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Densely Generated $2D$ q-Appell Polynomials of Bessel Type and q-addition Formulas *

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ABSTRACT: The article aims to introduce a densely generated class of $2D$ q-Appell polynomials of Bessel type and to investigate their properties. It is advantageous to consider the $2D$ q-Bernoulli, $2D$ q-Roger Szegö and $2D$ q-Al-Salam Carlitz polynomials of Bessel type as their special members and to derive the q-determinant forms and certain q-addition formulas for these polynomials. The article concludes with a brief view on discrete q-Bessel convolution of the $2D$ q-Appell polynomials.

Key Words: q-Bessel polynomials, $2D$ q-Appell polynomials, Generating equation, q-determinant forms.

Contents

1. Introduction

The q-analogues appear in the diverse areas of combinatorics and fluid mechanics, quantum group theory, group representation theory, number theory, statistical mechanics, quantum mechanics and also have an intimate connection with commutativity relations and Lie algebra and q-deformed super algebras. We begin this section by describing notations and definitions of q-Calculus following from [3].

The q-analogues of the shifted factorial $(a)_n$ are given by

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{m=0}^{n-1}(1-q^m a) \qquad (a,\ q \in \mathbb{R};\ n \in \mathbb{N}) \tag{1.1}$$

The q-analogues of a complex number a and of the factorial function are given by

$$[a]_q = \frac{1-q^a}{1-q} \qquad (q \in \mathbb{C} - \{1\};\ a \in \mathbb{C}), \tag{1.2}$$

$$[n]_q! = \prod_{m=1}^{n}[m]_q = [1]_q[2]_q \cdots [n]_q = \frac{(q;q)_n}{(1-q)^n} \qquad \big(q \neq 1;\ n \in \mathbb{N};\ [0]_q! = 1;\ q \in \mathbb{C};\ 0 < q < 1\big). \tag{1.3}$$

The Gauss q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} \qquad (k = 0,1,\ldots,n). \tag{1.4}$$

The q-analogue of the function $(x+y)^n$ is defined as:

$$(x+y)_q^n := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} x^{n-k} y^k,\ n \in \mathbb{N}_0. \tag{1.5}$$

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The q-analogues of exponential functions are given by

$$e_q(x)=\sum_{n=0}^{\infty}\frac{x^n}{[n]_q!}\qquad E_q(x)=\sum_{n=0}^{\infty}q^{n(n-1)/2}\,\frac{x^n}{[n]_q!}\qquad (0<|q|<1;\ x\in\mathbb{C}). \tag{1.6}$$

Consequently, we note that

$$e_q(x)E_q(-x)=1\qquad\text{and}\qquad e_q(-x)E_q(x)=1. \tag{1.7}$$

Moreover, the functions $e_q(x)$ and $E_q(x)$ satisfy the following properties:

$$D_q e_q(x)=e_q(x),\ \ D_q E_q(x)=E_q(qx), \tag{1.8}$$

where the q-derivative $D_q f$ of a function f at a point $0\neq z\in\mathbb{C}$ is defined as follows:

$$D_q f(z)=\frac{f(qz)-f(z)}{qz-z},\ \ 0<|q|<1. \tag{1.9}$$

For any two arbitrary functions $f(z)$ and $g(z)$, the following relation holds true:

$$D_{q,z}(f(z)g(z))=f(z)D_{q,z}g(z)+g(qz)D_{q,z}f(z). \tag{1.10}$$

The special polynomials of two variables are important from the point of view of applications in different branches of pure and applied mathematics and physics. These polynomials allow the derivation of a number of useful identities and relations in a fairly straight forward way and help in introducing new families of special polynomials.

