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Theorems on Analogous of Ramanujan's Remarkable Product of Theta-Function and Their Explicit Evaluations

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Abstract: In this article, we define $E_{m,n}$ for any positive real numbers m and n involving Ramanujan's product of theta-functions $\psi(-q)$ and $f(q)$, which is analogous to Ramanujan's remarkable product of theta-functions and establish its several properties by Ramanujan. We establish general theorems for the explicit evaluations of $E_{m,n}$ and its explicit values.

Key Words: Class invariant, Modular equation, Theta-function, Cubic continued fraction.

Contents

1. Introduction

Ramanujan's general theta-function [15] $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab|<1, \tag{1.1}$$

$$= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{1.2}$$

Three special cases of $f(a,b)$ are as follows:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1.3}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.4}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} = (q;q)_\infty, \tag{1.5}$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1.$$

On page 338 in his first notebook [4,15], Ramanujan defines

$$a_{m,n} = \frac{n e^{\frac{-(n-1)\pi}{4}\sqrt{\frac{m}{n}}}\psi^2(e^{-\pi\sqrt{mn}})\varphi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(e^{-\pi\sqrt{\frac{m}{n}}})\varphi^2(-e^{-2\pi\sqrt{\frac{m}{n}}})}. \tag{1.6}$$

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He then, on pages 338 and 339, offers a list of eighteen particular values. All these eighteen values have been established by Berndt, Chan and Zhang [5]. M. S. Mahadeva Naika and B. N. Dharmendra [7], also established some general theorems for explicit evaluations of the product of $a_{m,n}$ and found some new explicit values from it. Further results on $a_{m,n}$ are found by Mahadeva Naika, Dharmendra and K. Shivashankara [9], and Mahadeva Naika and M. C. Mahesh Kumar [10]. Recently Nipen Saikia [13] established new properties of $a_{m,n}$.

In [12], Mahadeva Naika et al. defined the product

$$b_{m,n} = \frac{n e^{\frac{-(n-1)\pi}{4}\sqrt{\frac{m}{n}}} \psi^2(-e^{-\pi\sqrt{mn}})\varphi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(-e^{-\pi\sqrt{\frac{m}{n}}})\varphi^2(-e^{-2\pi\sqrt{\frac{m}{n}}})}. \tag{1.7}$$

They established general theorems for explicit evaluation of $b_{m,n}$ and obtained some particular values. Mahadeva Naika et al. [11] established general formulas for explicit values of Ramanujan's cubic continued fraction $V(q)$ in terms of the products of $a_{m,n}$ and $b_{m,n}$ defined above, where

$$V(q) := \frac{q^{1/3}}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \frac{q^3+q^6}{1} + \cdots, \quad |q|<1, \tag{1.8}$$

and found some particular values of $V(q)$

In this paper, we define

$$E_{m,n} = \frac{f(e^{-\pi\sqrt{\frac{n}{m}}})\psi(-e^{-\pi\sqrt{mn}})}{e^{\frac{-\pi(1-m)}{12}\sqrt{\frac{n}{m}}} f(e^{-\pi\sqrt{mn}})\psi(-e^{-\pi\sqrt{\frac{n}{m}}})}, \tag{1.9}$$

where m and n are positive real numbers.

Let K, K', L and L' denote the complete elliptic integrals of the first kind associated with the moduli k, $k' := \sqrt{1-k^2}$, l and $l' := \sqrt{1-l^2}$ respectively, where $0 < k, l < 1$. For a fixed positive integer n, suppose that

$$n\frac{K'}{K} = \frac{L'}{L}. \tag{1.10}$$

Then a modular equation of degree n is a relation between k and l induced by (1.5). Following Ramanujan, set $\alpha = k^2$ and $\beta = l^2$. Then we say β is of degree n over α.
Define

$$\chi(q) := (-q;q^2)_\infty,$$

and

$$G_n := 2^{-\frac{1}{4}} q^{-\frac{1}{24}} \chi(q),$$

where

$$q = e^{-\pi\sqrt{r}}.$$

Moreover, if $q = e^{-\pi\sqrt{\frac{m}{n}}}$ and β has degree n over α, then

$$G_{\frac{n}{m}} = (4\alpha(1-\alpha))^{\frac{-1}{24}} \tag{1.11}$$

and

$$G_{nm} = (4\beta(1-\beta))^{\frac{-1}{24}}. \tag{1.12}$$

The main purpose of this paper is to obtain several general theorems for the explicit evaluations of analogous of Ramanujan's product of theta-function $E_{m,n}$ and also some new explicit evaluations from it.

