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Complexity Analysis of Interior Point Methods for Convex Quadratic Programming Based on a Parameterized Kernel Function *

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ABSTRACT: The kernel functions play an important role in the amelioration of the computational complexity of algorithms. In this paper, we present a primal-dual interior-point algorithm for solving convex quadratic programming based on a new parametric kernel function. The proposed kernel function is not logarithmic and not self-regular. We analysis a large and small-update versions which are based on a new kernel function. We obtain the best known iteration bound for large-update methods, which improves significantly the so far obtained complexity results. This result is the first to reach this goal.

Key Words: Convex quadratic programming, Interior point methods, Kernel function, Iteration bound.

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1. Introduction

Convex quadratic programs (CQP) appear in many areas of applications, for example in finance, agriculture, economics, optimal control, geometric problems and also as sub-problems in sequential quadratic programming.

There are a variety of solution approaches for CQP which have been studied intensively. Among them, the interior point methods ($IPMs$) gained more attention than others methods. Feasible primal-dual path following methods are the most attractive methods of $IPMs$ [13,14]. Their derived algorithms achieved important results such as polynomial complexity and numerical efficiency. However, in practice these methods don't always find a strictly feasible centered point to starting their algorithms. So, it is worth

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analyzing other cases when the starting points are not centered. Thus leads to define a new technique which is bases on finding initial strictly feasible point not necessarily centered.

Primal-dual interior point methods based on a kernel function were studied extensively by many authors for linear optimization (LO). Bai et al. [1] presented a large class of eligible kernel functions, which is fairly general and includes the classical logarithmic functions and the self-regular functions, as well as many non-self-regular functions as special cases. For some other related kernel function, we refer to [3,4,5,6,7,8,9,12].

In 2001, Peng et al. [9] introduced a new paradigm for primal-dual interior-point algorithms for LO, which has $O\left(qn^{\frac{q+1}{2q}}\log\frac{n}{\varepsilon}\right)$ complexity for large-update method with $q > 1$.

In 2002, Bai et al. [2] proposed a new parametric kernel function for LO, which has $O\left(qn\log\frac{n}{\varepsilon}\right)$ complexity for large-update method with $q > 1$.

In this paper, we propose a primal-dual interior-point method for solving CQP based on a new parametric kernel function, this function is used for determining the new search directions and for measuring the distance between the given iterate and the center. We present some complexity results for the generic algorithm and prove that the bound for large-update methods enjoys $O\left(pn^{\frac{p+3}{2(p+2)}}\log\frac{n}{\varepsilon}\right)$ if $p \geq 1$ and $O\left(n^{\frac{3}{4}}\log\frac{n}{\varepsilon}\right)$ if $p \in]0,1]$.

The paper is organized as follows: In section 2, the statement of the problem is presented and we recall the basic concepts of *IPMs* for CQP. Section 3 contains some properties of the kernel functions. An analysis of interior-point algorithm is described in section 4 as well as several properties and the growth behavior of the barrier function, the estimate of the step size and the decrease in behavior of barrier function. We derive the complexity bound of the algorithm in section 5. In section 6, we present a new kernel function and its properties. Section 7 contains some numerical experimentations and commentaries. In section 8, a conclusion is stated.

The following notations are used throughout the paper. Let $\Re^n$ be the n-dimensional Euclidean space with the inner product $\langle .,. \rangle$ and $\|.\|$ 2-norm. $\Re^n_+$ and $\Re^n_{++}$ denote the nonnegative orthant and positive orthant, respectively. For $x, z \in \Re^n$, $x_{\min}$ and $x_i z_i$ denote the smallest component of the vector x and the component wise product of the vector x and z, respectively. $X = diag(x)$ denotes a diagonal matrix that has components of the vector $x \in \Re^n$, e denotes the n-dimensional vector of ones. For $f, g : \Re^n_{++} \to \Re^n_{++}$, $f(x) = O(g(x))$ if $f(x) \leq C_1 g(x)$ for some positive constant C_1 and $f(x) = \Theta(g(x))$ if $C_2 g(x) \leq f(x) \leq C_3 g(x)$ for some positive constants C_2 and C_3.

2. Preliminaries

We consider the standard primal convex quadratic programming problem

$$\min\{\langle c, x\rangle + \frac{1}{2}\langle x, Qx\rangle : Ax = b, x \geq 0\}, \tag{P}$$

where Q is a given $n \times n$ real symmetric positive semidefinite matrix, A is a given $m \times n$ real matrix, $c \in \Re^n, b \in \Re^m, x \in \Re^n$. The dual problem of ($P$) can be formulated as

$$\max\{\langle b, y\rangle - \frac{1}{2}\langle x, Qx\rangle : A^t y + z - Qx = c, z \geq 0\}, \tag{D}$$

where $z \in \Re^n$ and $y \in \Re^m$.

2.1. Central path for CQP

Throughout the paper, we make the following assumptions:

(H1) The matrix A has full row rank ($rank(A) = m < n$).

(H2) (P) and (D) satisfy the interior-point condition (IPC), i.e., there exist $\left(x^0, y^0, z^0\right)$ such that:

$$Ax^0 = b, x^0 > 0, A^t y^0 + z^0 - Qx^0 = c, z^0 > 0. \tag{2.1}$$

It is well known that finding an optimal solution for (P) and (D) is equivalent to solve the nonlinear equations:

$$Ax = b, x \geq 0, A^t y + z - Qx = c, z \geq 0, xz = 0. \tag{2.2}$$

The basic idea of primal-dual *IPMs* is to replace the complementarity condition $xz = 0$ in (2.2) by the parameterized equation $xz = \mu e$, one obtains the following perturbed system:

$$Ax = b, x \geq 0, A^t y + z - Qx = c, z \geq 0, xz = \mu e, \tag{2.3}$$

where μ is a positive parameter. It is shown that, under our assumptions the system (2.3) has a unique solution $(x(\mu), y(\mu), z(\mu))$, for each $\mu > 0$. $x(\mu)$ and $(y(\mu), z(\mu))$ are called the μ-center of (P) and (D), respectively. The set of all μ-centers forms the so called central path for (P) and (D).