For $q\in\mathbb{C}$, $0<|q|<1$, the $2D$ q-Appell polynomials $A_{n,q}(x,y)$ are defined by the following generating function [7]:

$$A_q(t)e_q(xt)E_q(yt)=\sum_{n=0}^{\infty}A_{n,q}(x,y)\frac{t^n}{[n]_q!}, \tag{1.11}$$

where

$$A_q(t)=\sum_{n=0}^{\infty}A_{n,q}\frac{t^n}{[n]_q!};\ \ A_{0,q}=1;\ \ A_q(t)\neq 0. \tag{1.12}$$

The function $A_q(t)$ is analytic at $t=0$ and $A_{n,q}:=A_{n,q}(0,0)$. It is to be noted that

$$A_{n,q}(x,0)=A_{n,q}(x), \tag{1.13}$$

where $A_{n,q}(x)$ are the q-Appell polynomials [1].

We present the polynomials belonging to the $2D$ q-Appell family $A_{n,q}(x,y)$ (for appropriate choice of $A_q(t)$) in Table 1.

Table 1: Certain members belonging to the $2D$ q-Appell family

S.No.	**Name of the q-polynomiall**	$A_q(t)$	**Generating function**
I.	$2D$ q-Bernoulli polynomials [9]	$\left(\frac{t}{e_q(t)-1}\right)^{\alpha}$	$\left(\frac{t}{e_q(t)-1}\right)^{\alpha}e_q(xt)E_q(yt)=\sum\limits_{n=0}^{\infty}B_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}$
II.	$2D$ Roger-Szegö polynomials	$e_q(t)$	$e_q(t)e_q(xt)E_q(yt)=\sum\limits_{n=0}^{\infty}h_{n,q}(x,y)\frac{t^n}{[n]_q!}$
III.	$2D$ Al-salam Carlitz polynomials	$\frac{1}{e_q(t)e_q(at)}$	$\frac{e_q(xt)}{e_q(t)e_q(at)}E_q(yt)=\sum\limits_{n=0}^{\infty}U_{n,q}^{(a)}(x,y)\frac{t^n}{[n]_q!}$

The above polynomials for $y=0$ reduce to q-Bernoulli polynomials [2], q-Roger-Szegö polynomials [4] and q-Al-Salam Carlitz polynomials [8].

The importance of generalized Bessel functions stems from their wide use in applications. The scattering of free or weakly bounded electrons by intense laser fields is an example where generalized Bessel functions an important role. The analytical and numerical study of these functions has revealed their interesting properties, which led to their extension to two-dimensional domain. The relevance of these functions in mathematical physics has been emphasized, since they provide analytical solutions to partial differential equations such as the multi-dimensional diffusion equation, the Schrodinger and Klein-Gordon equation.

The two-dimensional Bessel functions are defined by the following generating function [6]:

$$\exp\left(\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2-\frac{1}{t^2}\right)\right)=\sum_{n=-\infty}^{\infty} J_n(x,y)t^n. \tag{1.14}$$

Motivated by the importance and relevance of the two-dimensional Bessel functions, in 2018 Riyasat and Khan introduce the two-dimensional (or $2D$) q-Bessel polynomials $p_{n,q}(x,y)$, which are defined by means of the following generating function:

$$e_q(x(1-\sqrt{1-2t}))E_q(y(1-\sqrt{1-2t}))=\sum_{n=0}^{\infty} p_{n,q}(x,y)\frac{t^n}{[n]_q!}. \tag{1.15}$$

Taking $y=0$ and $x=0$, consecutively in above equation, the two forms of q-Bessel polynomials $p_{n,q}(x)$ and $\mathcal{P}_{n,q}(x)$ are deduced, which are defined by the following generating functions:

$$e_q(x(1-\sqrt{1-2t}))=\sum_{n=0}^{\infty} p_{n,q}(x)\frac{t^n}{[n]_q!}. \tag{1.16}$$

$$E_q(y(1-\sqrt{1-2t}))=\sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}}\ \mathcal{P}_{n,q}(y)\frac{t^n}{[n]_q!}. \tag{1.17}$$

The q-Bessel polynomials $p_{n,q}(x)$ are the q-analogue of Carlitz type Bessel polynomials $p_n(x)$ [5].