2. Preliminary Results

In this section, we collect several identities which are useful in proving our main results.

Lemma 2.1. *[2, Ch. 17, Entry 11(ii) and Entry 12(i), pp. 123–124] We have,*

$$2^{1/2}\ e^{-y/8}\ \psi(-e^{-y}) = \sqrt{z_1}\ \{\alpha(1-\alpha)\}^{1/8}, \tag{2.1}$$
$$2^{1/2}\ e^{-my/8}\ \psi(-e^{-my}) = \sqrt{z_m}\ \{\beta(1-\beta)\}^{1/8}, \tag{2.2}$$
$$2^{1/6}\ e^{-y/24}\ f(e^{-y}) = \sqrt{z_1}\ \{\alpha(1-\alpha)\}^{1/24}, \tag{2.3}$$
$$2^{1/6}\ e^{-my/24}\ f(e^{-my}) = \sqrt{z_m}\ \{\beta(1-\beta)\}^{1/24}. \tag{2.4}$$

Lemma 2.2. *[2, Ch. 16, Entry 27(iii) and (iv), pp. 43] We have,*

$$e^{-\alpha/24}\ \sqrt[4]{\alpha}\ f(e^{-\alpha}) = e^{-\beta/24}\ \sqrt[4]{\beta}\ f(e^{-\beta}),\quad if\ \alpha\beta=\pi^2 \tag{2.5}$$
$$e^{-\alpha/12}\ \sqrt[4]{\alpha}\ f(-e^{-2\alpha}) = e^{-\beta/12}\ \sqrt[4]{\beta}\ f(-e^{-2\beta}),\quad if\ \alpha\beta=\pi^2. \tag{2.6}$$

Lemma 2.3. *[6, Theorem 2.1] We have,*

$$\frac{f^6(q)}{f^6(q^3)} = \frac{\psi^2(-q)}{\psi^2(-q^3)}\left\{\frac{\psi^4(-q)+9q\psi^4(-q^3)}{\psi^4(-q)+q\psi^4(-q^3)}\right\}. \tag{2.7}$$

Lemma 2.4. *[16, p. 56] [14] We have,*

$$\frac{f^3(q)}{f^3(q^5)} = \frac{\psi(-q)}{\psi(-q^5)}\left\{\frac{\psi^2(-q)+5q\psi^2(-q^5)}{\psi^2(-q)+\psi^2(-q^5)}\right\}. \tag{2.8}$$

Lemma 2.5. *[6, Theorem 2.2] We have,*

$$\frac{f^3(q)}{f^3(q^9)} = \frac{\psi(-q)}{\psi(-q^9)}\left\{\frac{\psi(-q)+3q\psi(-q^9)}{\psi(-q)+q\psi(-q^9)}\right\}^2. \tag{2.9}$$

Lemma 2.6. *[2, Chapter 19, entry 5(xii), page 231] We have,*
If $P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}$ *and* $Q := \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/4}$*, then*

$$Q+\frac{1}{Q} = 2\sqrt{2}\left(\frac{1}{P}-P\right). \tag{2.10}$$

Lemma 2.7. *[2, Chapter 19, entry 13(xiv), page 282] We have,*
If $P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}$ *and* $Q := \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/8}$*, then*

$$Q+\frac{1}{Q} = 2\left(\frac{1}{P}-P\right). \tag{2.11}$$

Lemma 2.8. *[2, Chapter 19, entry 19(ix), page 315] We have,*
If $P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}$ *and* $Q := \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/6}$*, then*

$$Q+\frac{1}{Q}+7 = 2\sqrt{2}\left(P+\frac{1}{P}\right). \tag{2.12}$$

Lemma 2.9. *[1, Theorem 5.1] We have,*
If $P = \frac{\psi(-q)}{q^{1/4}\psi(-q^3)}$ *and* $Q = \frac{\varphi(q)}{\varphi(q^3)}$*, then*

