{m+2})\},\ \frac{h_\emptyset(\mathcal{A}_{m+1},\ \mathcal{A}_{m+2}) + h_\emptyset(\mathcal{A}_{m+1},\ \mathcal{A}_{m+1})}{2\emptyset(\mathcal{A}_m,\ \mathcal{A}_{m+1})},$$
$$h_\emptyset(\mathcal{A}_{m+2},\ \mathcal{A}_{m+1}), h_\emptyset(\mathcal{A}_{m+2},\ \mathcal{A}_{m+1}), h_\emptyset(\mathcal{A}_{m+2},\ \mathcal{A}_{m+2})$$
$$= \max\{h_\emptyset(\mathcal{A}_m,\ \mathcal{A}_{m+1}),\ h_\emptyset(\mathcal{A}_{m+1},\ \mathcal{A}_{m+2})\}$$

In case,
$M_\mu(\mathcal{A}_m,\ \mathcal{A}_{m+1}) = h_\emptyset(\mathcal{A}_{m+1},\ \mathcal{A}_{m+2})$, we get

$$F(h_\emptyset(\mathcal{A}_{m+1}, \mathcal{A}_{m+2})) \leq F(h_\emptyset(\mathcal{A}_{m+1}, \mathcal{A}_{m+2})) - \mu h_\emptyset(\mathcal{A}_{m+1}, \mathcal{A}_{m+2})$$

a contradiction as $\mu h_\emptyset(\mathcal{A}_{m+1}, \mathcal{A}_{m+2}) > 0$
Therefore, $M_\mu(\mathcal{A}_m,\ \mathcal{A}_{m+1}) = h_\emptyset(\mathcal{A}_m, \mathcal{A}_{m+1})$, hence

$$F(h_\emptyset(\mathcal{A}_{m+1}, \mathcal{A}_{m+2})) \leq F(h_\emptyset(\mathcal{A}_m, \mathcal{A}_{m+1})) - \mu h_\emptyset(\mathcal{A}_m, \mathcal{A}_{m+1}) < F(h_\emptyset(\mathcal{A}_m, \mathcal{A}_{m+1}))$$

So, $\{h_\emptyset(\mathcal{A}_{m+1}, \mathcal{A}_{m+2})\}$ is decreasing and therefore convergent
Now, we show $\lim_{m\to\infty} h_\emptyset(\mathcal{A}_{m+1}, \mathcal{A}_{m+2}) = 0$. By the property of μ, $\exists\, c > 0$ with $n_0 \in \mathbb{N}$ $\ni h_\emptyset(\mathcal{A}_m,\ \mathcal{A}_{m+1}) > c$ for $m \geq n_0$. So,

$$F(h_\emptyset(\mathcal{A}_{m+1}, \mathcal{A}_{m+2})) \leq F(h_\emptyset(\mathcal{A}_m, \mathcal{A}_{m+1})) - \mu h_\emptyset(\mathcal{A}_m, \mathcal{A}_{m+1})$$

$$\leq F\left(h_{\emptyset}\left(\mathcal{A}_{m-1}, \mathcal{A}_m\right)\right)-\mu(h_{\emptyset}\left(\mathcal{A}_{m-1}, \mathcal{A}_m\right))-\mu h_{\emptyset}\left(\mathcal{A}_m, \mathcal{A}_{m+1}\right)$$

$$\leq \ldots$$

$$\leq h_{\emptyset}\left(\mathcal{A}_0, \mathcal{A}_1\right)-\left[\mu(h_{\emptyset}\left(\mathcal{A}_0, \mathcal{A}_1\right)+\mu(h_{\emptyset}\left(\mathcal{A}_1, \mathcal{A}_2\right))+\cdots+\mu h_{\emptyset}\left(\mathcal{A}_m, \mathcal{A}_{m+1}\right)\right]$$

$$\leq F\left(h_{\emptyset}\left(\mathcal{A}_0, \mathcal{A}_1\right)\right)-n_0$$

