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Constructing and Enumerating of Magic Squares

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ABSTRACT: A magic square of order n, where n is a positive integer, is an $n \times n$ square table , say A, filled with distinct positive numbers $1, 2, \dots, n^2$ such that all cells of A are distinct and the sum of the numbers in each row, column and diagonal of A is equal. Let $M(n, s)$ be the set of all $n \times n$ matrices with entries 0 or 1, say T, such that the number of 1 in every row and every column of T is equal to s. In this paper we introduce a new method for constructing magic squares of order $4k$, where k is a positive integer. We show that the number of magic squares of order $4k$ is at least $|M(2k, k)|$. In particular, we prove that the number of magic squares of order $4k$ is at least $\frac{1}{2}\binom{2k}{k}^2$.

Key Words: Magic square, Bipartite graphs.

Contents

1. Introduction

Let $n \geq 1$ be an integer. A magic square of order n is an $n \times n$ square table, say A, filled with distinct positive numbers $1, 2, \dots, n^2$ such that all cells of A are distinct and the sum of the numbers in each row, column and diagonal of A is equal. Note that if A is a magic square, then the sum of the numbers in each row, column and diagonal of A is equal to

$$\frac{1 + 2 + \cdots + n^2}{n} = \frac{n(n^2 + 1)}{2}.$$

More precisely, let $A = [a_{i,j}]$ be an $n \times n$ matrix such that for $1 \leq i, j \leq n$, $a_{i,j} \in \{1, 2, \dots, n^2\}$. Then A is a magic square of order n if and only if all $a_{i,j}$ are distinct and for every $r, s \in \{1, 2, \dots, n\}$,

$$\sum_{j=1}^{n} a_{r,j} = \sum_{i=1}^{n} a_{i,s} = \sum_{i=1}^{n} a_{i,i} = \sum_{i=1}^{n} a_{i,n-i+1} = \frac{n(n^2 + 1)}{2}.$$

It is well known that for every positive integer $n \neq 2$ there exists at least one magic square of order n. The study of magic squares has a long history. Magic squares have been the subject of interest among mathematicians for several centuries because of its magical properties. For more details on this topic and its applications see [1], [3], [4] and the references therein. The tables A and B are two magic squares of order 3 and 4, respectively.

$A =$

8	1	6
3	5	7
4	9	2

$B =$

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

A $(0, 1)$-*matrix* is a matrix whose entries are equal to 0 or 1. Let $M(n, s)$ be the set of all $n \times n$ $(0, 1)$-matrices, say T, such that the number of 1 in every row and column of T is equal to s. In this

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paper first we find a new method for constructing magic squares of order $4k$ where k is a positive integer. Finally, we show that the number of magic squares of order $4k$ is at least $|M(2k,k)|$ (where $|A|$ is the cardinality of the set A).

2. Constructing and enumerating magic squares

In this section for every positive integer k we construct $|M(2k,k)|$ magic squares of order $4k$. For every positive integer n, let $A_n = [a_{i,j}]$ be the $n \times n$ square table such that for $1 \le i,j \le n$, $a_{i,j} = (i-1)n + j$. For example A_4 is the following:

$A_4 =$

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Now for every matrix of $M(2k,k)$ we construct a magic square of order $4k$ using A_{4k} as follows. Let $E = [e_{i,j}] \in M(2k,k)$ and $n = 4k$. Let $\Phi(E)$ be the $n \times n$ square table that is obtained form A_n by the following two rules.

(i) If $e_{i,j} = 1$, then substitute $a_{i,j}$ with $a_{n+1-i,j}$ and substitute $a_{i,n+1-j}$ with $a_{n+1-i,n+1-j}$.

(ii) If $e_{i,j} = 0$, then substitute $a_{i,j}$ with $a_{i,n+1-j}$ and substitute $a_{n+1-i,j}$ with $a_{n+1-i,n+1-j}$.