$$Q^4+P^4Q^4 = 9+P^4. \tag{2.13}$$

Lemma 2.10. *[1, Theorem 5.1] We have,*
If $P = \dfrac{\psi(-q)}{q^{1/2}\psi(-q^5)}$ *and* $Q = \dfrac{\varphi(q)}{\varphi(q^5)}$, *then*

$$Q^2 + P^2Q^2 = 5 + P^2. \tag{2.14}$$

Lemma 2.11. *[8, Theorem 3.2] We have,*
If $P = \dfrac{\psi(-q)}{q\psi(-q^9)}$ *and* $Q = \dfrac{\varphi(q)}{\varphi(q^9)}$, *then*

$$Q + PQ = 3 + P. \tag{2.15}$$

3. Some Properties of $E_{m,n}$

In this section, we have establish some properties of $E_{m,n}$,

Theorem 3.1.

$$E_{m,n} = E_{n,m}. \tag{3.1}$$

Proof. Employing the equation (2.5) and (2.6), we deduce that

$$e^{-\alpha/8}\ \sqrt[4]{\alpha}\ \psi(-e^{-\alpha}) = e^{-\beta/8}\ \sqrt[4]{\beta}\ \psi(-e^{-\beta}),\quad if\ \alpha\beta = \pi^2. \tag{3.2}$$

Using the equations (2.5) and (3.2) in (1.9), we obtain (3.1). □

Theorem 3.2.

$$E_{m,n}E_{m,1/n} = 1. \tag{3.3}$$

Proof. Using the equations (2.5) and (3.2) in (1.9), we obtain (3.3). □

Corollary 3.3.

$$E_{m,1} = 1. \tag{3.4}$$

Proof. Putting $n = 1$ in the equation (3.3), we get (3.4) □

Remark 3.4. *By using the definition of* $\psi(q)$, $f(q)$ *and* $E_{m,n}$, *it can be seen that* $E_{m,n}$ *has positive real value and that the values of* $E_{m,n}$ *increases as* n *increase when* $m > 1$. *Thus by the above corollary,* $E_{m,n} > 1$ *for all* $n > 1$ *if* $m > 1$.

Theorem 3.5.

$$\frac{E_{km,n}}{E_{nm,k}} = E_{m,\frac{n}{k}}. \tag{3.5}$$

Proof. Employing the definition of $E_{m,n}$, we obtain

$$\frac{E_{km,n}}{E_{nm,k}} = e^{\frac{\pi\left(\sqrt{\frac{k}{mn}}-\sqrt{\frac{n}{mk}}\right)}{12}}\frac{f\left(e^{-\pi\sqrt{\frac{n}{mk}}}\right)\psi\left(-e^{\pi\sqrt{\frac{k}{mn}}}\right)}{f\left(e^{-\pi\sqrt{\frac{k}{mn}}}\right)\psi\left(-e^{-\pi\sqrt{\frac{n}{mk}}}\right)}. \tag{3.6}$$

Using the Lemma **2.2** in the above equation (3.6) and simplifying using the Theorems **3.1 and 3.2** , we obtain (3.5). □

Corollary 3.6.

$$E_{m^2,n} = E_{nm,n}E_{m,\frac{n}{m}}. \tag{3.7}$$

Proof. Putting $m = n$ in the above Theorem **3.5** and simplifying using the equation (3.3), we get

$$E_{m^2,k} = E_{mk,n}E_{m,\frac{k}{m}}. \tag{3.8}$$

Replace k by n, we obtain (3.7). □

Theorem 3.7. *If* $mn = rs$

$$\frac{E_{m,n}}{E_{kr,ks}} = \frac{E_{r,s}}{E_{km,kn}}. \tag{3.9}$$

Proof. Using the definition of $E_{m,n}$ and letting $mn = rs$ for positive real numbers m, n, r, s and k, we find that

$$\frac{E_{km,kn}}{E_{m,n}} = \frac{E_{kr,ks}}{E_{r,s}}. \tag{3.10}$$

On rearranging the above equation (3.10) we obtain the required result. □

Corollary 3.8. *If* $mn = rs$

$$E_{np,np} = E_{np^2,n}E_{p,p}. \tag{3.11}$$

Proof. Letting $m = p^2$, $n = 1$, $r = s = p$ and $k = n$ in above Theorem **3.7**, we deduced the equation (3.11). □