Let $\lim_{m \rightarrow \infty} F\left(h_{\emptyset}\left(\mathcal{A}_{m+1}, \mathcal{A}_{m+2}\right)\right)=-\infty$ which both with (1b) that means $\lim_{m \rightarrow \infty} h_{\emptyset}\left(\mathcal{A}_{m+1}, \mathcal{A}_{m+2}\right)=0$ now, by (1c), $\exists\, r \in(0,1)$ such that

$$\lim_{n \rightarrow \infty}\left[h_{\emptyset}\left(\mathcal{A}_{m+1}, \mathcal{A}_{m+2}\right)\right]^r F\left(h_{\emptyset}\left(\mathcal{A}_{m+1}, \mathcal{A}_{m+2}\right)\right)=0$$

Therefore,

$$\left[h_{\emptyset}\left(\mathcal{A}_m, \mathcal{A}_{m+1}\right)\right]^r F\left(h_{\emptyset}\left(\mathcal{A}_m, \mathcal{A}_{m+1}\right)\right)-\left[h_{\emptyset}\left(\mathcal{A}_m, \mathcal{A}_{m+1}\right)\right]^r F\left(h_{\emptyset}\left(\mathcal{A}_0, \mathcal{A}_1\right)\right)$$

$$\leq\left[h_{\emptyset}\left(\mathcal{A}_m, \mathcal{A}_{m+1}\right)\right]^r F\left(h_{\emptyset}\left(\mathcal{A}_0, \mathcal{A}_1\right)\right)-n_0-\left[h_{\emptyset}\left(\mathcal{A}_m, \mathcal{A}_{m+1}\right)\right]^r F\left(h_{\emptyset}\left(\mathcal{A}_0, \mathcal{A}_1\right)\right)$$

$$\leq-n_0\left[h_{\emptyset}\left(\mathcal{A}_m, \mathcal{A}_{m+1}\right)\right]^r$$

$$\leq 0$$

Since, $\lim_{m \rightarrow \infty} m\left[h_{\emptyset}\left(\mathcal{A}_{m+1}, \mathcal{A}_{m+2}\right)\right]^r=0$. So, $\lim_{m \rightarrow \infty} m^{1/r} h_{\emptyset}\left(\mathcal{A}_{m+1}, \mathcal{A}_{m+2}\right)=0$. Means that $\exists n_1 \in \mathbb{N}$ such that $m^{1/r} h_{\emptyset}\left(\mathcal{A}_{m+1}, \mathcal{A}_{m+2}\right) \leq 1 \;\; \forall\, m \geq n_1$, hence, $h_{\emptyset}\left(\mathcal{A}_{m+1}, \mathcal{A}_{m+2}\right) \leq \frac{1}{m^{1/r}} \; \forall\, m \geq n_1$.
For $m,\ n\ \in \mathbb{N}$ with $m>n \geq n_1$, we have
$h_{\emptyset}\left(\mathcal{A}_n, \mathcal{A}_m\right) \leq h_{\emptyset}\left(\mathcal{A}_n, \mathcal{A}_{n+1}\right)+h_{\emptyset}\left(\mathcal{A}_{n+1}, \mathcal{A}_{n+2}\right)+\cdots+h_{\emptyset}\left(\mathcal{A}_m, \mathcal{A}_{m+1}\right) \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}$, by the series $\sum_{i=n}^{\infty} \frac{1}{i^{1/r}}$, getting $h_{\emptyset}\left(\mathcal{A}_n, \mathcal{A}_m\right) \rightarrow 0$ as $n, m \rightarrow \infty$. Thus $\left\{\mathcal{A}_n\right\}$ is Cauchy in Ω. By completeness of $\left(h_{\emptyset}(\Omega), d_{\emptyset}\right)$, $\mathcal{A}_n \rightarrow \mathcal{V}$ as $n \rightarrow \infty$ for some $\mathcal{V} \in h_{\emptyset}(\Omega)$. Now, to show $\mathcal{V}$ is a fixed under $\mathcal{T}$, suppose that $h_{\emptyset}(\mathcal{V}, \mathcal{T}(\mathcal{V})) \neq 0$

$$\mu\left(\mathcal{M}_{\mathcal{T}}\left(\mathcal{A}_n, \mathcal{V}\right)\right)+\ F\left(h_{\emptyset}\left(\mathcal{A}_{n+1}, \mathcal{T}(\mathcal{V})\right)\right)=\mu+F(h_{\emptyset}\left(\mathcal{T}\left(\mathcal{A}_n\right), \mathcal{T}(\mathcal{V})\right)) \leq F\left(\mathcal{M}_{\mathcal{T}}\left(\mathcal{A}_n, \mathcal{V}\right)\right) .. \tag{3.1}$$