The principal idea of *IPMs* is to follow this central path and approach the optimal set of *CQP* as μ goes to zero.

From a theoretical point of view, the *IPC* can be assumed without loss of generality. In fact, we may assume that $\mu^0 = 1, x(1) = z(1) = e$ to simplify the theoretical contributions see [13].

2.2. The search directions determined by kernel function

Applying Newton's method in (2.3) for a given feasible point (x, y, z) then the Newton's direction $(\Delta x, \Delta y, \Delta z)$ at this point is the unique solution of the following linear system of equations:

$$A\Delta x = 0, A^t \Delta y + \Delta z - Q\Delta x = 0, z\Delta x + x\Delta z = \mu e - xz. \tag{2.4}$$

In this paper we follow [3], to reformulate the Newton's direction search in a different way. Let's introduce the following notation:

$$v = \sqrt{\frac{xz}{\mu}}, \qquad d = \sqrt{\frac{x}{z}}.$$

Note that if x is primal feasible and z is dual feasible then the pair (x, z) coincides with the μ-center $(x(\mu), z(\mu))$ if and only if $v = e$. And defining the scaled search directions d_x and d_z according to:

$$d_x = \frac{v\Delta x}{x}, \qquad d_z = \frac{v\Delta z}{z}. \tag{2.5}$$

System (2.4) can be rewritten as follows:

$$\bar{A}d_x = 0, \bar{A}^t \Delta y + d_z - \bar{Q}d_x = 0, d_x + d_z = v^{-1} - v, \tag{2.6}$$

where $\bar{A} = \frac{1}{\sqrt{\mu}} AD, \bar{Q} = DQD$ with $D = diag(d)$.

It is not difficult to verify that the right-hand side of the third equation in (2.6) equals minus the derivative of the classical logarithmic barrier function
$\Phi(v) : \Re^n_{++} \to \Re_+$ is defined as follows:

$$\Phi(v) = \Phi(x, z; \mu) = \sum_{i=1}^{n} \psi(v_i), \psi(v_i) = \frac{v_i^2 - 1}{2} - \log v_i. \tag{2.7}$$

Moreover, we call ψ the kernel function of the logarithmic barrier function $\Phi(v)$. The system (2.6) can be rewritten as follows:

$$\bar{A}d_x = 0, \bar{A}^t \Delta y + d_z - \bar{Q}d_x = 0, d_x + d_z = -\nabla\Phi(v). \tag{2.8}$$

We use $\Phi(v)$ as the proximity function to measure the distance between the current iterate and the μ-center for given $\mu > 0$. We also define the norm-based proximity measure, $\delta(v) : \Re^n_{++} \to \Re_+$, as follows:

$$\delta(v) = \frac{1}{2}\|\nabla\Phi(v)\| = \frac{1}{2}\|d_x + d_z\|. \tag{2.9}$$

The result of a Newton step with step size α is denoted as

$$x^+ = x + \alpha\Delta x, \qquad y^+ = y + \alpha\Delta y, \qquad z^+ = z + \alpha\Delta z, \tag{2.10}$$

where α satisfies $0 < \alpha \leq 1$.

2.3. The generic interior-point algorithm for CQP

It is clear from the above description that the closeness of (x,z) to $(x(\mu),z(\mu))$ is measured by the value of $\Phi(v)$ with $\tau>0$ as a threshold value. If $\Phi(v)\le\tau$, then we start a new outer iteration by performing a μ-update; otherwise, we enter an inner iteration by computing the search directions at the current iterates with respect to the current value of μ and apply (2.10) to get new iterates. If necessary, we repeat the procedure until we find iterates that are in the neighborhood of $(x(\mu),z(\mu))$. Then μ is again reduced by the factor $1-\theta$ with $0<\theta<1$, and we apply Newton's method targeting the new μ-centers, and so on. This process is repeated until μ is small enough, say until $n\mu\le\varepsilon$. At this stage, we have found an ε-approximate solution of CQP.

The parameters τ,θ and the step size α should be chosen in such a way that the algorithm is optimized in the sense that the number of iterations required by algorithm is as small as possible. Now, we give the generic form of the algorithm.

Generic primal-dual interior point algorithm for CQP

Input:
a proximity function $\Phi(v)$;
a threshold parameter $\tau>1$;
an accuracy parameter $\varepsilon>0$;
a barrier update parameter θ, $0<\theta<1$;
begin
$x=e, z=e, \mu=1$;
while $n\mu\ge\varepsilon$ do
begin (*outer iteration*)
  $\mu=(1-\theta)\mu$;
 while $\Phi(x,z,\mu)>\tau$ *do*
  begin (*inner iteration*)
   solve the system (2.8) *via* (2.5) *to obtain* $(\Delta x,\Delta y,\Delta z)$;
   compute the step size α *and put*:
   $x=x+\alpha\Delta x, y=y+\alpha\Delta y, z=z+\alpha\Delta z$;
   $v=\sqrt{\frac{xz}{\mu}}$;
  end
end
end.

---

3. Kernel functions and its properties

We call $\psi:\Re_{++}\to\Re_{+}$ a kernel function if ψ is twice differentiable and satisfies the following conditions:

$$\begin{aligned}\psi(1) &= \psi^{'}(1)=0,\\ \psi^{''}(t) &> 0,\forall t>0,\\ \lim_{t\to 0^+}\psi(t) &= \lim_{t\to+\infty}\psi(t)=+\infty.\end{aligned}$$

We call ψ eligible if and only if it satisfies the following conditions:

$$\psi^{'''}(t)<0,\ \forall t>0, \tag{3.1}$$

$$t\psi^{''}(t)-\psi^{'}(t)>0,\ \forall t>0, \tag{3.2}$$

$$t\psi^{''}(t)+\psi^{'}(t)>0,\ \forall t>0, \tag{3.3}$$

$$2\psi^{''}(t)^2-\psi^{'}(t)\,\psi^{'''}(t)>0, t<1. \tag{3.4}$$

$$\psi^{''}(t)\,\psi^{'}(\beta t)-\beta\psi^{'}(t)\,\psi^{''}(\beta t)>0,\ \forall\beta>1,\ \forall t>1. \tag{3.5}$$

The following lemma plays an important role in the analysis of the algorithm.