Theorem 3.9. *For all positive real numbers m, n, r and s, then*

$$E_{m/n,r/s} = \frac{E_{ms,nr}}{E_{mr,ns}}. \tag{3.12}$$

Proof. Employing the equation (3.3) in equation (3.5), we find that, for all positive real numbers m, n and k

$$E_{m/n,k} = E_{m,nk}E_{n,mk}^{-1}. \tag{3.13}$$

Letting $k = r/s$ and again using the equation (3.5) and (3.1) in (3.13), we get (3.12). □

Theorem 3.10.

$$E_{m/n,m/n} = E_{n,n}E_{m,m/n^2}. \tag{3.14}$$

Proof. Using the Theorems **3.2** and **3.9**, we get (3.14). □

Theorem 3.11.

$$E_{m,m}E_{m,n^2/m} = E_{n,n}E_{n,m^2/n}. \tag{3.15}$$

Proof. Putting $k = m/n$ in the equation (3.13) and Employing Theorems **3.2** and **3.10**, we obtain (3.15). □

Theorem 3.12.

$$E_{m,m}E_{n,m^2n} = E_{n,n}E_{m,mn^2}. \tag{3.16}$$

Proof. Employing the Theorems **3.1**, **3.2**, **3.10** and **3.11**, we obtain (3.16). □

4. Some General Theorems on $E_{m,n}$ and their explicit evaluations

In this section we establish some general theorems and their explicit evaluations of Ramanujan's remarkable product of theta functions involving $E_{m,n}$.

Theorem 4.1. *If n is any positive real* $P := \{G_{n/3}G_{3n}\}^3$ *and* $Q := E_{3,n}^3$, *then*

$$Q + \frac{1}{Q} = 2\sqrt{2}\left\{P - \frac{1}{P}\right\}. \tag{4.1}$$

Proof. Using the Lemma **2.1** with the definition of $E_{m,n}$, we obtain

$$E_{m,n} = \left\{ \frac{\beta(1-\beta)}{\alpha(1-\alpha)} \right\}^{1/12}. \tag{4.2}$$

Employing the above equation (4.2) and definition of class invariant (1.11), (1.12) in the Lemma **2.6** with $m = 3$, we obtain (4.1) □

Corollary 4.2.

$$E_{3,9} = \left\{ 1 + 2^{2/3} - 2^{4/3} \right\}^{1/3}. \tag{4.3}$$

Proof. Putting $n = 9$ in the above Theorem **4.1**, we obtain

$$E_{3,9}^3 + E_{3,9}^{-3} = 2\sqrt{2}\left\{ G_3^3 G_{27}^3 - G_3^{-3} G_{27}^{-3} \right\}. \tag{4.4}$$

Solving the above equation (4.4) with from the table of Chapter 34 of Ramanujan's notebooks [4, p.189,190] $G_3 = 2^{1/12}$ and $G_{27} = 2^{1/12}\left(\sqrt[3]{2}-1\right)^{-1/3}$, we obtain (4.3). □

Theorem 4.3. *If n is any positive real $P := \{G_{n/5} G_{5n}\}^2$ and $Q := E_{5,n}^{3/2}$, then*

$$Q + \frac{1}{Q} = 2\left\{ P - \frac{1}{P} \right\}. \tag{4.5}$$

Proof. Using the equation (4.2) and definition of class invariant (1.11), (1.12) in the Lemma **2.7** with $m = 5$, we obtain (4.5). □

Theorem 4.4. *If n is any positive real $P := \{G_{n/7} G_{7n}\}^3$ and $Q := E_{7,n}^2$, then*

$$Q + \frac{1}{Q} + 7 = 2\sqrt{2}\left\{ P + \frac{1}{P} \right\}. \tag{4.6}$$

Proof. Using the equation (4.2) and definition of class invariant (1.11), (1.12) in the Lemma **2.8** with $m = 7$, we obtain (4.6). □

Theorem 4.5.