In this article, the densely generated $2D$ q-Appell polynomials of Bessel type are defined by means of generating function and determinant form. The $2D$ q-Bernoulli, $2D$ q-Roger Szegö and $2D$ q-Al-Salam Carlitz polynomials of Bessel type are considered as their special members and corresponding determinant forms and q-addition formulas are derived.

2. $2D$ q-Appell polynomials of Bessel type

We introduce a dense form of generating function for the $2D$ q-Appell polynomials of Bessel type. For this, we prove the following result:

Theorem 2.1. *The generating function for the* $2D$ *q-Appell polynomials of Bessel type is given by*

$$A_q(1-\sqrt{1-2t})\ e_q(x(1-\sqrt{1-2t}))\ E_q(y(1-\sqrt{1-2t}))=\sum_{n=0}^{\infty} {}_Ap_{n,q}(x,y)\frac{t^n}{[n]_q!}. \tag{2.1}$$

Proof. Expanding the exponential function $e_q(x(1-\sqrt{1-2t}))$ and then replacing the powers $x^0,\dots,x^n$ by the sequences $A_{0,q}(x,y)$, $A_{1,q}(x,y)$, $\dots$, $A_{n,q}(x,y)$ in the l.h.s. and replacing x by $A_{1,q}(x,y)$ in the r.h.s. of equation (1.16), if we sum up the terms in the l.h.s. of the resulting equation, we find that

$$\sum_{n=0}^{\infty} A_{n,q}(x,y)\frac{(1-\sqrt{1-2t})^n}{[n]_q!}=\sum_{n=0}^{\infty} p_{n,q}(A_{1,q}(x+0,y))\ \frac{t^n}{[n]_q!}, \tag{2.2}$$

which, on using equation (1.11) in the l.h.s. and denoting the resulting polynomials in the r.h.s. by

$${}_Ap_{n,q}(x,y):=p_{n,q}(A_{1,q}(x,y)), \tag{2.3}$$

yields the assertion (2.1). □

- It is to be noted that for $x=y=0$, we have

$$A_q(1-\sqrt{1-2t})=\sum_{n=0}^{\infty}{}_Ap_{n,q}\frac{t^n}{[n]_q!}. \tag{2.4}$$

- Again, for $y=0$, we have the q-Appell polynomials of Bessel type defined by

$$A_q(1-\sqrt{1-2t})\ e_q(x(1-\sqrt{1-2t}))=\sum_{n=0}^{\infty}{}_Ap_{n,q}(x,0)\frac{t^n}{[n]_q!}. \tag{2.5}$$

Theorem 2.2. *The 2D q-Appell polynomials of Bessel type satisfy the following q-differential recurrence relations:*

$$D_{x,q}\{{}_Ap_{n,q}(x,y)\}={}_Ap_{n,q}(x,y)-\sum_{k=0}^{n}\frac{[n]_q!}{(n-k)![k]_q!}\Big(-\frac{1}{2}\Big)_{n-k}2^{n-k}{}_Ap_{k,q}(x,y), \tag{2.6}$$

$$D_{y,q}\{{}_Ap_{n,q}(x,y)\}={}_Ap_{n,q}(x,qy)-\sum_{k=0}^{n}\frac{[n]_q!}{(n-k)![k]_q!}\Big(-\frac{1}{2}\Big)_{n-k}2^{n-k}{}_Ap_{k,q}(x,qy). \tag{2.7}$$

Proof. Differentiating generating equation (2.1) with respect to x and y, respectively using equation (1.8) and then using formula:

$$(1-z)^{-\alpha}=\sum_{k=0}^{\infty}(\alpha)_k\frac{t^k}{k!}, \tag{2.8}$$

in resultant equations and after rearranging the summations, we get recurrence relations (2.6) and (2.7). □

Next, we find the determinant form for the $2D$ q-Appell polynomials of Bessel type by proving the following theorem:

Theorem 2.3. *The following determinant form for the 2D q-Appell polynomials of Bessel type holds true:*

$${}_Ap_{n+1,q}\Big(\frac{1}{x},\frac{1}{y}\Big)=\frac{(-1)^n}{\prod_{i=0}^{n}b^{-1}_{i,i,q}\ A_{n+1,q}(x,y)}\begin{vmatrix} 1 & A_{1,q}(x,y) & \cdots & A_{n-1,q}(x,y) & A_{n,q}(x,y)\\ b^{-1}_{0,0;q} & b^{-1}_{1,0;q} & \cdots & b^{-1}_{n-1,0;q} & b^{-1}_{n,0;q}\\ 0 & b^{-1}_{1,1;q} & \cdots & b^{-1}_{n-1,1;q} & b^{-1}_{n,1;q}\\ 0 & 0 & \cdots & b^{-1}_{n-1,2;q} & b^{-1}_{n,2;q}\\ . & . & \cdots & . & .\\ . & . & \cdots & . & .\\ 0 & 0 & \cdots & b^{-1}_{n-1,n-1;q} & b^{-1}_{n,n-1;q}\end{vmatrix}, \tag{2.9}$$

where the coefficients $b^{-1}_{i,j;q}$ are given by

$$\begin{cases} b^{-1}_{n,n;q}=\frac{2^n\ [n]_q!}{[2n]_q!}\\ b^{-1}_{i,j;q}=-\sum\limits_{k=j}^{i-1}\begin{bmatrix} i\\ k\end{bmatrix}_q 2^{i-k}\frac{[i+k]_q!}{[2i]_q!}\ b^{-1}_{k,j;q}=\frac{(-1)^{i-j}\ 2^i\ [i]_q!\ [2i+1]_q!}{[i-1]_q!\ [i+j+1]_q!}, \qquad i,\ j=0,1,\ldots;\ i>j.\end{cases} \tag{2.10}$$

Proof. We recall the following determinant form for the q-Bessel polynomial sequences $\{p_{n,q}(x)\}_{n\in\mathbb{N}}$ from [10]:

$$p_{n+1,q}\left(\tfrac{1}{x}\right) = \frac{(-1)^n}{\prod\limits_{i=0}^{n} b_{i,i,q}^{-1}\ x^{n+1}} \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} & x^n \\ b_{0,0;q}^{-1} & b_{1,0;q}^{-1} & b_{2,0;q}^{-1} & \cdots & b_{n-1,0;q}^{-1} & b_{n,0;q}^{-1} \\ 0 & b_{1,1;q}^{-1} & b_{2,1;q}^{-1} & \cdots & b_{n-1,1;q}^{-1} & b_{n,1;q}^{-1} \\ 0 & 0 & b_{2,2;q}^{-1} & \cdots & b_{n-1,2;q}^{-1} & b_{n,2;q}^{-1} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & b_{n-1,n-1;q}^{-1} & b_{n,n-1;q}^{-1} \end{vmatrix}, \tag{2.11}$$

where the coefficients $b_{i,j;q}^{-1}$ are given by equation (2.10).

Now, replacing the powers x^m by the q-polynomials $A_{m,q}(x,y)$ for $m = 0, 1, \ldots, n, n+1$ in the r.h.s. and replacing $\frac{1}{x}$ by $A_{1,q}\left(\frac{1}{x}, \frac{1}{y}\right)$ in the l.h.s. of equation (2.11) and then using relation:

$${}_{A}p_{n+1,q}\left(\frac{1}{x}, \frac{1}{y}\right) = p_{n+1,q}\left(A_{1,q}\left(\frac{1}{x}, \frac{1}{y}\right)\right) \tag{2.12}$$

in the l.h.s. of resultant equation, yields assertion (2.9). □

We now consider the following interesting remarks, which may give several important polynomials and corresponding results related to the $2D$ q-Appell polynomials of Bessel type.