Lemma 3.1. *([10]) Given a function ψ that is twice differentiable, then the following properties are equivalent*

(i) $\psi(\sqrt{t_1t_2})\leq\frac{\psi(t_1)+\psi(t_2)}{2}$.

(ii) *the function ϕ defined by $\phi(\xi)=\psi(e^{\xi})$ is convex.*

(iii) $t\psi^{''}(t)+\psi^{'}(t)>0,\ t>0.$

4. Analysis of the interior-point algorithm for *CQP*

4.1. Upper bound of $\Phi(v)$ after each outer iteration

During the course of the algorithm the largest values of $\Phi(v)$ occur just after the updates of μ. In this subsection we derive an estimate for the effect of a μ-update on the value of $\Phi(v)$.

We offer important theorem, which is valid for all kernel functions that satisfy (3.5).

Theorem 4.1. *([1]) Let $\sigma:[0,+\infty[\to[1,+\infty[$ be the inverse function of $\psi(t)$ for $t\geq1$ which satisfies (3.5). Then we have:*

$$\Phi(\beta v)\leq n\psi(\beta\sigma(\frac{\Phi(v)}{n})),v\in\Re^n_{++},\ \beta>1.$$

Now let v be the variance vector of (x,z) with respect to μ. Then one easily understands that the variance vector v^+ of (x,z) with respect to $\mu^+=(1-\theta)\mu$ is given by $v^+=\frac{v}{\sqrt{1-\theta}}$.

By the theorem 4.1 with $\beta=\frac{1}{\sqrt{1-\theta}}>1$ and if $\Phi(v)\leq\tau$, we obtain:

$$\Phi(v^+)\ \leq\ n\psi\left(\frac{\sigma\left(\frac{\Phi(v)}{n}\right)}{\sqrt{1-\theta}}\right)\leq n\psi\left(\frac{\sigma\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}\right). \tag{4.1}$$

4.2. Decrease of the barrier function during an inner iteration

In this subsection, we compute a default step size α and the resulting decrease of the barrier function. After a damped step we have:

$$x^+=x+\alpha\Delta x,\qquad y^+=y+\alpha\Delta y,\qquad z^+=z+\alpha\Delta z.$$

Using (2.5), we obtain:

$$x^+=\frac{x}{v}\left(v+\alpha d_x\right),z^+=\frac{z}{v}\left(v+\alpha d_z\right).$$

So, we have:

$$v_+=\sqrt{\frac{x^+z^+}{\mu}}=\sqrt{\left(v+\alpha d_x\right)\left(v+\alpha d_z\right)}.$$

Define, for $\alpha>0,f(\alpha)=\Phi(v_+)-\Phi(v)$. Then $f(\alpha)$ is the difference of proximities between a new iterate and a current iterate for fixed μ. By (3.3) and Lemma 3.1, we have:

$$\Phi(v_+)=\Phi(\sqrt{\left(v+\alpha d_x\right)\left(v+\alpha d_z\right)})\leq\frac{1}{2}\left(\Phi\left(v+\alpha d_x\right)+\Phi\left(v+\alpha d_z\right)\right).$$

Therefore, we have:

$$f(\alpha)\leq f_1(\alpha)=\frac{1}{2}\left(\Phi\left(v+\alpha d_x\right)+\Phi\left(v+\alpha d_z\right)\right)-\Phi(v). \tag{4.2}$$

Obviously, $f(0) = f_1(0) = 0$. Taking the first two derivatives of $f_1(\alpha)$ with respect to α, we have:

$$f_1^{'}(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\left(\psi^{'}\left(v_i + \alpha\left[d_x\right]_i\right)\left[d_x\right]_i + \psi^{'}\left(v_i + \alpha\left[d_z\right]_i\right)\left[d_z\right]_i\right),$$

$$f_1^{''}(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\left(\psi^{''}\left(v_i + \alpha\left[d_x\right]_i\right)\left[d_x\right]_i^2 + \psi^{''}\left(v_i + \alpha\left[d_z\right]_i\right)\left[d_z\right]_i^2\right). \tag{4.3}$$

Using (2.8) and (2.9), we have

$$f_1^{'}(0) = \frac{1}{2}\left\langle \nabla\Phi(v), (d_x + d_z)\right\rangle = -\frac{1}{2}\left\langle \nabla\Phi(v), \nabla\Phi(v)\right\rangle = -2\left(\delta(v)\right)^2.$$

For convenience, we denote $v_{\min} = \min_i(v_i)$, $\delta = \delta(v)$ and $\Phi = \Phi(v)$.

The next lemma is valid for all kernel function that satisfy (3.1) which the same lemma in the *LO* case (see [1]).

Lemma 4.2. *Let $f_1(\alpha)$ be as defined in (4.2) and δ be as defined in (2.9). Then we have:*

$$f_1^{''}(\alpha) \leq 2\delta^2\psi^{''}\left(v_{\min} - 2\alpha\delta\right).$$

Proof. According to the system (2.8), we observe that

$$\left(d_x\right)^t d_z = d_x^t\left(\bar{Q}d_x - \bar{A}^t\Delta y\right) = d_x^t\bar{Q}d_x \geq 0,$$

this implies that

$$4\delta^2 = \left\|d_x + d_z\right\|^2 = \left\|d_x\right\|^2 + \left\|d_z\right\|^2 + \left(d_x\right)^t d_z \geq \left\|d_x\right\|^2 + \left\|d_z\right\|^2,$$

so $\|d_x\| \leq 2\delta, \|d_z\| \leq 2\delta$. We have:

$$v_i + \alpha\left[d_x\right]_i \geq v_{\min} - 2\alpha\delta, v_i + \alpha\left[d_z\right]_i \geq v_{\min} - 2\alpha\delta, i = 1,..,n.$$

According to (3.1) ($\psi^{''}$ is strictly decreasing) and (4.3), we obtain:

$$\begin{aligned} f_1^{''}(\alpha) &\leq \frac{1}{2}\psi^{''}\left(v_{\min} - 2\alpha\delta\right)\sum_{i=1}^{n}\left(\left[d_x\right]_i^2 + \left[d_z\right]_i^2\right), \\ &\leq 2\delta^2\psi^{''}\left(v_{\min} - 2\alpha\delta\right). \end{aligned}$$

□

From this stage on we can apply exactly the same arguments as in the *LO* case to obtain the following results which require no further proof.