$$E_{3,n} = \frac{f(q)\psi(-q^3)}{q^{-1/6} f(q^3)\psi(-q)}; \quad q := e^{-\pi\sqrt{\frac{n}{3}}} \tag{4.7}$$

If

$$P := \frac{\psi(-q)}{q^{1/4}\psi(-q^3)} \quad \textit{and} \quad Q := \frac{f(q)}{q^{1/12} f(q^3)}, \quad \textit{then} \tag{4.8}$$

$$E_{3,n}^6 = \frac{P^4 + 9}{P^4(1+P^4)}, \quad \textit{if} \quad P^4 \neq -1. \tag{4.9}$$

Proof. Employing the definition of $E_{m,n}$ with $m = 3$, we get

$$E_{3,n} = \frac{f(q)\psi(-q^3)}{q^{-1/6} f(q^3)\psi(-q)}. \tag{4.10}$$

Raising the power by 6 in the above equation (4.10) with the Lemma **2.3**, we deduce that

$$E_{3,n}^6 = \frac{f^6(q)\psi^6(-q^3)}{q^{-1} f^6(q^3)\psi^6(-q)}, \tag{4.11}$$

$$E_{3,n}^6 = \frac{P^2 \left\{ \dfrac{P^4 + 9}{1 + P^4} \right\}}{P^6}. \tag{4.12}$$

On simplifying the above equation (4.12), we obtain (4.9). □

Corollary 4.6.

$$E_{3,3} = \left\{2-\sqrt{3}\right\}^{1/3}. \tag{4.13}$$

Proof. Putting $n = 3$ in the equation (4.8) and from Ramanujan's Notebooks [4, p. 327], we have

$$\frac{\varphi(e^{-\pi})}{\varphi(e^{-3\pi})} = \sqrt[4]{6\sqrt{3}-9}. \tag{4.14}$$

Employing the equation (2.13) and (4.14), we obtain

$$P := \frac{\psi(-e^{-\pi})}{\psi(-e^{-3\pi})} = \sqrt[4]{9+6\sqrt{3}}. \tag{4.15}$$

Substituting (4.15) in (4.9), we obtain the required result. □

Theorem 4.7.

$$E_{5,n} = \frac{f(q)\psi(-q^5)}{q^{-1/3}f(q^5)\psi(-q)}; \quad q := e^{-\pi\sqrt{\frac{n}{5}}}. \tag{4.16}$$

If

$$P := \frac{\psi(-q)}{q^{1/2}\psi(-q^5)} \quad \textit{and} \quad Q := \frac{f(q)}{q^{1/6}f(q^5)}, \quad \textit{then} \tag{4.17}$$

$$E_{5,n}^3 = \frac{P^2+5}{P^2(P^2+1)}, \quad \textit{if} \quad P^2 \neq -1. \tag{4.18}$$

Proof. Employing the definition of $E_{m,n}$ with $m = 5$, we get

$$E_{5,n} = \frac{f(q)\psi(-q^5)}{q^{-1/3}f(q^5)\psi(-q)}. \tag{4.19}$$

Raising the power by 3 in the above equation (4.19) with the Lemma **2.4**, we deduce that

$$E_{5,n}^3 = \frac{f^3(q)\psi^3(-q^5)}{q^{-1}f^3(q^5)\psi^3(-q)}, \tag{4.20}$$

$$E_{5,n}^3 = \frac{P\left\{\dfrac{5+P^2}{P^2+1}\right\}}{P^3}. \tag{4.21}$$

On simplifying the above equation (4.21), we obtain (4.18). □

Corollary 4.8.

$$E_{5,5} = \left\{9-4\sqrt{5}\right\}^{2/3}. \tag{4.22}$$

Proof. Putting $n = 5$ in the equation (4.17) and from Ramanujan's Notebooks [4, p. 327], we have

$$\frac{\varphi(e^{-\pi})}{\varphi(e^{-5\pi})} = \sqrt{5\sqrt{5}-10}. \tag{4.23}$$

Employing the equation (2.14) and (4.23), we obtain

$$P := \frac{\psi(-e^{-\pi})}{\psi(-e^{-5\pi})} = \sqrt{5\sqrt{5}+10}. \tag{4.24}$$

Substituting (4.24) in (4.18), we obtain the required result. □

Theorem 4.9.