where,

$$\mathcal{M}_{\mathcal{T}}\left(\mathcal{A}_n, \mathcal{V}\right)=\max \left\{h_{\emptyset}\left(\mathcal{A}_n, \mathcal{V}\right), h_{\emptyset}\left(\mathcal{A}_n, \mathcal{T}\left(\mathcal{A}_n\right)\right), h_{\emptyset}(\mathcal{V}, \mathcal{T}(\mathcal{V}))\ , \frac{h_{\emptyset}\left(\left(\mathcal{A}_n,\ \mathcal{T}(\mathcal{V})\right)+h_{\emptyset}\left(\mathcal{V},\ \mathcal{T}(\mathcal{A}_n)\right)\right.}{2 \emptyset\left(\mathcal{A}_n,\ \mathcal{A}_{n+1}\right)},\right.$$

$$\left.h_{\emptyset}\left(\mathcal{T}^2\left(\mathcal{A}_n\right), \mathcal{T}\left(\mathcal{A}_n\right)\right), h_{\emptyset}\left(\mathcal{T}^2\left(\mathcal{A}_n\right), \mathcal{V}\right), h_{\emptyset}(\mathcal{T}^2\left(\mathcal{A}_n\right), \mathcal{T}(\mathcal{V}))\right\}$$

$$=\max \{h_{\emptyset}\left(\mathcal{A}_n,\ \mathcal{V}\right)\ , h_{\emptyset}(\mathcal{A}_n,\ \mathcal{A}_{n+1}), h_{\emptyset}(\mathcal{V}, \mathcal{T}(\mathcal{V}))\ \frac{h_{\emptyset}\left(\mathcal{A}_n,\ \mathcal{V}\right)+h_{\emptyset}\left(\mathcal{V},\ \mathcal{A}_{n+1}\right)}{2 \emptyset\left(\mathcal{A}_n,\ \mathcal{A}_{n+1}\right)},$$

$$h_{\emptyset}(\mathcal{A}_{n+2},\ \mathcal{A}_{n+1}), h_{\emptyset}(\mathcal{A}_{n+2},\ \mathcal{V}), h_{\emptyset}(\mathcal{A}_{n+2},\ \mathcal{T}(\mathcal{V}))\}$$

Now we show the following cases:

1. If $\mathcal{M}_{\mathcal{T}}\left(\mathcal{A}_n, \mathcal{V}\right)=h_{\emptyset}\left(\mathcal{A}_n, \mathcal{V}\right)$, then $n \longrightarrow \infty$ in (3.1), we obtain

$\lim_{n \rightarrow \infty} \inf \mu(h_{\emptyset}\left(\mathcal{A}_n, \mathcal{V}\right))+F(h_{\emptyset}(\mathcal{V}, \mathcal{T}(\mathcal{V})) \leq F(h_{\emptyset}(\mathcal{V}, \mathcal{V}))$. This is a contradiction as $\lim inf_{t \rightarrow 0}\ \mu(t)>0, \forall t \geq 0$.

1. In case $\mathcal{M}_{\mathcal{T}}\left(\mathcal{A}_n, \mathcal{V}\right)=h_{\emptyset}\left(\mathcal{A}_n, \mathcal{A}_{n+1}\right)$, then $n \longrightarrow \infty$, we have

$\lim_{n \rightarrow \infty} \inf \mu(h_{\emptyset}\left(\mathcal{A}_n, \mathcal{A}_{n+1}\right))+F(h_{\emptyset}(\mathcal{T}(\mathcal{V}), \mathcal{V})) \leq F(h_{\emptyset}(\mathcal{V}, \mathcal{V}))$. This is a contradiction.