Lemma 4.3. *([1]) If α satisfies the inequality*

$$-\psi^{'}\left(v_{\min} - 2\alpha\delta\right) + \psi^{'}\left(v_{\min}\right) \leq 2\delta, \tag{4.4}$$

then

$$f_1^{'}(\alpha) \leq 0.$$

Let $\rho : [0, +\infty[\to]0, 1]$ be the inverse function of $\frac{-1}{2}\psi^{'}(t)$ for all $t \in]0, 1]$, then we have the following lemma.

Lemma 4.4. *([1]) The largest step size $\bar{\alpha}$ holding (4.4) is given by*

$$\bar{\alpha} = \frac{\left(\rho(\delta) - \rho(2\delta)\right)}{2\delta}.$$

Lemma 4.5. *([1]) Let $\bar{\alpha}$ be as defined in Lemma 4.4. Then*

$$\bar{\alpha} \geq \frac{1}{\psi^{''}\left(\rho(2\delta)\right)}.$$

Lemma 4.6. *([1]) If the step size α satisfies $\alpha \leq \bar{\alpha}$, then $f(\alpha) \leq -\alpha\delta^2$.*

5. Iteration bound

We need to count how many inner iterations are required to return to the situation where $\Phi \leq \tau$. Let $(\Phi)_0$ is an upper bound for $\Phi(v^+)$ during the process of the algorithm, the subsequent values in the same outer iteration are denoted as $(\Phi)_k$, $k = 1, 2, .., K$, where K denotes the total number of inner iterations in the outer iteration.

According to Lemma 4.6 with $\alpha \leq \bar{\alpha}$, we have:

$$f(\alpha) = \Phi_{k+1} - \Phi_k \leq -\alpha\delta^2,$$

then we suppose that they exist $\bar{\kappa} > 0$ and $\gamma \in]0, 1]$, such as

$$\Phi_{k+1} - \Phi_k \leq -\bar{\kappa}\Phi_k^{1-\gamma}. \tag{5.1}$$

Lemma 5.1. *([10]) Let $t_0, t_1, .., t_K$ be a sequence of positive numbers that verifies:*

$$t_{k+1} \leq t_k - \bar{\kappa}t_k^{1-\gamma}, k = 0, 1, .., K-1,$$

such that $\bar{\kappa} > 0$ and $\gamma \in]0, 1]$, then:

$$K \leq \frac{(t_0)^\gamma}{\bar{\kappa}\gamma}.$$

According to (5.1) and using Lemma 5.1 for $t_k = \Phi_k - \tau > 0$ then K as follows:

$$K \leq \frac{((\Phi)_0 - \tau)^\gamma}{\bar{\kappa}\gamma} \leq \frac{(\Phi_0)^\gamma}{\bar{\kappa}\gamma}. \tag{5.2}$$

The number of outer iterations is bounded above by $\frac{\log \frac{n}{\varepsilon}}{\theta}$ (see [13]). Through multiplying the number of outer iterations by the number of inner iterations and according to (4.1), we get an upper bound for the total number of iterations, namely:

$$\frac{\left(n\psi\left(\frac{\sigma\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}\right)\right)^\gamma}{\theta\bar{\kappa}\gamma} \log\frac{n}{\epsilon} \tag{5.3}$$

Remark 5.2. *After the analysis of complexity and the inequality (5.3)(The same inequality in linear case see inequality 2.4.3 in [15]), it may be clear that for any given kernel function ψ in the CQP case will yield the same complexity results as in the LO case.*

We summarize the complexity result of large-update methods using some polynomial kernel functions in Table 1.

Table 1: Complexity result of large-update methods using some polynomial kernel functions

i	ψ_i	Large-update methods	reference
1	$t - 1 + \frac{t^{1-q}}{q-1}, q > 1$	$O\left(qn \log \frac{n}{\varepsilon}\right)$	[9]
2	$\frac{t^2-1}{2} + \frac{t^{1-q}-1}{q-1}, q > 1$	$O\left(qn^{\frac{q+1}{2q}} \log \frac{n}{\varepsilon}\right)$	[2]

6. New kernel function

We define a new kernel function $\psi(t)$ as follows:

$$\psi(t) = t^2 - t + \frac{t^{-(p+1)} - 1}{p+1}, \quad p > 0. \tag{6.1}$$

6.1. Eligibility of the new kernel

We give the first three derivatives with respect to t as follows:

$$\psi^{'}(t) = 2t - 1 - t^{-(p+2)}, \tag{6.2}$$

$$\psi^{''}(t) = 2 + (p+2)t^{-(p+3)}, \tag{6.3}$$

$$\psi^{'''}(t) = -(p+2)(p+3)t^{-(p+4)}. \tag{6.4}$$

Obviously, we have:

$$\psi^{''}(t) > 2 > 0, \ \forall t > 0, \tag{6.5}$$

$$\psi(1) = \psi^{'}(1) = 0. \tag{6.6}$$

Next lemma serves to prove some properties of eligibility of our new kernel function (6.1).

