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On Explicit Evaluation of Ratio's of Theta Function Which is Analogous to Ramanujan's Function $a_{m,n}$

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ABSTRACT: In this article, Ramanujan defined $a_{m,n}$ [3], B. N. Dharmendra and S. Vasanth Kumar defined $E_{m,n}$ [5] for any positive real numbers m and n involving Ramanujan's product of theta-functions. We established new relation between $a_{m,n}$ and $E_{m,n}$ and explicit evaluations of $E_{m,n}$.

Key Words: Modular equation, Theta-function.

Contents

1. Introduction

The Ramanujan's general theta function [11] is defined by

$$\begin{aligned} f(a,b) :&= \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1, \\ &= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty \end{aligned} \tag{1.1}$$

where,

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q| < 1.$$

Three special cases of $f(a,b)$ are defined as follows:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.3}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} = (q;q)_\infty. \tag{1.4}$$

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On page 338 in his first notebook [11], Ramanujan defines

$$a_{m,n} = \frac{ne^{\frac{-(n-1)\pi}{4}\sqrt{\frac{m}{n}}}\psi^2(e^{-\pi\sqrt{mn}})\varphi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(e^{-\pi\sqrt{\frac{m}{n}}})\varphi^2(-e^{-2\pi\sqrt{\frac{m}{n}}})}, \tag{1.5}$$

where m and n are positive real numbers.

In [3], on pages 337 - 338, Ramanujan has listed eighteen particular values. Berndt, Chan and Zhang [4] have been established all these values. For some general theorems and explicit evaluation on $a_{m,n}$ one can refer [6,7,8,10].
Following the above definition [9], Mahadeva Naika et al. defined a new function $b_{m,n}$ and in [5], B. N. Dharmendra and S. Vasanth Kumar defined the Ramanujan theta function $E_{m,n}$. They established new properties of $b_{m,n}$ and $E_{m,n}$ and find its explicit values.

In [9], defined the theta function

$$b_{m,n} = \frac{ne^{\frac{-(n-1)\pi}{4}\sqrt{\frac{m}{n}}}\psi^2(-e^{-\pi\sqrt{mn}})\varphi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(-e^{-\pi\sqrt{\frac{m}{n}}})\varphi^2(-e^{-2\pi\sqrt{\frac{m}{n}}})}. \tag{1.6}$$

In [5], B. N. Dharmendra and S. Vasanth Kumar defined the Ramanujan theta function

$$E_{m,n} = \frac{f(e^{-\pi\sqrt{\frac{n}{m}}})\psi(-e^{-\pi\sqrt{mn}})}{e^{\frac{-\pi(1-m)}{12}\sqrt{\frac{n}{m}}}f(e^{-\pi\sqrt{mn}})\psi(-e^{-\pi\sqrt{\frac{n}{m}}})}. \tag{1.7}$$

The main purpose of this paper to be establish new relation between $a_{m,n}$ and $E_{m,n}$ and explicit evaluation of $E_{m,n}$.

2. Preliminary Results

In this section, we tend to collect many identities that square measure helpful in proving our main results.

Lemma 2.1. *[6] If m is any positive rational,*

$$a_{m,3} = \frac{3q^{1/2}\psi^2(-q^3)\varphi^2(q^3)}{\psi^2(-q)\varphi^2(q)}, \tag{2.1}$$

$$P = \frac{\psi(-q)}{q^{1/4}\psi(-q^3)} \quad \text{and} \quad Q = \frac{\varphi(q)}{\varphi(q^{1/3})}, \tag{2.2}$$

then we have,

$$a^2_{m,3} = \frac{9(1+P^4)}{P^4(9+P^4)} = \frac{9(1-Q^4)}{Q^4(Q^4-9)}, \quad Q^4 \neq 9. \tag{2.3}$$

Lemma 2.2. *[5] If n is any positive rational,*

$$E_{3,n} = \frac{f(q)\psi(-q^3)}{q^{-1/6}f(q^3)\psi(-q)}; \quad q := e^{-\pi\sqrt{\frac{n}{3}}}. \tag{2.4}$$

$$P := \frac{\psi(-q)}{q^{1/4}\psi(-q^3)} \quad \text{and} \quad Q := \frac{f(q)}{q^{1/12}f(q^3)}, \tag{2.5}$$

then we have,

$$E^6_{3,n} = \frac{P^4+9}{P^4(1+P^4)}. \tag{2.6}$$

Lemma 2.3. *[6] If m is any positive rational,*

$$a_{m,5} = \frac{5q\psi^2(-q^5)\varphi^2(q^5)}{\psi^2(-q)\varphi^2(q)}, \tag{2.7}$$

$$P = \frac{\psi(-q)}{q^{1/2}\psi(-q^5)} \quad and \quad Q = \frac{\varphi(q)}{\varphi(q^5)}, \tag{2.8}$$

then we have,

$$a_{m,5} = \frac{5(1+P^2)}{P^2(5+P^2)} = \frac{5(1-Q^2)}{Q^2(Q^2-5)}, \quad Q \neq \sqrt{5}. \tag{2.9}$$