$$E_{9,n} = \frac{f(q)\psi(-q^9)}{q^{-2/3} f(q^9)\psi(-q)}; \quad q := e^{-\pi\sqrt{\frac{n}{9}}}. \tag{4.25}$$

If

$$P := \frac{\psi(-q)}{q\psi(-q^9)} \quad and \quad Q := \frac{f(q)}{q^{1/3} f(q^9)}, \quad then \tag{4.26}$$

$$E_{9,n}^3 = \left\{ \frac{P+3}{P(P+1)} \right\}^2, \quad if \quad P \neq -1. \tag{4.27}$$

Proof. Employing the definition of $E_{m,n}$ with $m = 9$, we get

$$E_{9,n} = \frac{f(q)\psi(-q^9)}{q^{-2/3} f(q^9)\psi(-q)}. \tag{4.28}$$

Raising the power by 3 in the above equation (4.28) with the Lemma **2.5**, we deduce that

$$E_{9,n}^3 = \frac{f^3(q)\psi^3(-q^9)}{q^{-2} f^3(q^9)\psi^3(-q)}, \tag{4.29}$$

$$E_{9,n}^3 = \frac{P\left\{\frac{P+3}{P+1}\right\}^2}{P^3}. \tag{4.30}$$

On simplifying the above equation (4.30), we obtain (4.27). □

Corollary 4.10.

$$E_{9,9} = \left\{ \frac{\left[33s^2 - (39+\sqrt{3})s - 21\sqrt{3} + 6\right]\left[54 - 31\sqrt{3}\right]}{33} \right\}^{1/3}. \tag{4.31}$$

where $s = (2\sqrt{3}+2)^{1/3}$

Proof. Putting $n = 9$ in the equation (4.26) and from Ramanujan's Notebooks [4, p. 327] we have,

$$P := \frac{\varphi(e^{-\pi})}{\varphi(e^{-9\pi})} = \frac{3}{1 + \sqrt[3]{2(\sqrt{3}+1)}}. \tag{4.32}$$

Employing the equation (2.15) and (4.32), we obtain

$$P := \frac{\psi(-e^{-\pi})}{\psi(-e^{-9\pi})} = \frac{(s^2 + 2s + \sqrt{3} + 1)(3 + \sqrt{3})}{2}. \tag{4.33}$$

Substituting (4.33) in (4.27), we obtain the required result. □

Theorem 4.11.

$$E_{m,n} = \left\{ \frac{G_{n/m}}{G_{mn}} \right\}^2. \tag{4.34}$$

Proof. Employing the Lemma **2.1** in the definition of $E_{m,n}$, we obtain

$$E_{m,n} = \left\{ \frac{\beta(1-\beta)}{\alpha(1-\alpha)} \right\}^{1/12}. \tag{4.35}$$

Using the equation (1.11) and (1.12), we get

$$\frac{G_{nm}}{G_{n/m}} = \left\{ \frac{\alpha(1-\alpha)}{\beta(1-\beta)} \right\}^{1/24}. \tag{4.36}$$

By observing the equations (4.35) and (4.36), we obtain (4.34). □

Corollary 4.12.

$$E_{n,n} = G_{n^2}^{-2}. \tag{4.37}$$

Proof. Setting $m = n$ in the above Theorem **4.7** with the value $G_1 = 1$, we obtain required result. □

Corollary 4.13.

$$(i)\ E_{2,2} = 2^{3/8}(1+\sqrt{2})^{-1/2}, \tag{4.38}$$

$$(ii)\ E_{3,3} = \left\{2-\sqrt{3}\right\}^{1/3}, \tag{4.39}$$

$$(iii)\ E_{5,5} = \frac{3-\sqrt{5}}{2}, \tag{4.40}$$

$$(iv)\ E_{9,9} = \left\{\frac{\left[2(\sqrt{3}+1)\right]^{1/3}+1}{\left[2(\sqrt{3}-1)\right]^{1/3}-1}\right\}^{-2/3}. \tag{4.41}$$

Proof. For (i), we use the values of G_4 from [3, p.114, Theorem 6.2.2(ii)]. For $(ii)-(iv)$, we use corresponding values of G_n from [2, p.189-193]. □

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