Lemma 6.1. *Let $\psi(t)$ be as defined in (6.1). Then,*

$$\psi^{'''}(t) < 0, \ \forall t > 0, \tag{6.7}$$

$$t\psi^{''}(t) - \psi^{'}(t) > 0, \ \forall t > 0, \tag{6.8}$$

$$t\psi^{''}(t) + \psi^{'}(t) > 0, \ \forall t > 0, \tag{6.9}$$

$$\psi^{''}(t)\,\psi^{'}(\beta t) - \beta\psi^{'}(t)\,\psi^{''}(\beta t) > 0, \ \forall \beta > 1, \ \forall t > 1. \tag{6.10}$$

Proof: For (6.7), using (6.4), we obtain $\psi^{'''}(t) < 0, \forall t > 0$. For (6.8) and (6.9), we use (6.2), (6.3), the positivity of t and p, we obtain:

$$t\psi^{''}(t) - \psi^{'}(t) \;=\; 1 + (p+3)t^{-(p+2)} > 0.$$

Now, we provide that (6.9) holds. For this purpose, let $0 < t \leq 1$, we have:

$$t\psi^{''}(t) + \psi^{'}(t) \;=\; 4t + (p+1)t^{-(p+2)} - 1 > (p+1) - 1 = p > 0.$$

On the other hand, for $t > 1$, we have:

$$\begin{aligned} t\psi^{''}(t) + \psi^{'}(t) &= 4t + (p+1)t^{-(p+2)} - 1 > 4 + (p+1)t^{-(p+2)} - 1, \\ &= 3 + (p+1)t^{-(p+2)} > 0. \end{aligned}$$

For (6.10), ψ check (6.7) and (6.8). Let $t > 1$ be considered

$$f(\beta) = \psi^{''}(t)\,\psi^{'}(\beta t) - \beta\psi^{'}(t)\,\psi^{''}(\beta t), \ \forall \beta > 1,$$

we have $f^{'}(\beta) > 0, \ \forall \beta > 1$ and $f(1) = 0$ then $f(\beta) > 0, \ \forall \beta > 1$. □

As a preparation for later, we present some technical results of our new kernel function.

Lemma 6.2. *For $\psi(t)$, we have*

$$(t-1)^2 \leq \psi(t) \leq \frac{1}{4}\left[\psi^{'}(t)\right]^2, \ t > 0. \tag{6.11}$$

$$\psi(t) \leq \frac{1}{2}\psi^{''}(1)\,(t-1)^2 = \frac{p+4}{2}\,(t-1)^2, \ t \geq 1. \tag{6.12}$$

Proof: For (6.11), using (6.5), we have:

$$\psi(t)=\int_1^t\int_1^x \psi''(y)\,dydx \geq \int_1^t\int_1^x 2dydx=(t-1)^2,$$

and

$$\begin{aligned}\psi(t) &= \int_1^t\int_1^x \psi''(y)\,dydx\\ &\leq \frac{1}{2}\int_1^t\int_1^x 2\psi''(y)\,dydx\\ &\leq \frac{1}{2}\int_1^t\int_1^x \psi''(x)\,\psi''(y)\,dydx\\ &\leq \frac{1}{2}\int_1^t \psi''(x)\,\psi'(x)\,dx=\frac{1}{4}\left[\psi'(t)\right]^2.\end{aligned}$$

For (6.12), using (6.6), (6.7) and Taylor's development, we have for some $\zeta\,(1\leq\zeta\leq t)$:

$$\begin{aligned}\psi(t) &= \psi(1)+\psi'(1)(t-1)+\frac{1}{2}\psi''(1)(t-1)^2+\frac{1}{6}\psi'''(\zeta)(\zeta-1)^3,\\ &\leq \frac{1}{2}\psi''(1)(t-1)^2=\frac{p+4}{2}(t-1)^2.\end{aligned}$$

□

Lemma 6.3. *Let σ and ρ, the inverse functions of $\psi(t), t\geq 1$ and $\frac{-1}{2}\psi'(t)$, as defined below, we have:*

$$1+\sqrt{\frac{2}{p+4}s}\leq\sigma(s)\leq 1+\sqrt{s},\, s\geq 0. \tag{6.13}$$

$$\rho(z)>\frac{1}{(2z+2)^{\frac{1}{p+2}}},\, z\geq 0. \tag{6.14}$$

Proof: For (6.13), let $\psi(t)=s$, $t\geq 1$, i.e., $t=\sigma(s)$. By (6.11), we have $(t-1)^2\leq\psi(t)=s$ this implies that $t=\sigma(s)\leq 1+\sqrt{s}$. By (6.12), we have $s=\psi(t)\leq\frac{p+4}{2}(t-1)^2, t\geq 1$, this implies $1+\sqrt{\frac{2}{p+4}s}\leq\sigma(s)=t$.

For (6.14), let $z=\frac{-1}{2}\psi'(t)$, $t\in]0,1]$, i.e., $\rho(z)=t$. By (6.2), we have:

$$z=\frac{-1}{2}\left(2t-1-t^{-(p+2)}\right)>\frac{1}{2}(-2+t^{-(p+2)}),$$

this implies that

$$t=\rho(z)>\frac{1}{(2z+2)^{\frac{1}{p+2}}},\, z\geq 0.$$

□

Lemma 6.4. *Let $0<\theta<1, v^+=\frac{v}{\sqrt{1-\theta}}$. If $\Phi(v)\leq\tau$, then we have:*

$$\Phi(v^+)\leq\frac{(\tau+2\sqrt{n\tau}+\theta n)}{(1-\theta)}. \tag{6.15}$$

Proof: By the Theorem 4.1 with $\beta = \frac{1}{\sqrt{1-\theta}} > 1$, (6.13) and $\psi(t) \leq t^2 - 1, \forall t \geq 1$, we obtain:

$$\begin{aligned}
\Phi(v^+) &\leq n\psi\left(\frac{\sigma\left(\frac{\Phi(v)}{n}\right)}{\sqrt{1-\theta}}\right) \leq n\psi\left(\frac{\sigma\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}\right), \\
&\leq n\left(\left(\frac{\sigma\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}\right)^2 - 1\right) = \frac{n}{(1-\theta)}\left(\left(\sigma\left(\frac{\tau}{n}\right)\right)^2 - (1-\theta)\right), \\
&\leq \frac{n}{(1-\theta)}\left(\left(1 + \sqrt{\left(\frac{\tau}{n}\right)}\right)^2 - (1-\theta)\right), \\
&\leq \frac{n}{(1-\theta)}\left(\frac{\tau}{n} + 2\sqrt{\left(\frac{\tau}{n}\right)} + \theta)\right) = \frac{(\tau + 2\sqrt{n\tau} + \theta n)}{(1-\theta)}.
\end{aligned}$$

□

Corollary 6.5. *For any positive vector v, if $\Phi(v) \leq \tau$ and $\beta > 1$, we have:*

$$\Phi(\beta v) \leq \frac{n}{2}\psi''(1)\left(\beta\sigma\left(\frac{\tau}{n}\right) - 1\right)^2.$$

Proof: For any positive vector v, If $\Phi(v) \leq \tau$ and $\beta > 1$ then by the theorem 4.1, we have:

$$\Phi(\beta v) \leq n\psi(\beta\sigma(\frac{\Phi(v)}{n})) \leq n\psi(\beta\sigma(\frac{\tau}{n})).$$

ψ satisfies (6.12) and $\beta\sigma(\frac{\tau}{n}) > 1$ then $\psi(\beta\sigma(\frac{\tau}{n})) \leq \frac{\psi''(1)}{2}\left(\beta\sigma(\frac{\tau}{n}) - 1\right)^2$.