Lemma 2.4. *[5] If n is any positive rational,*

$$E_{5,n} = \frac{f(q)\psi(-q^5)}{q^{-1/3}f(q^5)\psi(-q)}; \quad q := e^{-\pi\sqrt{\frac{n}{5}}} \tag{2.10}$$

$$P := \frac{\psi(-q)}{q^{1/2}\psi(-q^5)} \quad and \quad Q := \frac{f(q)}{q^{1/6}f(q^5)}. \tag{2.11}$$

then we have,

$$E_{5,n}^3 = \frac{P^2+5}{P^2(P^2+1)}. \tag{2.12}$$

Lemma 2.5. *[5] We have,*

$$a_{m,n} = a_{n,m}$$

and

$$E_{m,n} = E_{n,m}.$$

3. Modular relation between $a_{m,n}$ and $E_{m,n}$

Theorem 3.1. *If $x := E_{m,3}$ and $y := a_{m,3}$ then*

$$x^3 - \frac{1}{x^3} = 3\left(y - \frac{1}{y}\right). \tag{3.1}$$

Proof. From Lemma (2.1), we obtain

$$P^4 := \frac{9 - 9y + 3\sqrt{9y^2 - 14y + 9}}{2y}, \tag{3.2}$$

where,

$$y := a_{m,3}^2.$$

Employing the above equation (3.2) in Lemma (2.2), we obtain

$$(x^3(yx^3 - 3 + 3y^2) - y)(x^3(yx^3 + 3 - 3y^2) - y) = 0 \tag{3.3}$$

By examining the behavior of the above factors near $q = 0$, we can find a neighborhood about the origin, where the second factor is zero; whereas another factor is not zero in this neighborhood. By the Identity Theorem second factor vanishes identically. This completes the proof. □

Theorem 3.2. *If $x := E_{m,5}$ and $y := a_{m,5}$ then*

$$\left(x^3 + \frac{1}{x^3}\right) + 8 = 5\left(y + \frac{1}{y}\right). \tag{3.4}$$

Proof. From Lemma (2.3), we obtain

$$P^2 := \frac{5 - 5y + \sqrt{25y^2 - 30y + 25}}{2y}. \tag{3.5}$$

Employing the above equation (3.5) in Lemma (2.4), we get

$$x^3(5 - x^3y - 8y + 5y^2) - y = 0 \tag{3.6}$$

By examining the behavior of the above term near $q = 0$. This completes the proof. □

4. Explicit evaluation of $E_{m,n}$

Corollary 4.1. *Explicit values of $E_{3,n}$*

Table 1: $E_{3,n}$

Sr. No	$a_{3,n}$	$E_{3,n}$
1	$a_{3,2} = \frac{\sqrt{\sqrt{3}-1}(1-\sqrt{3}+\sqrt{6})}{2}$	$E_{3,2} = \frac{\left((-4\sqrt{3}\sqrt{2}+12-12\sqrt{3}+16\sqrt{2})\sqrt{\sqrt{3}-1}\right)^{\frac{1}{3}}}{2}$
2	$a_{3,3} = \frac{1}{\sqrt{3}}$	$E_{3,3} = (2-\sqrt{3})^{\frac{1}{3}}$
3	$a_{3,5} = \frac{3-\sqrt{5}}{2}$	$E_{3,5} = \frac{(28-12\sqrt{5})^{\frac{1}{3}}}{2}$
4	$a_{3,7} = 2-\sqrt{3}$	$E_{3,7} = \frac{\sqrt{7}-\sqrt{3}}{2}$
5	$a_{3,9} = \frac{1}{(2^{1/3}+1)^2}$	$E_{3,9} = (1-(2)2^{\frac{1}{3}}+2^{\frac{2}{3}})^{\frac{1}{3}}$
6	$a_{3,11} = 2\sqrt{3}-\sqrt{11}$	$E_{3,11} = (10-3\sqrt{11})^{\frac{1}{3}}$
7	$a_{3,15} = \frac{2-\sqrt{3}}{3}$	$E_{3,15} = ((2-\sqrt{5})(\sqrt{15}-4))^{\frac{1}{3}}$
8	$a_{3,19} = 2\sqrt{19}-5\sqrt{3}$	$E_{3,19} = 2-\sqrt{3}$
9	$a_{3,31} = \sqrt{2-\sqrt{3}}3$	$E_{3,31} = \frac{\sqrt{31}-3\sqrt{3}}{2}$
10	$a_{3,35} = 4\sqrt{21}+10\sqrt{3}-8\sqrt{5}-3\sqrt{35}$	$E_{3,35} = \left((4\sqrt{5}-9)(\sqrt{35}-6)\right)^{\frac{1}{3}}$
11	$a_{3,55} = 3\sqrt{11}-104\sqrt{3}-7$	$E_{3,55} = \frac{31(\sqrt{5}-3)(\sqrt{11}-\sqrt{15})}{124}$
12	$a_{3,59} = 102\sqrt{3}-23\sqrt{59}$	$E_{3,59} = (530-69\sqrt{59})^{\frac{1}{3}}$