Remark 2.4. *For the choice of*

$$A_q(1-\sqrt{1-2t}) = \left(\frac{1-\sqrt{1-2t}}{e_q(1-\sqrt{1-2t})-1}\right)^{\alpha},$$

we have the $2D$ *q-Bernoulli polynomials of Bessel type (2DqBPoBT) defined by*

$$\left(\frac{1-\sqrt{1-2t}}{e_q(1-\sqrt{1-2t})-1}\right)^{\alpha} e_q(x(1-\sqrt{1-2t}))\ E_q(y(1-\sqrt{1-2t})) = \sum_{n=0}^{\infty} {}_{B}p_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}, \tag{2.13}$$

which for $x = y = 0$ *gives*

$$\left(\frac{1-\sqrt{1-2t}}{e_q(1-\sqrt{1-2t})-1}\right)^{\alpha} = \sum_{n=0}^{\infty} {}_{B}p_{n,q}^{(\alpha)}\frac{t^n}{[n]_q!}, \tag{2.14}$$

The determinant form for the $2D$ *q-Bernoulli polynomials of Bessel type is given by*

$${}_{B}p_{n+1,q}^{(\alpha)}\left(\tfrac{1}{x}, \tfrac{1}{y}\right) = \frac{(-1)^n}{\prod\limits_{i=0}^{n} b_{i,i,q}^{-1}\ B_{n+1,q}^{(\alpha)}(x,y)} \begin{vmatrix} 1 & B_{1,q}^{(\alpha)}(x,y) & \cdots & B_{n-1,q}^{(\alpha)}(x,y) & B_{n,q}^{(\alpha)}(x,y) \\ b_{0,0;q}^{-1} & b_{1,0;q}^{-1} & \cdots & b_{n-1,0;q}^{-1} & b_{n,0;q}^{-1} \\ 0 & b_{1,1;q}^{-1} & \cdots & b_{n-1,1;q}^{-1} & b_{n,1;q}^{-1} \\ 0 & 0 & \cdots & b_{n-1,2;q}^{-1} & b_{n,2;q}^{-1} \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & b_{n-1,n-1;q}^{-1} & b_{n,n-1;q}^{-1} \end{vmatrix}. \tag{2.15}$$

Remark 2.5. *For the choice of*

$$A_q(1-\sqrt{1-2t}) = e_q(1-\sqrt{1-2t}),$$

we have the 2D q-Roger-Szegö polynomials of Bessel type (2DqRSPoBT) defined by

$$e_q(1-\sqrt{1-2t})\ e_q(x(1-\sqrt{1-2t}))\ E_q(y(1-\sqrt{1-2t})) = \sum_{n=0}^{\infty} {}_hp_{n,q}(x,y)\frac{t^n}{[n]_q!}, \tag{2.16}$$

which for $x = y = 0$ gives

$$e_q(1-\sqrt{1-2t}) = \sum_{n=0}^{\infty} {}_hp_{n,q}\frac{t^n}{[n]_q!}. \tag{2.17}$$

The determinant form for the 2D q-Roger-Szegö polynomials of Bessel type is given by

$${}_hp_{n+1,q}\left(\frac{1}{x},\frac{1}{y}\right) = \frac{(-1)^n}{\prod_{i=0}^{n} b_{i,i,q}^{-1}\ h_{n+1,q}(x,y)} \begin{vmatrix} 1 & h_{1,q}(x,y) & \cdots & h_{n-1,q}(x,y) & h_{n,q}(x,y) \\ b_{0,0;q}^{-1} & b_{1,0;q}^{-1} & \cdots & b_{n-1,0;q}^{-1} & b_{n,0;q}^{-1} \\ 0 & b_{1,1;q}^{-1} & \cdots & b_{n-1,1;q}^{-1} & b_{n,1;q}^{-1} \\ 0 & 0 & \cdots & b_{n-1,2;q}^{-1} & b_{n,2;q}^{-1} \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & b_{n-1,n-1;q}^{-1} & b_{n,n-1;q}^{-1} \end{vmatrix}. \tag{2.18}$$