For example let $k = 1$ (so $n = 4$) and $E = \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array}$. Since $e_{1,1} = 1$, so we replace $a_{1,1}$ with $a_{4,1}$ (and $a_{4,1}$ with $a_{1,1}$) and replace $a_{1,4}$ with $a_{4,4}$ (and $a_{4,4}$ with $a_{1,4}$). In fact

$$A_4 = \begin{array}{|c|c|c|c|}\hline a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ \hline a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ \hline a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ \hline a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \\ \hline \end{array} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \\ \hline 13 & 14 & 15 & 16 \\ \hline \end{array}$$

and

$$\Phi(E) = \begin{array}{|c|c|c|c|}\hline a_{4,1} & a_{1,3} & a_{1,2} & a_{4,4} \\ \hline a_{2,4} & a_{3,2} & a_{3,3} & a_{2,1} \\ \hline a_{3,4} & a_{2,2} & a_{2,3} & a_{3,1} \\ \hline a_{1,1} & a_{4,3} & a_{4,2} & a_{1,4} \\ \hline \end{array} = \begin{array}{|c|c|c|c|}\hline 13 & 3 & 2 & 16 \\ \hline 8 & 10 & 11 & 5 \\ \hline 12 & 6 & 7 & 9 \\ \hline 1 & 15 & 14 & 4 \\ \hline \end{array}$$

Now we prove that $\Phi(E)$ is a magic square of order n. For $1 \le i,j \le n$, let R_i and C_j be the summation of all entries of i^{th} row and j^{th} column of A_n, respectively. In other words $R_i = \sum_{k=1}^{n} a_{i,k}$ and $C_j = \sum_{k=1}^{n} a_{k,j}$. Since for $1 \le i,j \le n$, $a_{i,j} = (i-1)n + j$ we obtain that

$$R_i = (i-1)n^2 + \frac{n(n+1)}{2} \quad \text{and} \quad C_j = jn + \frac{n^2(n-1)}{2}.$$

Let R'_i and C'_j be the summation of all entries of the i^{th} row and j^{th} column of $\Phi(E)$, respectively. We show for every $1 \le i \le n$, $R'_i = C'_i = \frac{n(n^2+1)}{2}$. First we prove that for every $i \in \{1,\dots,n\}$, $R'_i = \frac{n(n^2+1)}{2}$. We note that by applying the rule (ii) the summation of the entries of any row does not change. Thus it remains to consider the effect of the first rule. Assume that $e_{i,j_1} = e_{i,j_2} = \cdots = e_{i,j_k} = 1$ where $j_1 < j_2 < \cdots < j_k$. Hence the entries of the i^{th} row of $\Phi(E)$ is

$$a_{i,1},\dots,a_{i,j_1-1},a_{n+1-i,j_1},a_{i,j_1+1},\dots,a_{i,j_2-1},a_{n+i-1,j_2},a_{i,j_2+1},\dots,a_{i,j_k-1},a_{n+1-i,j_k},a_{i,j_k+1},\dots,a_{i,n}.$$

This shows that $R'_i = R_i + \sum_{t=1}^{k}(a_{n+1-i,j_t} - a_{i,j_t})$. Since for every $1 \le r, s \le n$, $a_{r,s} = (r-1)n + s$ we obtain that $R'_i = \frac{n(n^2+1)}{2}$. Similarly one can see that for every $i \in \{1, \ldots, n\}$, $C'_i = \frac{n(n^2+1)}{2}$.

Now we investigate the summation of entries of the diagonals (main and secondary diagonals) of $\Phi(E)$. First note that the summation of all entries of the diagonals of A_n is $\frac{n(n^2+1)}{2}$. In other words $a_{1,1} + a_{2,2} + \cdots + a_{n,n} = a_{1,n} + a_{2,n-1} + \cdots + a_{n,1} = \frac{n(n^2+1)}{2}$. Assume that $e_{r_1,r_1} = \cdots = e_{r_p,r_p} = 0$ and $e_{s_1,s_1} = \cdots = e_{s_q,s_q} = 1$. Hence the numbers of the main diagonal of $\Phi(E)$ are $a_{r_1,n+1-r_1}, \ldots, a_{r_p,n+1-r_p}$, $a_{n+1-r_1,r_1}, \ldots, a_{n+1-r_p,r_p}$, $a_{n+1-s_1,s_1}, \ldots, a_{n+1-s_q,s_q}$ and $a_{s_1,n+1-s_1}, \ldots, a_{s_q,n+1-s_q}$. This shows that the summation of all entries of the main diagonal of $\Phi(E)$ is $\frac{n(n^2+1)}{2}$ (note that $p + q = \frac{n}{2} = 2k$). Similarly one can see that the summation of all entries of the secondary diagonal of $\Phi(E)$ is also $\frac{n(n^2+1)}{2}$.