1. When $\mathcal{M}_{\mathcal{T}}\left(\mathcal{A}_n, \mathcal{V}\right)=h_{\emptyset}(\mathcal{V}, \mathcal{T}(\mathcal{V}))$, then we obtain

$\mu(h_{\emptyset}(\mathcal{T}(\mathcal{V}), \mathcal{V})))+F(h_{\emptyset}(\mathcal{T}(\mathcal{V}), \mathcal{V})) \leq F(h_{\emptyset}(\mathcal{V}, \mathcal{T}(\mathcal{V})))$. Which is not true as the $\mu(h_{\emptyset}(\mathcal{V}, \mathcal{T}(\mathcal{V})))>0$

1. If $\mathcal{M}_{\mathcal{T}}(\mathcal{A}_n, \mathcal{V}) = \frac{h_\emptyset((\mathcal{A}_n,\ \mathcal{T}(\mathcal{V}))+h_\emptyset(\mathcal{V},\ \mathcal{T}(\mathcal{A}_{n+1}))}{2\emptyset(\mathcal{A}_n,\ \mathcal{A}_{n+1})}$, then $n \longrightarrow \infty$, we get

$$\lim_{n\to\infty} \inf\mu(\frac{h_\emptyset((\mathcal{A}_n,\ \mathcal{T}(\mathcal{V}))+h_\emptyset(\mathcal{V},\ \mathcal{T}(\mathcal{A}_{n+1}))}{2\emptyset(\mathcal{A}_n,\ \mathcal{A}_{n+1})}) + F(h_\emptyset(\mathcal{V}, \mathcal{T}(\mathcal{V})))$$
$$\le F(\frac{h_\emptyset((\mathcal{V},\ \mathcal{T}(\mathcal{V}))+\mathcal{H}_\emptyset(\mathcal{V},\ \mathcal{V}))}{2\emptyset(\mathcal{A}_n,\ \mathcal{A}_{n+1})}) = F(\frac{h_\emptyset((\mathcal{V},\ \mathcal{T}(\mathcal{V}))}{2\emptyset(\mathcal{A}_n,\ \mathcal{A}_{n+1})})$$

This is a contradiction (1a)

1. In case $\mathcal{M}_{\mathcal{T}}(\mathcal{A}_n, \mathcal{V}) = h_\emptyset(\mathcal{A}_{n+2}, \mathcal{A}_{n+1})$, we have

$\lim_{n\to\infty} \inf\mu(h_\emptyset(\mathcal{A}_{n+2}, \mathcal{A}_{n+1})) + F(h_\emptyset(\mathcal{T}(\mathcal{V}), \mathcal{V})) \le F(h_\emptyset(\mathcal{V}, \mathcal{V}))$ .Obtains a contradiction.

1. If $\mathcal{M}_{\mathcal{T}}(\mathcal{A}_n, \mathcal{V}) = h_\emptyset(\mathcal{A}_{n+2}, \mathcal{V})$, then $n \longrightarrow \infty$, we obtain

$\lim_{n\to\infty} \inf\mu(h_\emptyset(\mathcal{A}_{n+2}, \mathcal{V})) + F(h_\emptyset(\mathcal{T}(\mathcal{V}), \mathcal{V})) \le F(h_\emptyset(\mathcal{V}, \mathcal{V}))$. This implies a contradiction.

1. If $\mathcal{M}_{\mathcal{T}}(\mathcal{A}_n, \mathcal{V}) = h_\emptyset(\mathcal{A}_{n+2}, \mathcal{T}(\mathcal{V}))$, then $n \longrightarrow \infty$, then

$\lim_{n\to\infty} \inf\mu(h_\emptyset(\mathcal{A}_{n+2}, \mathcal{V})) + F(h_\emptyset(\mathcal{T}(\mathcal{V}), \mathcal{V})) \le F(h_\emptyset(\mathcal{V}, \mathcal{T}(\mathcal{V})))$, also a contradiction.
Consequently, $\mathcal{V}$ is invariant by $\mathcal{T}$. For uniqueness, fix $\mathcal{T}(\mathcal{V}) = \mathcal{V}$, $\mathcal{T}(\mathcal{W}) = \mathcal{W}$ where $h_\emptyset(\mathcal{V},\ \mathcal{W}) \ne 0$. Since $\mathcal{T}$ is a F-contraction, then