So $\Phi(\beta v) \leq \frac{n}{2}\psi''(1)\left(\beta\sigma\left(\frac{\tau}{n}\right) - 1\right)^2$. □

By the corollary 6.5, with $\beta = \frac{1}{\sqrt{1-\theta}} > 1$ and (6.13), we obtain:

$$\begin{aligned}
\Phi(v^+) &\leq \frac{n}{2}\psi''(1)\left(\frac{\sigma\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}} - 1\right)^2, \\
&= \frac{n(p+4)}{2(1-\theta)}\left(\sigma\left(\frac{\tau}{n}\right) - \sqrt{1-\theta}\right)^2, \\
&\leq \frac{n(p+4)}{2(1-\theta)}\left(1 + \sqrt{\left(\frac{\tau}{n}\right)} - \sqrt{1-\theta}\right)^2, \\
&\leq \frac{n(p+4)}{2(1-\theta)}\left(\theta + \sqrt{\left(\frac{\tau}{n}\right)}\right)^2, \ (1 - \sqrt{1-\theta} \leq \theta, \ 0 < \theta < 1), \\
&= \frac{(p+4)}{2(1-\theta)}\left(\theta\sqrt{n} + \sqrt{\tau}\right)^2.
\end{aligned}$$

So

$$\Phi(v^+) \leq \frac{(p+4)}{2(1-\theta)}\left(\theta\sqrt{n} + \sqrt{\tau}\right)^2. \tag{6.16}$$

Lemma 6.6. *Let $\delta(v)$ be as defined in (2.9). Then we have:*

$$\delta(v) \geq \sqrt{\Phi(v)}. \tag{6.17}$$

Proof: Using (6.11), we have:

$$\Phi(v) = \sum_{i=1}^{n}\psi(v_i) \leq \sum_{i=1}^{n}\frac{1}{4}\left[\psi'(v_i)\right]^2 = \frac{1}{4}\| \bigtriangledown \Phi(v)\|^2 = \delta(v)^2,$$

so $\delta(v) \geq \sqrt{\Phi(v)}$. □

Remark 6.7. *Throughout the paper, we assume that $\tau \geq 1$. Using Lemma 6.6, the assumption that $\Phi(v) \geq \tau$, we have $\delta(v) \geq 1$.*

Lemma 6.8. *Let $\bar{\alpha}$ be that defined in Lemma 4.4. If $\Phi = \Phi(v) \geq \tau \geq 1$, so we have:*

$$\bar{\alpha} \geq \frac{1}{2 + (p+2)(4\delta + 2)^{\frac{p+3}{p+2}}}.$$

Proof: Using Lemma 4.5, we have $\bar{\alpha} \geq \frac{1}{\psi''(\rho(2\delta))}$ and (6.3) so:

$$\bar{\alpha} \geq \frac{1}{\psi''\left(\rho(2\delta)\right)} = \frac{1}{2 + (p+2)\left(\rho(2\delta)\right)^{-(p+3)}},$$

we have also according to (6.14) :

$$\bar{\alpha} > \frac{1}{2 + (p+2)\left(4\delta + 2\right)^{\frac{p+3}{p+2}}}.$$

□

Denoting:

$$\tilde{\alpha} = \frac{1}{2 + (p+2)\left(4\delta + 2\right)^{\frac{p+3}{p+2}}}. \tag{6.18}$$

$\tilde{\alpha}$ is the step of displacement and $\tilde{\alpha} \leq \bar{\alpha}$.

Lemma 6.9. *For the displacement step, defined in (6.18), and taking $\Phi(v) \geq 1$. So:*

$$f(\tilde{\alpha}) < \frac{-\Phi^{\frac{p+1}{2(p+2)}}}{36(p+4)}. \tag{6.19}$$

Proof: Using Lemma 4.6 with $\alpha = \tilde{\alpha}$ and (6.17), we obtain:

$$\begin{aligned}
f(\tilde{\alpha}) &\leq \frac{-\delta^2}{2 + (p+2)(4\delta + 2)^{\frac{p+3}{p+2}}}, \\
&< \frac{-\delta^2}{(p+4)(4\delta + 2)^{\frac{p+3}{p+2}}}, \\
&\leq \frac{-\delta^2}{(p+4)\left(6\delta\right)^{\frac{p+3}{p+2}}}, \delta \geq 1, \\
&\leq \frac{-\Phi^{\frac{p+1}{2(p+2)}}}{36(p+4)}.
\end{aligned}$$

□

6.2. Complexity of algorithm

Our aim is to compute iteration bounds for large and small-update methods based on our new kernel function. Large-update methods are characterized by $\tau = O(n)$ and $\theta = \Theta(1)$ and small-update methods are characterized by $\tau = O(1)$ and $\theta = \Theta(\frac{1}{\sqrt{n}})$.

Using (6.19) and (5.1), we have:

$$\gamma = 1 - \frac{p+1}{2(p+2)} = \frac{p+3}{2(p+2)}, \qquad \bar{\kappa} = \frac{1}{36(p+4)}.$$

According to (5.2), we obtain an upper bound for the total number of iterations by the following equation:

$$\frac{72(p+2)(p+4)}{(p+3)} (\Phi)_0^{\frac{p+3}{2(p+2)}} \frac{\log \frac{n}{\varepsilon}}{\theta}. \tag{6.20}$$

Using (6.15) for large-update methods, we have $(\Phi)_0 = O(n)$. We distinguish the two cases:
– If $p \in [1, +\infty[$, we have $O\left(pn^{\frac{p+3}{2(p+2)}} \log \frac{n}{\varepsilon}\right)$ iterations complexity.
– If $p \in]0, 1]$, we have $O\left(n^{\frac{3}{4}} \log \frac{n}{\epsilon}\right)$ iterations complexity.