For example we construct a magic square of order 8. Let

$$L = \begin{array}{|c|c|c|c|} \hline 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ \hline \end{array}$$

By applying L on A_8 we obtain that

$$\Phi(L) = \begin{array}{|r|r|r|r|r|r|r|r|} \hline 57 & 7 & 59 & 5 & 4 & 62 & 2 & 64 \\ \hline 16 & 50 & 14 & 52 & 53 & 11 & 55 & 9 \\ \hline 41 & 23 & 43 & 21 & 20 & 46 & 18 & 48 \\ \hline 32 & 34 & 30 & 36 & 37 & 27 & 39 & 25 \\ \hline 40 & 26 & 38 & 28 & 29 & 35 & 31 & 33 \\ \hline 17 & 47 & 19 & 45 & 44 & 22 & 42 & 24 \\ \hline 56 & 10 & 54 & 12 & 13 & 51 & 15 & 49 \\ \hline 1 & 63 & 3 & 61 & 60 & 6 & 58 & 8 \\ \hline \end{array}$$

The summation of the numbers of every row, column and diagonal of $\Phi(L)$ is $\frac{8(64+1)}{2} = 260$.

Remark 2.1. *As we explained above, by every matrix of $M(2k, k)$ we can construct a magic square of order $4k$. This shows that the number of magic squares of order $4k$ is at least the cardinality of the set $M(2k, k)$.*

Let $G = (V, E)$ be a simple graph. For a vertex v of G, the *degree* of v is the number of edges incident with v. A k–regular graph is a graph such that every vertex of that has degree k. For two disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *disjoint union* of G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. An *independent* set S of G is a subset of vertices of G such that there is no edge between every two vertices of S. A *bipartite graph* is a graph, say G, whose vertices can be divided into two disjoint and independent sets, say X and Y, such that every edge of G has an end point vertex in X and an end point vertex in Y. By $K_{m,n}$ we mean the complete bipartite graph with part sizes m and n. To see the relevant definitions related to graph theory see [2]. In the next result we find a lower bound for the number of magic squares of order $4k$. Our method shows that for every positive integer k there exist at least $\frac{\binom{2k}{k}^2}{2}$ magic squares of order $4k$.

Theorem 2.2. *For every positive integer k, the number of magic squares of order $4k$ is at least $\frac{\binom{2k}{k}^2}{2}$.*

Proof. Since (as we mentioned before) by every matrix in $M(2k, k)$ we can construct a magic square of order $4k$, to complete the proof it suffices to show that $|M(2k, k)| \ge \frac{\binom{2k}{k}^2}{2}$. Let $X = \{x_1, \ldots, x_{2k}\}$ and $Y = \{y_1, \ldots, y_{2k}\}$. Let $\mathcal{H}$ be the set of all k-regular bipartite graphs G with parts X and Y. In other words, $\mathcal{H}$ is the set of all k-regular graphs G with vertex set $X \cup Y$ such that X and Y are independent sets. We claim that the cardinality of $\mathcal{H}$ is at least $\frac{\binom{2k}{k}^2}{2}$.

For instance $K_{k,k} \cup K_{k,k}$ (the disjoint union of two complete bipartite graphs) is one the graphs belonging to $\mathcal{H}$. To prove the claim consider a subset $A \subseteq X$ with $|A| = k$ and a subset $B \subseteq Y$ with $|B| = k$. Then consider the disjoint union of two complete bipartite graphs; one of them construct with parts A and B and the other construct with parts $X \setminus A$ and $Y \setminus B$. By this method we can construct $\frac{\binom{2k}{k}^2}{2}$ graphs belonging to $\mathcal{H}$ (note that these graphs are isomorphic to $K_{k,k} \cup K_{k,k}$). Therefore the claim is proved.

Let G be a k-regular bipartite graph with parts X and Y, that is $G \in \mathcal{H}$. Let $C(G) = [c_{i,j}]$ be the $2k \times 2k$ $(0,1)-$matrix such that $c_{i,j} = 1$ if and only if x_i and y_j are adjacent and $c_{i,j} = 0$, otherwise. Obviously $C(G) \in M(2k,k)$. This shows that $|M(2k,k)| \geq |\mathcal{H}|$. On the other hand $|\mathcal{H}| \geq \frac{\binom{2k}{k}^2}{2}$. Hence $|M(2k,k)| \geq \frac{\binom{2k}{k}^2}{2}$. $\square$

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