$$\mu(\mathcal{M}_{\mathcal{T}}(\mathcal{V}, \mathcal{W})) + F(\mathcal{H}_\emptyset(\mathcal{V},\ \mathcal{W})) = \mu(\mathcal{M}_{\mathcal{T}}(\mathcal{V}, \mathcal{W})) + F(\mathcal{H}_\emptyset(\mathcal{T}(\mathcal{V}),\ \mathcal{T}(\mathcal{W})))$$
$$\le F(\mathcal{M}_{\mathcal{T}}(\mathcal{V}, \mathcal{W}))$$

where

$$\mathcal{M}_{\mathcal{T}}(\mathcal{V}, \mathcal{W}) = \max\{h_\emptyset(\mathcal{V},\ \mathcal{W})\ ,\ h_\emptyset(\mathcal{V},\ \mathcal{T}(\mathcal{V})), h_\emptyset(\mathcal{W},\ \mathcal{T}(\mathcal{W})), \frac{h_\emptyset((\mathcal{V},\ \mathcal{T}(\mathcal{W}))+h_\emptyset(\mathcal{W},\ \mathcal{T}(\mathcal{V}))}{2\emptyset(\mathcal{A}_n,\ \mathcal{A}_{n+1})}$$
$$h_\emptyset(\mathcal{T}^2(\mathcal{V}), \mathcal{T}(\mathcal{W})), h_\emptyset(\mathcal{T}^2(\mathcal{V}), \mathcal{W}), h_\emptyset(\mathcal{T}^2(\mathcal{V}), \mathcal{T}(\mathcal{W}))\}$$
$$= \max\{h_\emptyset(\mathcal{V}, \mathcal{W}\ ), h_\emptyset(\mathcal{V},\ \mathcal{V}), h_\emptyset(\mathcal{W}, \mathcal{W})\ \frac{h_\emptyset(\mathcal{V},\ \mathcal{W})+h_\emptyset(\mathcal{W},\ \mathcal{V})}{2\emptyset(\mathcal{A}_n,\ \mathcal{A}_{n+1})},$$
$$h_\emptyset(\mathcal{V},\ \mathcal{V}), h_\emptyset(\mathcal{V},\ \mathcal{W}), h_\emptyset(\mathcal{V}, \mathcal{W})\} = h_\emptyset(\mathcal{V}, \mathcal{W})$$

that is

$$\mu(\mathcal{M}_{\mathcal{T}}(\mathcal{V}, \mathcal{W})) + F(h_\emptyset(\mathcal{V},\ \mathcal{W})) \le F(h_\emptyset(\mathcal{V},\ \mathcal{W}))$$

as $\mu h_\emptyset(\mathcal{V},\ \mathcal{W}) > 0$, it is a contradiction. So $\mathcal{T}$ has a unique fixed point $\mathcal{V} \in h_\emptyset(\Omega)$.

□

Remark 3.8. *In theorem 2.6, if $h_\emptyset(\Omega) =$ the collection of all singleton subsets of Ω and $f_n = f$ for each n, where $\boldsymbol{\mu} = f_i$ for any $i \in \{1, 2, 3, \ldots k\}$, then the mapping $\mathcal{T}$ becomes $\mathcal{T}(p) = f(p)$.*

The following is another fixed point result

Corollary 3.9. *Let $\{\Omega:\ f_n,\ n = 1, 2, 3, \ldots, k\}$ a generalized iterated function system and $f:\ \Omega \longrightarrow \Omega$ as in Remark (3.8). If there exist some $F \in F$ and $\mu \in \boldsymbol{\mu}$ ∋for any $p, q \in \mathcal{H}(\Omega)$ with $d_\emptyset(f(p), f(q)) \ne 0$ the following holds:*

$$\mu((p, q)) + F(d_\emptyset(fp, fq)) \le F(M_f(p, q)),$$

where

$$M_{\mathcal{T}}(p, q) = \max\{d_\emptyset(p, q), d_\emptyset(fp, fq), d_\emptyset(qM_f, fp), \frac{d_\emptyset(p, fp) + d_\emptyset(q, fp)}{2\emptyset},$$
$$d_\emptyset(f^2p, q), d_\emptyset(f^2p, fp,), d_\emptyset(f^2p, fq))\}$$

Then $f$has a unique fixed point in Ω. Further for initial $p_0 \in \Omega$, the sequence $\{p_0, fp_0,\ f^2p_0, \ldots\}$ approaching to a fixed point of f.

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