Remark 2.6. *For the choice of*

$$A_q(1-\sqrt{1-2t}) = \frac{1}{e_q(1-\sqrt{1-2t})e_q(a(1-\sqrt{1-2t}))},$$

we have the 2D q-Al-Salam Carlitz polynomials of Bessel type (2DqACPoBT) defined by

$$\frac{e_q(x(1-\sqrt{1-2t}))}{e_q(1-\sqrt{1-2t})e_q(a(1-\sqrt{1-2t}))}\ E_q(y(1-\sqrt{1-2t})) = \sum_{n=0}^{\infty} {}_Up_{n,q}^{(a)}(x,y)\frac{t^n}{[n]_q!}, \tag{2.19}$$

which for $x = y = 0$ gives

$$\frac{1}{e_q(1-\sqrt{1-2t})e_q(a(1-\sqrt{1-2t}))} = \sum_{n=0}^{\infty} {}_Up_{n,q}^{(a)}\frac{t^n}{[n]_q!}. \tag{2.20}$$

The determinant form for the 2D q-Al-Salam Carlitz polynomials of Bessel type is given by

$${}_Up_{n+1,q}^{(a)}\left(\frac{1}{x},\frac{1}{y}\right) = \frac{(-1)^n}{\prod_{i=0}^{n} b_{i,i,q}^{-1}\ U_{n+1,q}^{(a)}(x,y)} \begin{vmatrix} 1 & U_{1,q}^{(a)}(x,y) & \cdots & U_{n-1,q}^{(a)}(x,y) & U_{n,q}^{(a)}(x,y) \\ b_{0,0;q}^{-1} & b_{1,0;q}^{-1} & \cdots & b_{n-1,0;q}^{-1} & b_{n,0;q}^{-1} \\ 0 & b_{1,1;q}^{-1} & \cdots & b_{n-1,1;q}^{-1} & b_{n,1;q}^{-1} \\ 0 & 0 & \cdots & b_{n-1,2;q}^{-1} & b_{n,2;q}^{-1} \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & b_{n-1,n-1;q}^{-1} & b_{n,n-1;q}^{-1} \end{vmatrix}. \tag{2.21}$$

In the next section, we establish the q-addition formulas for the members belonging to the $2D$ q-Appell polynomials of Bessel type.

3. q-addition formulas

To prove some q-addition formulas for the $2D$ q-Bernoulli, $2D$ q-Roger-Szegö and $2D$ q-Al-Salam Carlitz polynomials of Bessel type, we prove the following results:

Theorem 3.1. *For the* $2D$ *q-Bernoulli polynomials of Bessel type, the following q-addition formula holds:*

$$ {}_Bp_{n,q}^{(\alpha+\beta)}(x+z,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q {}_Bp_{k,q}^{(\alpha)}(x,y)\, {}_Bp_{n-k,q}^{(\beta)}(z,0). \tag{3.1} $$

Proof. In view of equation (2.13), we have

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_Bp_{n,q}^{(\alpha+\beta)}(x+z,y)\frac{t^n}{[n]_q!} &= \left(\frac{1-\sqrt{1-2t}}{e_q(1-\sqrt{1-2t})-1}\right)^{\alpha+\beta} e_q(x+z(1-\sqrt{1-2t}))\ E_q(y(1-\sqrt{1-2t})) \\ &= \left(\sum_{n=0}^{\infty} {}_Bp_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}\right)\left(\sum_{n=0}^{\infty} {}_Bp_{n,q}^{(\beta)}(z,0)\frac{t^n}{[n]_q!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q {}_Bp_{k,q}^{(\alpha)}(x,y)\, {}_Bp_{n-k,q}^{(\beta)}(z,0)\right)\frac{t^n}{[n]_q!}, \end{aligned} $$