Using (6.16) for small-update methods, we distinguish the two cases:

– If $p \in [1, +\infty[$, we have $O\left(p^2\sqrt{n} \log \frac{n}{\varepsilon}\right)$ iterations complexity such that $(\Phi)_0 = O(p)$.
– If $p \in]0, 1]$, we have $O\left(\sqrt{n} \log \frac{n}{\varepsilon}\right)$ iterations complexity such that $(\Phi)_0 = O(1)$.

If we take $p = \frac{\log n}{2} - 2$, we obtain the best know complexity bound for large-update methods namely $O\left(\sqrt{n} \log n \log \frac{n}{\varepsilon}\right)$ which is the minimum of $O\left((p+2)n^{\frac{p+3}{2(p+2)}} \log \frac{n}{\varepsilon}\right)$ iterations complexity.

7. Numerical results

To prove the effectiveness of our new kernel function and evaluate its effect on the behavior of the algorithm, we conduct a numerical comparative tests with the kernel function 1 and 2. Our new kernel function is noted by B.

The generic primal-dual algorithm described in section 2 was implemented in MATLAB, we have taken $\varepsilon = 10^{-4}, \mu^0 = 1, \theta \in \{0.01, 0.9\}, p = 1$ for our new kernel function, $q = 2$ for the kernel function 1 and 2, $\tau = 5n$ and $\tau = 10n$.
We choose a step size α, satisfy $0 < \alpha < \bar{\alpha}$: we take, respectively,
$\alpha_1 = \frac{1}{1+q(4\delta+1)^{\frac{q+1}{q}}}, \alpha_2 = \frac{1}{q(4\delta+1)^{\frac{q+1}{q}}}, \alpha_B = \frac{1}{2+(p+2)(4\delta+2)^{\frac{p+3}{p+2}}}$, which match with the notation of precedent kernel functions.
In the tables of results: m is the number of constraints, n is the number of variables, $Itr(A)$ and $(time\, A(s))$ represent the number of inner iterations and the computation time by second respectively, to obtain the optimal solution using the kernel function A. Examples are stated under the following form:

$$\begin{cases} \min c^t x + \frac{1}{2} x^t Q x, \\ Ax = b, \\ x \geq 0, \end{cases}$$

Example 1

$$Q = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, c = \begin{pmatrix} -2 \\ -4 \\ 0 \end{pmatrix}, A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The initial strictly feasible interior point is:
$x^0 = \begin{pmatrix} 0.3262 & 1.3261 & 0.3477 \end{pmatrix}^t, y^0 = \begin{pmatrix} 0 & -2.0721 \end{pmatrix}^t,$
$z^0 = \begin{pmatrix} 0.7247 & 0.7247 & 2.0722 \end{pmatrix}^t.$
The obtained primal-dual optimal solution is:
$x^* = \begin{pmatrix} 0.4999 & 1.4998 & 0.0003 \end{pmatrix}^t, y^* = \begin{pmatrix} 0 & -1 \end{pmatrix}^t.$

Example 2

$$Q = \begin{pmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, c = \begin{pmatrix} -4 \\ -6 \\ 0 \\ 0 \end{pmatrix}, A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

The initial strictly feasible interior point is:

$x^0 = \begin{pmatrix} 0.9683 & 0.5775 & 0.4543 & 1.1444 \end{pmatrix}^t, y^0 = \begin{pmatrix} -0.9184 & -1.1244 \end{pmatrix}^t,$
$z^0 = \begin{pmatrix} 0.7612 & 0.9141 & 0.9185 & 1.1244 \end{pmatrix}^t.$
The obtained primal-dual optimal solution is:
$x^* = \begin{pmatrix} 1.1288 & 0.7742 & 0.0971 & 0.0003 \end{pmatrix}^t, y^* = \begin{pmatrix} -0.0017 & -1.0318 \end{pmatrix}^t.$

Example 3

$$Q = \begin{pmatrix} 20 & 1.2 & 0.5 & 0.5 & -1 \\ 1.2 & 32 & 1 & 1 & 1 \\ 0.5 & 1 & 14 & 1 & 1 \\ 0.5 & 1 & 1 & 15 & 1 \\ -1 & 1 & 1 & 1 & 16 \end{pmatrix}, c = \begin{pmatrix} 1 \\ -1.5 \\ 2 \\ 1.5 \\ 3 \end{pmatrix}, A = \begin{pmatrix} 1 & 1.2 & 1 & 1.8 & 0 \\ 3 & -1 & 1.5 & -2 & 1 \\ -1 & 2 & -3 & 4 & 2 \end{pmatrix},$$

$b = \begin{pmatrix} 9.31 & 5.45 & 6.60 \end{pmatrix}^t.$
The initial strictly feasible interior point is:
$x^0 = \begin{pmatrix} 2.4539 & 0.7875 & 1.5838 & 2.4038 & 1.3074 \end{pmatrix}^t, y^0 = \begin{pmatrix} 20.5435 & 9.4781 & 4.3927 \end{pmatrix}^t,$
$z^0 = \begin{pmatrix} 7.1215 & 7.9763 & 8.3150 & 6.8686 & 7.9750 \end{pmatrix}^t.$
The obtained primal-dual optimal solution is:
$x^* = \begin{pmatrix} 2.6321 & 0.7019 & 1.3994 & 2.4643 & 1.0846 \end{pmatrix}^t, y^* = \begin{pmatrix} 25.2686 & 11.7725 & 5.2567 \end{pmatrix}^t.$