Proof. In Ramanujan notebook Part V [3] he recorded many values of $a_{3,n}$. In particularly, he recorded for $n = 3, 5, 7, 9, 11, 15, 19, 31, 59$.
Then, M. S. Mahadeva Naika , B. N. Dharmendra and K. Shivashankar [7] also evaluated the values of $a_{3,n}$ for $n = 2, 35, 55$.
Noting all these values of n, we have established the values for $E_{3,n}$
If

$$n = 3$$

then, we find in [3], $a_{3,3} = \frac{1}{\sqrt{3}}$,
substituting this value in (3.3) we obtain an equation

$$-2x^3 + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}x^6$$

and solving for x we get the desired result.
i.e.,

$$E_{3,3} = (2-\sqrt{3})^{\frac{1}{3}}.$$

Similarly we can obtain for remaining values of n which is mentioned in the above table1. □

Corollary 4.2. *Explicit evaluation of* $E_{5,n}$

Table 2: $E_{5,n}$

Sr.No	$a_{5,n}$	$E_{5,n}$
1	$a_{5,2} = (\sqrt{2}+1)(\sqrt{5}-2)$	$E_{5,2} = \frac{(\sqrt{5}+1)(\sqrt{2}-1)}{2}$
2	$a_{5,5} = \frac{1}{5}$	$E_{5,5} = \frac{3-\sqrt{5}}{2}$
3	$a_{5,9} = \left(2-\sqrt{3}\right)^2$	$E_{5,9} = \left(31-8\sqrt{15}\right)^{\frac{1}{3}}$
4	$a_{5,11} = \left(\frac{\sqrt{7+\sqrt{5}}-\sqrt{\sqrt{5}-1}}{8}\right)^8$	$E_{5,11} = \frac{\left(1+\sqrt{5}\right)\left(12-\sqrt{14+2\sqrt{5}}\sqrt{-2+2\sqrt{5}}\right)}{16}$
5	$a_{5,13} = \left(\frac{\sqrt{9+\sqrt{65}}}{2}-\sqrt{7+\sqrt{652}}\right)^2$	$E_{5,13} = \frac{(\sqrt{5}-1)(\sqrt{13}-3)}{4}$
6	$a_{5,21} = 32+3\sqrt{105}-4\sqrt{123+12\sqrt{105}}$	$E_{5,21} = \left((\sqrt{35}-6)(15\sqrt{3}-26)\right)^{\frac{1}{3}}$
7	$a_{5,29} = \left(\sqrt{49+4\sqrt{145}}-\sqrt{48+4\sqrt{145}}\right)^2$	$E_{5,29} = \frac{13+\sqrt{145}+(7-\sqrt{145})\sqrt{12+\sqrt{145}}}{4}$
8	$a_{5,33} = \left(2-\sqrt{3}\right)^2\left(2\sqrt{3}-\sqrt{11}\right)^2$	$E_{5,33} = \left((9-4\sqrt{5})(89-12\sqrt{55})\right)^{\frac{1}{3}}$
9	$a_{5,69} = \frac{\left(5-\sqrt{23}\right)^2\left(7\sqrt{5}-\sqrt{11}\right)^2}{4}$	$E_{5,69} = \left((1126-105\sqrt{115})(26-15\sqrt{3})\right)^{\frac{1}{3}}$
10	$a_{5,77} = 11303+576\sqrt{385}-1524\sqrt{55}-4272\sqrt{7}$	$E_{5,77} = \frac{63}{4}+\frac{3}{4}\sqrt{55}\sqrt{7}$ $+\left(\frac{189}{3040}-\frac{743}{608}\sqrt{55}\sqrt{7}\right)\sqrt{99794330+5085990\sqrt{55}\sqrt{7}}$

Proof. In [3] Ramanujan has recorded many values of $a_{5,n}$ for $n = 9, 11, 13, 29$.
Then [7], M. S. Mahadeva Naika et al. also evaluated the values of $a_{5,n}$ for $n = 2, 5, 9, 33, 69, 77$.
Noting all these values of n we have established the values for $E_{5,n}$
If

$$n = 5$$

then, $a_{5,5} = \frac{1}{5}$, [3] substituting this value in (3.6) we obtain an equation

$$\frac{18}{5}x^3 - \frac{1}{5}x^6 - \frac{1}{5} = 0$$

and solving for x we get the desired result.
i.e.,

$$E_{5,5} = \frac{3-\sqrt{5}}{2}.$$

Similarly we can obtain for remaining values of n which is mentioned in the above table2.

□

Conclusion: Finally in this article we established new relation between $a_{m,n}$ and $E_{m,n}$ and explicit evaluations of $E_{3,n}$ and $E_{5,n}$ by setting particular values to n, similarly we can also obtain for other values of m.

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