which proves assertion (3.1). □

Theorem 3.2. *For the* $2D$ *q-Bernoulli polynomials of Bessel type, the following formula holds:*

$$ {}_Bp_{n,q}^{(\alpha)}(x,y) = \sum_{l=0}^{n}\sum_{s=0}^{l} \begin{bmatrix} n \\ l \end{bmatrix}_q \begin{bmatrix} l \\ s \end{bmatrix}_q q^{s(s-1)/2}\, {}_Bp_{n-l,q}^{(\alpha)}\ p_{l-s,q}(x)\ \mathcal{P}_{s,q}(y). \tag{3.2} $$

Proof. By making use of equations (2.14), (1.16) and (1.17) in (2.13), it follows that

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_Bp_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!} &= \left(\sum_{n=0}^{\infty} {}_Bp_{n,q}^{(\alpha)}\frac{t^n}{[n]_q!}\right)\left(\sum_{l=0}^{\infty} p_{l,q}(x)\frac{t^l}{[l]_q!}\right)\left(\sum_{s=0}^{\infty} q^{s(s-1)/2}\mathcal{P}_{s,q}(y)\frac{t^s}{[s]_q!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{s=0}^{l} \begin{bmatrix} n \\ l \end{bmatrix}_q \begin{bmatrix} l \\ s \end{bmatrix}_q q^{s(s-1)/2}\, {}_Bp_{n-l,q}^{(\alpha)} p_{l-s,q}(x)\mathcal{P}_{s,q}(y)\right)\frac{t^n}{[n]_q!}, \end{aligned} $$

which yields formula (3.2). □

Theorem 3.3. *For the* $2D$ *q-Roger-Szegö polynomials of Bessel type, the following formulas hold:*

$$ {}_hp_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{(n-k)(n-k-1)}{2}}\, {}_hp_{k,q}(x,0)\mathcal{P}_{n-k,q}(y), \tag{3.3} $$

$$ {}_hp_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q p_{k,q}(x,y)p_{n-k,q}(1). \tag{3.4} $$

Proof. Combined use of equation (1.17) with (2.16) gives

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_hp_{n,q}(x,y)\frac{t^n}{[n]_q!} &= \left(\sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}}\ \mathcal{P}_{n,q}(y)\frac{t^n}{[n]_q!}\right)\left(\sum_{n=0}^{\infty} {}_hp_{n,q}(x,0)\frac{t^n}{[n]_q!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{(n-k)(n-k-1)}{2}}\, {}_hp_{k,q}(x,0)\mathcal{P}_{n-k,q}(y)\right)\frac{t^n}{[n]_q!}, \end{aligned} $$

which proves assertion (3.3). The proof for assertion (3.4) can be easily done. Thus, we omit it. □

Theorem 3.4. *For the 2D q-Al-Salam Carlitz polynomials of Bessel type, the following formulas hold:*

$$ {}_{U}p_{n,q}^{(a)}(x-1,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \, {}_{U}p_{k,q}^{(a)}(x) p_{n-k,q}(-1,y), \tag{3.5} $$

$$ p_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \, {}_{U}p_{k,q}^{(a)}(x,y) h_{n-k,q}(a). \tag{3.6} $$

Proof. In view of equations (1.15) and (2.19), we have

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_{U}p_{n,q}^{(a)}(x-1,y) \frac{t^n}{[n]_q!} &= \left(\sum_{n=0}^{\infty} p_{n,q}(-1,y) \frac{t^n}{[n]_q!} \right) \left(\sum_{n=0}^{\infty} {}_{U}p_{n,q}^{(a)}(x,0) \frac{t^n}{[n]_q!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \, {}_{U}p_{k,q}^{(a)}(x,0) p_{n-k,q}(-1,y) \right) \frac{t^n}{[n]_q!}, \end{aligned} $$