Example 4

$$Q = \begin{pmatrix} 30 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 21 & 0 & 1 & -1 & 1 & 0 & 1 & 0.5 & 1 \\ 1 & 0 & 15 & -0.5 & -2 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & -0.5 & 30 & 3 & -1 & 1 & -1 & 0.5 & 1 \\ 1 & -1 & -2 & 3 & 27 & 1 & 0.5 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 16 & -0.5 & 0.5 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0.5 & -0.5 & 8 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 0.5 & 1 & 24 & 1 & 1 \\ 1 & 0.5 & 1 & 0.5 & 1 & 0 & 1 & 1 & 39 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 11 \end{pmatrix}, c = \begin{pmatrix} -0.5 \\ -1 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ -1 \\ -0.5 \\ -1 \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & -1 & 1.9 & 1.25 & 1.2 & 0.4 & -0.7 & 1.06 & 1.5 & 1.05 \\ 1.3 & 1.2 & 0.15 & 2.15 & 1.25 & 1.5 & 0.4 & 1.52 & 1.3 & 1 \\ 1.5 & -1.1 & 3.5 & 1.25 & 1.8 & 2 & 1.95 & 1.2 & 1 & -1 \end{pmatrix}, b = \begin{pmatrix} 11.651 \\ 16.672 \\ 21.295 \end{pmatrix}.$$

The initial strictly feasible interior point is:
$x^0 = \begin{pmatrix} 0.949 & 0.612 & 1.847 & 1.811 & 1.251 & 2.521 & 1.506 & 1.565 & 0.820 & 1.128 \end{pmatrix}^t,$
$y^0 = \begin{pmatrix} 4.3800 & 19.9367 & 4.5679 \end{pmatrix}^t,$
$z^0 = \begin{pmatrix} 3.890 & 4.462 & 3.978 & 3.660 & 3.901 & 3.556 & 3.876 & 3.719 & 3.913 & 4.339 \end{pmatrix}^t.$
The obtained primal-dual optimal solution is:
$x^* = \begin{pmatrix} 0.963 & 0.509 & 1.739 & 1.904 & 1.243 & 2.626 & 1.322 & 1.617 & 0.824 & 0.897 \end{pmatrix}^t,$
$y^* = \begin{pmatrix} 4.2429 & 22.3605 & 5.1916 \end{pmatrix}^t.$

Example 5

$$n = 2m,\ A(i,j) = \begin{cases} 0 \text{ If } i \neq j \text{ and } j \neq i+m \\ 1 \text{ If } i = j \text{ and } j = i+m \end{cases}$$

$c(i) = -1, c(i+m) = 0, b(i) = 2$, for $i = 1, ..., m$. $Q(i,j) = 0$, for $i, j = 1, ..., n$.
The initial strictly feasible interior point is:
$x^0(i) = x^0(i+m) = 1, y^0(i) = -2, z^0(i) = 1, z^0(i+m) = 2$ for $i = 1, ..., m$.

We summarize the numerical results for the examples 1, 2, 3, 4 in Tables 2, 3 and the example 5 in the Tables 4 and 5.

Table 2: A numerical results for $\theta = 0.01$ and $\tau = 10n$.

Example	$Itr(1)$	$time\,1(s)$	$Itr(2)$	$time\,2(s)$	$Itr(B)$	$time\,B(s)$
1	987	2.41	1180	2.59	901	2.26
2	1289	3.63	1617	4.04	1141	3.36
3	2156	6.5	2952	7.87	1836	5.81
4	3493	18.55	4656	22.59	2794	15.55

Table 3: A numerical results for $\theta = 0.9$ and $\tau = 10n$.

Example	$Itr(1)$	$time\,1(s)$	$Itr(2)$	$time\,2(s)$	$Itr(B)$	$time\,B(s)$
1	2804	4.19	2739	3.97	2275	3.50
2	3520	6.64	3604	6.64	2800	5.05
3	5721	13.16	6056	13.63	4475	10.64
4	8739	38.91	8470	36.62	6269	28.72

Table 4: Comparison of examples for $\theta = 0.01$ and $\tau = 5n$.

(m, n)	$Itr(1)$	$time\,1(s)$	$Itr(2)$	$time\,2(s)$	$Itr(B)$	$time\,B(s)$
(10,20)	3724	38.92	3999	39.80	3027	32.65
(20,40)	6597	155.97	6881	144.98	5027	115.37
(30,60)	9215	330.57	9488	309.12	6768	249.33
(40,80)	11687	590.71	11943	553.57	8364	437.29
(50,100)	14046	898.96	14280	881.86	9852	662.63
(75,150)	19612	2489.01	19786	3534.63	13269	1719.50
(100,200)	24860	7564.33	24973	7914.69	16397	4496.71

Table 5: Comparison of examples for $\theta = 0.9$ and $\tau = 5n$.

(m, n)	$Itr(1)$	$time\,1(s)$	$Itr(2)$	$time\,2(s)$	$Itr(B)$	$time\,B(s)$
(10,20)	11219	98.97	8884	76.30	8047	75.36
(20,40)	18732	376.05	14134	293.94	12658	276.26
(30,60)	25314	790.31	18675	656.45	16526	548.84
(40,80)	31352	1412	22818	1070.31	19973	983.69
(50,100)	37018	2229.21	26691	1650.28	23141	1530.53
(75,150)	60020	7418.47	43962	5430.86	36149	4914.11

Comments

The realized numerical experiments show the effectiveness of our new efficient kernel function. We note that when the dimension of the problem becomes large, the difference between our new kernel function, that of Peng et al. [9] and that of Bai et al. [2] becomes large in terms of number of inner iteration and computation time. These numerical results consolidate and confirm our theoretical results.

8. Conclusion

In this paper, we proposed a new kernel function consisting of a polynomial function in its barrier term defined by (6.1) for the primal-dual interior point methods for convex quadratic programming. A simple analysis for the primal-dual *IPMs* based on the proximity function induced by the new kernel function is provided. The proposed kernel function is not logarithmic and not self-regular. We proved that the iteration bound of interior point method based on this function for large-update method is

$O\left(pn^{\frac{p+3}{2(p+2)}}\log\frac{n}{\varepsilon}\right)$ iterations complexity if $p\geq 1$ and $O\left(n^{\frac{3}{4}}\log\frac{n}{\varepsilon}\right)$ if $p\in]0,1]$. This bound improves significantly the results obtained by Peng et al. in [9] and Bai et al. in [2]. These results are important contributions to improve the computational complexity of the studied problem to the best knowledge. We are the first to reach the best known complexity for large-update with a polynomial barrier term.

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