which proves assertion (3.5).
Similarly, we have

$$ \begin{aligned} \sum_{n=0}^{\infty} p_{n,q}(x,y) \frac{t^n}{[n]_q!} &= \left(\sum_{n=0}^{\infty} h_{n,q}(a) \frac{t^n}{[n]_q!} \right) \left(\sum_{n=0}^{\infty} {}_{U}p_{n,q}^{(a)}(x,y) \frac{t^n}{[n]_q!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \, {}_{U}p_{k,q}^{(a)}(x,y) h_{n-k,q}(a) \right) \frac{t^n}{[n]_q!}, \end{aligned} $$

which gives formula (3.6). □

4. Concluding remarks

The convolution can be defined for functions on groups other than Euclidean space. For example, periodic functions, such as the discrete-time Fourier transform, can be defined on a circle and convolved by periodic convolution. A discrete convolution can be defined for functions on the set of integers. It has applications that include probability, statistics, computer vision, natural language processing, image and signal processing, engineering and differential equations, numerical analysis and numerical linear algebra *et cetera*. It will be interesting to define the following polynomials from the convolution aspect:

The $2D$ q-Appell polynomials of Bessel type are defined as the discrete q-Bessel convolution of the $2D$ q-Appell polynomials given by

$$ {}_{A}p_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q {}_{A}p_{n-k,q}(0,y) \; p_{k,q}(x), \tag{4.1} $$

or,

$$ {}_{A}p_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q {}_{A}p_{n-k,q} \; \mathcal{P}_{k,q}(x,y). \tag{4.2} $$

The series representation for the $2D$ q-Appell polynomials of Bessel type is given by

$$ {}_{A}p_{n,q}(x,y) = \sum_{k=0}^{n-1} \frac{[n-1+k]_q!}{[n-1-k]_q! \; [k]_q!} \frac{A_{n-k,q}(x,y)}{2^k}, \tag{4.3} $$

which in view of following expansion:

$$ A_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} A_{n-k,q}(x) \; y^k \tag{4.4} $$

becomes

$$ {}_{A}p_{n,q}(x,y)=\sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\begin{bmatrix} n-k \\ l \end{bmatrix}_q q^{l(l-1)/2}\frac{[n-1+k]_q!}{[n-1-k]_q!\,[k]_q!}\frac{A_{n-k-l,q}(x)y^l}{2^k}. \tag{4.5} $$

Next, we find the series representations for the 2DqBPoBT, 2DqRSPoBT and 2DqACPoBT. These are given in Table 2.

Table 2: Series expansions for the 2DqBPoBT, 2DqRSPoBT and 2DqACPoBT

S.No.	**Name of the q-polynomial**	**Series expansions**
I.	2DqBPoBT	${}_{B}p^{(\alpha)}_{n,q}(x,y)=\sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\begin{bmatrix} n-k \\ l \end{bmatrix}_q q^{l(l-1)/2}\frac{[n-1+k]_q!}{[n-1-k]_q!\,[k]_q!}\frac{B^{(\alpha)}_{n-k-l,q}(x)y^l}{2^k}$
II.	2DqRSPoBT	${}_{h}p_{n,q}(x,y)=\sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\begin{bmatrix} n-k \\ l \end{bmatrix}_q q^{l(l-1)/2}\frac{[n-1+k]_q!}{[n-1-k]_q!\,[k]_q!}\frac{h_{n-k-l,q}(x)y^l}{2^k}$
III.	2DqACPoBT	${}_{U}p^{(a)}_{n,q}(x,y)=\sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\begin{bmatrix} n-k \\ l \end{bmatrix}_q q^{l(l-1)/2}\frac{[n-1+k]_q!}{[n-1-k]_q!\,[k]_q!}\frac{U^{(a)}_{n-k-l,q}(x)y^l}{2^k}$

The algebraic properties for the dense class of $2D$ q-Appell polynomials of Bessel and for their corresponding members, the $2D$ q-Roger Szegö polynomials and $2D$ q-Al-Salam Carlitz polynomials may shown to be helpful from numerical and computation point of views.

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